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ALMOST CONSTANT SEQUENCES

by

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Rauzy has characterized all real sequences (c_n) such that for any uniformly distributed sequence (x_n) the sequence $(c_n + x_n)$ is again u.d. modulo 1, [6]. A new proof of this result was given in [10], together with a generalization to uniform distribution in compact metric groups and to R^n. The aim of this paper is to consider the corresponding questions for locally compact Abelian groups with respect to several concepts of uniform distribution. Our methods admit also generalizations to the non-Abelian case. For general references we refer to [1]. For other generalizations see [7] Ch. IV and [2].

1. Definition 1: *If G is a locally compact group let M(G) be the set of (Hartman) uniformly distributed sequences (x_n) in G, i.e.:*

$$\lim_{N\to\infty} N^{-1} \sum_{n\leq N} U(x_n) = 0$$

hold for all non-trivial irreducible finite-dimensional continuous unitary representations U of G.

Definition 2: a) $C(G) = \{(c_n) : x_n \in M(G) \Rightarrow (c_n x_n) \in M(G)\}$

b) $C_o(G) = \{(c_n) : \exists a > 1$ *such that* $c_n = c_m$ *if* $a^k \leq n,m < a^{k+1}, k = 0,1,2,\ldots\}$

If G is metrizable let $d(x,y)$ be a bounded left-invariant metric on G and define for arbitrary sequences

$$(x_n),(y_n) : g(x_n,y_n) = \overline{\lim}\, N^{-1} \sum_{n\leq N} d(x_n,y_n)$$

If A is a family of sequences, A^- will denote the "closure" with respect to the pseudo-metric g. If G is non-metrizable consider the family of all left-invariant pseudo-metrics d_i, $i \in I$, the according pseudo-metrics g_i, $i \in I$ induce a topology on the space of all sequences on G.

Theorem 1: If G is a locally compact separable abelian group then

$$C(G) = C_o(G)^-$$

Remarks: If G is not separable, i.e. there exists no countable dense subset it can easily be seen that $M(G)$ is void. For metric groups Th. 1 has been announced in [11], but our methods admit generalizations to non-Abelian groups.

Proof:

Lemma 1: Let $h: G \to H$ be a continuous group homomorphism, G l.c. separable, H compact metric, such that $h(G)^- = H$. If $(y_n) \in M(H)$ then there exists $(x_n) \in M(G)$ such that $\lim d_H(h(x_n), y_n) = 0$ (d_H = metric on H).

Remarks: If h is surjective it is possible to achieve $h(x_n) = y_n$ if G is metric and abelian or compact, [8], in the compact case it is sufficient that G is separable, [3]; if H is only separable even for $G = H \times Z_2$, $Z_2 = \{-1, 1\}$ this is no longer true in general, [4], and open even in the case $G = R \times Z_2$, $H = R$. For $G = Z$, $H = R/Z$, $h(z) = za$, a irrational the lemma above cannot be strengthened (e.g.: $(h^{-1}(2az)) = (2z)$ is not u.d. in Z).

Proof: Let (z_n) be an arbitrary u.d. sequence in G (which exists, [9], Th. 1). For $k = 1, 2, \ldots$ let (V_{ik}) $(i = 1, 2, \ldots, m_k)$ be a partition of H into sets of diameter less than $1/k$ such that the boundary of each V_{ik} has measure 0 and each V_{ik} has positive measure. Put λ for the normalized Haarmeasure on H and C_V for the characteristic function of a set V. We can construct a sequence of indices (N_k) such that $N_{k+1} \geq 2N_k$ and for all $N \geq N_k$ and $i \leq m_k$:

$$\left| N^{-1} \sum_{n \leq N} C_{V_{ik}}(y_n) - \lambda(V_{ik}) \right| < 1/2^k m_k$$

and

$$|N^{-1}\sum_{n\le N} c_{V_{ik}}(h(z_n))-\lambda(V_{ik})|<1/2^k m_k$$

By induction we define a map p: $N \to N$ in the following way: If $n<N_1$ put $p(n)=n$. If $N_k\le n<N_{k+1}$, $y_n\in V_{ik}$, let $p(n)$ be the smallest number m such that $h(z_m)\in V_{ik}$ and $m\neq p(s)$ for $s<n$. We put $x_n=z_{p(n)}$. Since $d_H(h(z_{p(n)}),y_n)<1/k$ for $n\ge N_k$ we have $\lim d_H(h(x_n),y_n)=0$.

Let $|A|$ be the number of elements of a finite set A, put $D_p(N)=$ $=|p([1,N])\setminus[1,N]|$. If $N>N_k$, it is easily seen that

$$D_p(N)\le D_p(N_k)+\sum_{i\le m_k}|\sum_{N_{k+1}\le n\le N} c_{V_{ik}}(y_n)-c_{V_{ik}}(h(z_n))|<D_p(N_k)+$$
$$+4Nm_k/2^k m_k = D_p(N_k)+N/2^{k-1}.$$

In particular we get by induction:

$$D_p(N_k)\le \sum_{2\le j\le k} N_j/2^{j-3}\le N_k\cdot k/2^{k-3}$$

and consequently $D_p(N)\le N(k+2)2^{k-3}=o(N)$.

Since p is by definition injective, we have $|[1,N]\setminus p([1,N])|=D_p(N)$, i.e. the symmetric difference between the two sets is $2D_p(N)=o(N)$. It follows immediately that the sequence $(x_n)=(z_{p(n)})$ is u.d. like the sequence (z_n). q.e.d.

As any representation U (in Def. 1) is a homomorphism into a compact group and because the homomorphic image of a u.d. sequence is u.d. in the closure of the image (this follows easily from the Definition of u.d.) we obtain

<u>Proposition 1</u>: $(c_n)\in C(G)$ <u>iff</u> $U(c_n)\in U(G)^-$ <u>for any representation</u> U.

Remarks: If G is Abelian, the U's are just the elements of $\hat{G}$ the character group of G; we have $U(G)\subseteq T$ the 1-dimensional torus.

In order to prove that $(c_n) \in C(G)$ it is sufficient to know that $(U(c_n)) \in C(T)$ for all U from a subset of $\hat{G}$ which separates the points of G and is either non meagre or has positive measure. This follows from the observation that C(T) is a group and consequently the set of all U for which $(U(c_n)) \in C(T)$ a subgroup of $\hat{G}$. As a consequence of Proposition 1 we obtain also for non-compact H:

Corollary: *If* $h: G \to H$ *is a continuous epimorphism,* $(c_n) \in C(G)$ *then* $(h(c_n)) \in C(H)$.

In order to prove Theorem 1 it suffices to consider metric groups. (G is the projective limit of metric (even Lie-)groups: G/N_i, $i \in I$, and by the corollary we have for $(c_n) \in C(G)$ and $h_i: G \to G/N_i$: $(h_i(c_n)) \in C(G/N_i)$, and the metrics d_i from G/N_i determine the topology on G).

In order to prove Theorem 1 it is sufficient to prove the following Lemma (see [10], esp. Lemma 4).

Lemma 2: *If* $(c_n) \in C(G)$ *then* $\lim N^{-1} \cdot \sum_{n \leq N} d(c_n, c_{n+1}) = 0$

Proof: For any $U \in \hat{G}$ we have by Prop. 1 that $U(c_n) \in C(U(G)^-)$. As Th. 1 is known for $U(G)^-$ (already proved by Rauzy, see also [10]) it follows that

$$\lim N^{-1} \sum_{n \leq N} |U(c_n) - (c_{n+1})| = 0 \text{ for all } U \in \hat{G}.$$

Take $\varepsilon > 0$ and put $W = \{x \in G: d(x,e) < \varepsilon\}$ W is an open neighbourhood of the unit element of G. If V is an open symmetric neighbourhood of e with $V + V \subseteq W$ then $f = (1/\lambda(V))\, C_V * C_V$ is a positive definite continuous function, satisfies $f(e) = 1$ and vanishes outside W. By Bochner's theorem there exists a probability measure μ on $\hat{G}$ such that

$$f(x) = \int U(x)\,d\mu(U) \text{ for all } x \in G.$$

By Lebesgue's dominated convergence theorem we conclude that

$$\lim N^{-1} \sum_{n\leq N} |f(-c_{n+1}+c_n)-1| \leq \lim \int N^{-1} \sum_{n\leq N} |U(-c_{n+1}+c_n)-1|\, d\mu(U) = 0.$$

It follwos that the set $\{n: -c_{n+1}+c_n \notin W\} = \{n: d(c_n,c_{n+1}) \geq \epsilon\}$ has density 0 in N. Since d is bounded we get $\overline{\lim}\ 1/N \cdot \sum_{n\leq N} d(c_n,c_{n+1}) < \epsilon$ for all $\epsilon > 0$. q.e.d.

Remarks: Theorem 1 holds also for non-abelian groups with the property that all irreducible unitary representations are finite dimensional (same proof). If the finite dimensional unitary representations U do not separate points of G, i.e. if there exists $x \neq e$ such that $U(x) = U(e)$ for all U, then the sequence e,x,e,x,e,x,... belongs to C(G) but not to $C_o(G)^-$.

Denote by S_3 the symmetric group and by A_3 the alternating group of 3 elements and consider the discrete group G of all sequences $g = g^{(i)}$, $i = 1,2,\ldots, g^{(i)} \in S_3$, $g^{(i)} \neq e$ for at most finitely many i. If U is a fine-dimensional representation of G, denote by $U^{(i)}$ the restriction to the i-th component. From the fact the $U^{(i)}$ commutes with $U^{(j)}$ $i \neq j$ it can be derived that $U^{(i)}$ restricted to A_i is the trivial representation for all but at most finitely many i ($\leq \dim U/2$). It follows that the sequence $(c_n) : c_n^{(i)} = e$ if $i \neq n$, $c_n^{(n)} = (123) \in A_3$ belongs to C(G) but not to $C_o(G)^-$. Nevertheless G has a separating family of 2-dimensional representations.

2. Now we want to consider concepts of uniform distribution connected with infinite dimensional representation. For $1 \leq p < \infty$ put $L^p(G) = \{f: (\int |f(x)|^p\, dx)^{1/p} = \|f\|_p < \infty\}$, "dx" denotes a left Haar measure. Consider the left regular representation

$$f \to L_y f, \quad L_y f(x) = f(y^{-1}x), \quad x,y \in G$$

Definition 3: A sequence (x_n) in G is called L^p-uniformly distributed if $\|N^{-1}\cdot \sum_{n\leq N} L_{x_n} f\|_p \to 0$ for all $f\in L^1\cap L^p$ with $\int f=0$. $M(L^p(G))$ shall denote the set of all L^p-u.d. sequences.

Definition 4: $C(L^p(G))=\{(c_n):(x_n)\in M(L^p(G)) \Rightarrow (c_n x_n)\in M(L^p(G))$

Remark: It is known that for compact groups $M(G)=M(L^p(G))$ for all p and $M(L^1(G)\subseteq M(G)$ in general, [9]. $M(L^1(G)\neq\emptyset$ if and only if G is amenable and separable (see [9] and [5]).

Theorem 2: If G is a locally compact separable abelian group then $C(L^1(G))=C(G)$.

Proof: The same proof as in Lemma 1 shows the following result: if $h: G\to H$ is a continuous group homomorphism onto a dense subgroup of H, and if $(y_n)\in M(H)$ then there exists $(x_n)\in M(L^1(G))$ such that $d_H(h(x_n),y_n)\to 0$ (use the relation $M(L^1(G))\subseteq M(G)$). It follows that $(c_n)\in C(L^1(G))$ implies $(U(c_n))\in C(U(G)^-)$ for all $U\in\hat{G}$ and by Proposition 1 that $(c_n)\in C(G)$. On the other hand it follows easily from the characterization of C(G) in Theorem 1 that $C(G)\subseteq C(L^1(G))$. q.e.d.

Remarks: Again the result can be extended to groups having only finite dimensional representations which are known to be amenable. $C(L^1(G)\subseteq C(G)$ for any separable amenable group (same proof as above). In the general non-abelian case several pathologies may appear:

If $G=P_1$ the projective group then $M(L^1(P_1))=\emptyset$. P_1 is not amenable, $M(P_1)=P_1^{\mathbb{N}}$ the set of all sequences as P_1 is minimal almost periodic i.e. has no non-trivial finite dimensional unitary representation. We have $C(L^1(P_1))=C(P_1)=P_1^{\mathbb{N}}$!

Let G be the group of all permutations p of an uncountable set such that $E_p=\{x:p(x)\neq x\}$ is finite then it is known that G is amenable and minimal almost periodic. We have again $M(L^1(G)) \neq \emptyset$ (G is not separable), $M(G) = G^N$, $C(L^1(G)) = C(G) = G^N$. Replacing G by $G \times Z_2$ we obtain $C(L^1(G \times Z_2)) \neq C(G \times Z_2)$.

If G is the group of affine transformations of the line it can be shown that $C(L^1(G))$ contains a sequence (c_n) such that $(x_n c_n) \notin M(L^1(G))$ for some $(x_n) \in M(L^1(G))$, [9], Satz 9.

It can be shown that for a connected separable l.c. group G the set $C(L^1(G))$ coincides with $C(G)$ if and only if either $G \cong R^n \times K$, K compact or G is non amenable and minimal almost periodic. The essential part of the proof is the classical theorem of Freudenthal which implies that the only connected groups such that the finite dim. unitary representations separate points of G are isomorphic to $R^n \times K$.

$G = R$, $G = Z$ are typical for the abelian non-compact case. It should be noted that in both cases $M(L^1(G))$ is a proper subset of $M(G)$. It is already a consequence of results of Weyl that the sequence $(\sqrt{2}n^2+n)$ is u.d. in R and the according sequence of integers (z_n)

$$(z_n < \sqrt{2}\, n^2 + n < z_n + 1)$$

is u.d. in Z. Therefore Theorem 2 does not follow from Theorem 1.

3. In this section we study the case of L^2-uniform distribution which is quite different from the preceding two cases. The results hold for arbitrary locally compact groups ($M(L^p(G) = M(G)$ if G is compact as mentioned above). For non-compact groups $L^p \cap L^1(G)$ is dense in $L^p(G)$ and we can choose any $f \in L^p(G)$ in Definition 3 ($p > 1$).

Theorem 3: Let (x_n) be a sequence in G (non-compact), K a compact subset of G with nonempty interior. The following statements are equivalent:

a) (x_n) is L^2-uniformly distributed

b) (x_n) is L^p-u.d. for some p with $1 < p < \infty$

c) (x_n) is L^p-u.d. for all p with $1 < p < \infty$

d) For any $\epsilon > 0$ there exists $N_o(\epsilon)$ such that

$$1/N\,|\{n:\ 1 \leq n \leq N,\ x_n \in xK\}| < \epsilon \text{ for all } x \in G,\ N \geq N_o(\epsilon)$$

Proof: Assume a), b) or c) holds and that $1/N|\{n:\ 1 \leq n \leq N, x_n \in xK\}| \geq \epsilon$ for arbitrary large N. Put $f = C_{K^{-2}} \in L^p(G)(1 < p < \infty)$. If $x_n \in xK$ then $x_n^{-1} y \in K^{-2}$ for all $y \in xK^{-1}$. It follows that $1/N \sum_{n \leq N} L_{x_n} f \geq \epsilon$ on xK^{-1}. Since f is non-negative we conclude that

$$\|1/N \sum_{n\leq N} L_{x_n} f\|_p \geq \epsilon\ (\lambda(K^{-1}))^{1/p}$$

which contradicts the assumption that (x_n) is L^p-u.d. (note the remark before Th. 3).

Now we assume that d) holds and we will prove c). It is easily seen that d) can be extended to arbitrary compact sets with non-empty interior. Put $f = C_{K^{-1}}$. If $x_n^{-1} \cdot y \in K^{-1}$ then $x_n \in yK$. It follows that

$$1/N \sum_{n\leq N} L_{x_n} f(y) = 1/N|\{n: 1 \leq n \leq N,\ x_n \in yK| < \epsilon \text{ for } N \geq N_o(\epsilon).$$

Consequently: $\|1/N \sum_{n\leq N} L_{x_n} f\|_p \leq \epsilon^{(p-1)/p} \|1/N \sum_{n\leq N} L_{x_n} f\|_1^{1/p} =$

$$= \epsilon^{(p-1)/p}\,\lambda(K^{-1})^{1/p}.$$

This shows that $\lim \|1/N \sum_{n\leq N} L_{x_n} f\|_p = 0$ for all $f = C_{K^{-1}}$. Since these functions generate a dense subspace of $L^p(G)$ it follows that (x_n) is L^p-u.d.

The first two theorems have shown that (for abelian groups) C(G) and $C(L^1(G))$ consists of all sequences which are "almost constant" in a certain sense (closure of $C_o(G)$). Here the situation is quite different:

Definition 5: $C_f(G) = \{(c_n) \subseteq G, \exists b \in N: |\{c_n: 2^i < n \leq 2^i\}| \leq b$ for $i = 1,2,..$ If d_i, $i \in I$ is the family of all bounded left invariant metrics on G we can define the closure of $C_f(G)$ as before.

Theorem 4: For any non-compact l.c. group $C(L^2(G)) = C_f(G)^-$

Proof: It follows easily from Theorem 3 d) that $C_f(G) \subseteq C(L^2(G))$ and consequently $C_f(G)^- \subseteq C(L^2(G))$.

Conversely assume that (d_n) does not belong to the closure of $C_f(G)$, i.e. there exists $\delta > 0$ and a pseudo-metric g such that $g((c_n),(d_n)) \geq \delta$ for all $(c_n) \in C_f(G)$. Denote by d the corresponding pseudo-metric of G, let U be a compact neighbourhood of e which is contained in

$$\{x: d(x,e) < \delta/2\}$$

and choose a compact symmetric neighbourhood V such that $V^4 \subseteq U$. There exists a sequence (x_n) such that x_nU covers G and $x_n \notin x_iV^2$ for $i < n$. Take $b, i \in N$. Let $x_{i1}(b)$ be that element x_j such that

$$|\{n: 2^i < n \leq 2^{i+1}, d_n^{-1} \in x_jU\}|$$

is maximal. Similar choose $x_{i2}(b)$ in such a way that

$$|\{n: 2^i < n \leq 2^{i+1}, d_n^{-1} \in x_{i2}(b)U, d_n^{-1} \notin x_{i1}(b)U\}|$$

is maximal, and so on. In this way we get elements $x_{i1}(b),\dots,x_{ib}(b)$ for $i = 1,2,\dots$ Now define $c_n = x_{i1}(b)$ for those n such that $2^i < n \leq 2^{i+1}$, $d_n^{-1} \in x_{i2}(b)U$. Similarly define $c_n = x_{i2}(b)$ for those n such that $2^i < n \leq 2^{i+1}$, $d_n^{-1} \in x_{i2}(b)U$, $d_n^{-1} \notin x_{i1}(b)U$ and so on. Let I_1 be the set of indices for which c_n is defined as above. For $n \in I_2 = N \setminus I_1$ put $c_n = e$. Since we may assume that d is bounded by one we find that $\delta \leq \overline{\lim}_{N\to\infty} 1/N \sum_{n\leq N} d(c_n,d_n) < \delta/2 + \overline{\lim}\ 1/N\ |I_2 \cap [1,N]| = \delta/2 + \bar{d}(I_2)$ ($\bar{d}$ denotes the outer density). It follows that $\bar{d}(I_2) > \delta/2$. We make this construction for each b and find sets $I_2(b)$. Now choose an increasing sequence of indices $N_1 = 0 < N_2 < \dots$ such that $1/(N_{k+1} - N_k)\ |I_2(k) \cap (N_k, N_{k+1}]| > \delta/2$ and each N_k is of the form 2^{j_k}.

For $n \in I_2(b)$ with $N_b \leq 2^i < n \leq 2^{i+1} \leq N_{b+1}$ choose the smallest number $j \in N$ such that $d_n^{-1} \in x_j U$ and put $y_n = x_j$. If $x \in G$, $x_j, x_k \in xV$, then $j = k$ by the definition of the sequence (x_n). It follows that $\{n\colon 2^i < n \leq 2^{i+1},\ n \in I_2(b),\ y_n \in xV\} \subseteq \{n\colon 2^i < n \leq 2^{i+1}, n \in I_2(b), d_n^{-1} \in x_j U, d_n^{-1} \notin x_k U\ \forall k < j\}$. By construction of the x_j this set has at most $2^i/(b+1)$ elements. Put $\bar{I}_2 = \bigcup_{k=1}^{\infty} (N_k, N_{k+1}] \cap I_2(k)$. The preceding argument shows that for any $\gamma > 0$ and $N \geq N(\gamma)$ we have $1/N|\{n \leq N : n \in \bar{I}_2 : y_n \in xV\}| < \gamma$ for all $x \in G$. Now choose a sequence $\{y_n\colon n \in N \setminus \bar{I}_2\}$ such that the sets $\{y_n U\colon n \in N \setminus \bar{I}_2\}$ are pairwise disjoint. Then it is easily seen that the sequence (y_n) is L^2-u.d. in G. On the other hand $d_n y_n \in U^{-1}$ for $n \in \bar{I}_2$. Since $\bar{I}_2$ has positive outer density it follows that $(d_n y_n)$ is not L^2-u.d. in G, consequently $(d_n) \notin C(L^2(G))$. q.e.d.

Remarks: Theorem 3 implies that $M(L^2(G)) \neq \emptyset$ for any non compact group. $M(L^2(G)) \subseteq M(G)$ just for compact groups (then equality holds, also in the non-separable case (both sets are empty)). Denote by $N(G)$ the intersection of all kernels of finite dimensional unitary representations then we can show: If $M(G) \neq \emptyset$ then $M(G)$ is not a subset of $M(L^2(G))$ if and only if G is not compact and $G/N(G)$ is compact: If $G/N(G)$ is compact then replacing $(x_n) \in M(G)$ by (y_n) such that $x_n y_n^{-1} \in N(G)$ and that $(y_n) \subseteq K$ compact we have $(y_n) \in M(L^2(G)) \setminus M(G)$ if G is not compact (Th. 3). If G/N is not compact any compact set K in G has measure 0 considered as a subset of the Bohr compactification of G. Then it is easy to see that for $(x_n) \in M(G)$ $1/N \sum_{n \leq N} C_{xK}(x_n) \to 0$ uniformly in $x \in G$ i.e. $(x_n) \in M(L^2(G))$ by Th. 3.

If $(c_n) \in C_f(G)$ then we are able to prove that $(x_n c_n) \in M(L^2(G)$ for any $(x_n) \in M(L^2(G)$ iff there exists a compact neighbourhood of e such that $\bigcup_n c_n V c_n^{-1}$ has compact closure in G. The closure of this subset of $C_f(G)$ consists exactly of these (c_n) such that $(x_n) \in M(L^2(G))$ implies that $(x_n c_n) \in M(L^2(G))$ (see also [2]). It follows that $(x_n c_n) \in M(L^2(G)$ for any $(c_n) \in C(L^2(G))$, $(x_n) \in M(L^2(G))$ if and only if G has a compact invariant neighbourhood U ($x^{-1}Ux = U$ for all $x \in G$). For M(G) this is true in general, [9] Satz 11. For $C(L^1(G))$ the result above is not true even for groups having a compact invariant neighbourhood (see [9] p. 222) but holds if G has a basis at e of invariant neighbourhoods, [9], Satz 12.

References

[1] Kuipers, L., Niederreiter, H.: Uniform distribution of sequences, John Wiley & Sons, New York (1974).

[2] Losert, V.: Almost constant sequences of transformations, Monatsh. Math. 85, 105-113 (1978).

[3] Losert, V.: Uniformly distributed sequences on compact, separable non metrizable groups,
Acta Sci. Math. 40 Fasc. 1-2, 107-110 (1978).

[4] Losert, V., Rindler, H.: Teilfolgen gleichverteilter Folgen, Crelle J., to appear.

[5] Losert, V., Rindler, H.: Uniform distribution and the mean ergodic theorem, Inventiones Math., to appear.

[6] Rauzy, G.: Etude de quelques ensembles de fonction définis par des proprietés de moyenne, Théorie des Nombres, Univ. Bordeaux, 1972/73, Exp. 20.

[7] Rauzy, G.: Propriétés statistiques de suites arithmétiques, Presses Univeritaires de France, le mathematicien 15 (1976).

[8] Rindler, H.: Uniformly distributed sequences in quotient groups, Acta Sci. Math. 38, 153-156 (1976).

[9] Rindler, H.: Gleichverteilte Folgen in lokalkompakten Gruppen, Monatsh. f. Math. 82, 207-235 (1976).

[10] Rindler, H.: Fast konstante Folgen, Acta Arithmetica, to appear.

[11] Rindler, H.: Fast konstante Folgen II, Anzeiger Österr.Akad.Wiss. to appear.

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