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## **On topological and measure entropies of semigroups**

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On topological and measure entropies of semigroups

by

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The present paper contains a generalization of the theory of topological and measure entropies to the case of an action of an arbitrary subsemigroup of  $Z^N$ . Some ideas were suggested to the author by M. Misiurewicz.

1. Definitions of the topological and measure entropies.

A subset  $\tilde{\Lambda} \subset \mathbb{R}^N$  will be called a cone in  $\mathbb{R}^N$  if  $\forall x \in \tilde{\Lambda} \forall t > 0 \quad t \cdot x \in \tilde{\Lambda}$  and  $\tilde{\Lambda} \cap B(0,1)$  is of positive Jordan measure, where  $B(0,1)$  is the unit-ball in  $\mathbb{R}^N$ . The set  $\Lambda$  of the form  $\Lambda = \tilde{\Lambda} \cap Z^N$ , where  $\tilde{\Lambda}$  is a cone in  $\mathbb{R}^N$ , will be called a cone in  $Z^N$ .

If  $G$  is a semigroup in  $Z^N$  then  $G$  generates a subgroup of  $Z^N$  isomorphic to  $Z^{N'}$  for some  $N' \in \mathbb{N} / \mathbb{N}$  as usually denotes the set of positive integers  $\mathbb{N}$ . Thus without loss of generality, we can restrict ourselves to the study of these semigroups in  $Z^N$  which generate  $Z^N$ . It is easy to prove the following.

Proposition 1. A semigroup  $G \subset Z^N$  generates  $Z^N$  iff  $G$  contains a cone in  $Z^N$ .

Commencing from now  $G$  is a fixed semigroup in  $Z^N$  containing a cone  $\Lambda$  in  $Z^N$ .

We introduce the following notations:

For  $r_1 = (r_1^1, \dots, r_1^N)$ ,  $r_2 = (r_2^1, \dots, r_2^N) \in \mathbb{R}^N$

the relation  $r_1 < r_2 / r_1 < r_2 /$  means that  $r_1^i < r_2^i / r_1^i < r_2^i /$   
for  $i = 1, \dots, N$ .

$\mathbb{R}_+^N \stackrel{\text{df}}{=} \{x \in \mathbb{R}^N : x \geq 0\}$ .

For  $\varrho \in \mathbb{R}_+^N$ ,  $J_\varrho \stackrel{\text{df}}{=} \{x \in \mathbb{R}_+^N : x < \varrho\} \cdot J_\varrho + s$ , where  
 $s \in \mathbb{R}^N$ , will be called a rectangle in  $\mathbb{R}^N$ .

$\mathbb{Z}_+^N \stackrel{\text{df}}{=} \{z \in \mathbb{Z}^N : z \geq 0\}$ .

For  $w \in \mathbb{Z}_+^N$ ,  $I_w \stackrel{\text{df}}{=} \{x \in \mathbb{Z}_+^N : x < w\} \cdot I_w + z$ , where  
 $z \in \mathbb{Z}^N$ , will be called a rectangle in  $\mathbb{Z}^N$ .

$X$  is a non-empty, compact Hausdorff (probability) space.

$T$  is an action of  $G$  in  $X$  (it is not assumed that  $T^0 = \text{id}_X$ ).

$\mathcal{A}$  denotes an open cover (a finite measurable partition) of  $X$ .

For every subset  $B$  of  $G$  we set  $\mathcal{A}_B \stackrel{\text{df}}{=} \bigvee_{g \in B} (Tg)^{-1} \mathcal{A}$ .

$H(\mathcal{A}_B)$  stands for the topological (measure) entropy of the  
cover (partition)  $\mathcal{A}_B$ .

For  $n \in \mathbb{N}$  we set  $\Lambda^n \stackrel{\text{df}}{=} \Lambda \cap B(0, n)$ , where  $B(0, n)$  is  
the ball with center 0 and radius  $n$ .

Theorem 1.  $\lim_n \frac{1}{\text{card } \Lambda^n} H(\mathcal{A}_{\Lambda^n})$  exists and does not  
depend on the choice of  $\Lambda \subset G$ .

Lemma 1. Let  $\delta$  be an arbitrary positive number. If  
 $\Lambda$  is a cone in  $\mathbb{Z}^N$  and  $(n_1)$  is a sequence of positive

integers such that  $\lim_l n_l = +\infty$  then there exist

(i) positive integers  $l_1, \dots, l_k, t_1, \dots, t_k$

(ii)  $w \in \mathbb{Z}_+^N$

(iii)  $z_{i,j} \in I_w, j=1, \dots, t_i, i=1, \dots, k$

such that  $I_w = \bigcup_{j=1}^{t_1} (\Lambda^{n_{l_1}} + z_{1,j}) \cup \dots \cup$

$\dots \cup \bigcup_{j=1}^{t_k} (\Lambda^{n_{l_k}} + z_{k,j}) \cup I'_w$  where all the sets in the

above sum are pairwise disjoint and  $\frac{\text{card } I'_w}{\text{card } I_w} < \delta$ .

Proof: By assumption,  $\Lambda = \tilde{\Lambda} \cap \mathbb{Z}^N, \Lambda^{n_l} = \Lambda \cap B(0, n_l) = \tilde{\Lambda} \cap B(0, n_l) \cap \mathbb{Z}^N$  for  $l \in \mathbb{N}$ . Let  $\tilde{\Lambda}^{n_l} \stackrel{\text{df}}{=} \tilde{\Lambda} \cap B(0, n_l) \subset \mathbb{R}^N$ .

Fix  $\varepsilon > 0$ . If  $|\cdot|$  denotes the Jordan measure on  $\mathbb{R}^N$  then

$$(1) \quad \lim_l \frac{\text{card}(\tilde{\Lambda}^{n_l} \cap \mathbb{Z}^N)}{|\tilde{\Lambda}^{n_l}|} = 1,$$

by definition of Jordan measure.

Let  $J \subset \mathbb{R}^N$  be a rectangle with vertices belonging to  $\mathbb{Z}^N$  such that  $\tilde{\Lambda}^1 \subset J$ . Denote

$$(2) \quad \beta \stackrel{\text{df}}{=} \frac{|\tilde{\Lambda}^1|}{|J|}$$

$I_w$  can be constructed inductively. The idea is the following. We chose  $l_1 \in \mathbb{N}$  such that  $n_{l_1} \cdot J \setminus \tilde{\Lambda}^{n_{l_1}}$  can be covered by pairwise disjoint translates of  $n_{l_1} \cdot J$  by vectors with integer coordinates so precisely that if we denote the covered part of  $n_{l_1} \cdot J$  by  $(n_{l_1} \cdot J)_c$  then

$$(3) \quad \frac{|(n_{1_1} \cdot J)_c|}{|n_{1_1} \cdot J \setminus \tilde{\Lambda}^{n_{1_1}}|} > 1 - \varepsilon .$$

Then,  $n_{1_1} \cdot J$  contains both  $\tilde{\Lambda}^{n_{1_1}}$  and the translates of  $\tilde{\Lambda}^{n_{1_1}}$ . Now, if  $(n_{1_1} \cdot J)_{\tilde{\Lambda}}$  denotes the sum of and these translates then, in virtue of (2) and (3),

$$(4) \quad \frac{|(n_{1_1} \cdot J)_{\tilde{\Lambda}}|}{|n_{1_1} \cdot J|} > \beta + (1 - \varepsilon)(1 - \beta) \cdot \beta .$$

Now, we chose  $l_2 \in \mathbb{N}$  such that  $n_{1_2} \cdot J \setminus \tilde{\Lambda}^{n_{1_2}}$  can be covered pairwise disjoint translates of  $n_{1_1} \cdot J$  by vectors with integer coordinates, so precisely that if we denote the covered part of  $n_{1_2} \cdot J$  by  $(n_{1_2} \cdot J)_c$  then

$$(5) \quad \frac{|(n_{1_2} \cdot J)_c|}{|n_{1_2} \cdot J \setminus \tilde{\Lambda}^{n_{1_2}}|} > 1 - \varepsilon .$$

Then,  $n_{1_2} \cdot J$  contains both  $\tilde{\Lambda}^{n_{1_2}}$  and the translates of  $\tilde{\Lambda}^{n_{1_1}}$  and  $\tilde{\Lambda}^{n_{1_1}}$ . Now, if  $(n_{1_2} \cdot J)_{\tilde{\Lambda}}$  denotes the sum of  $\tilde{\Lambda}^{n_{1_2}}$  and these translates then by (2), (4) and

(5) we have

$$(6) \quad \frac{|(n_{1_2} \cdot J)_{\tilde{\Lambda}}|}{|n_{1_2} \cdot J|} > \beta + (1 - \varepsilon)(1 - \beta) \cdot \frac{|(n_{1_1} \cdot J)_{\tilde{\Lambda}}|}{|n_{1_1} \cdot J|}$$

Continuing this procedure, after the  $k$ -th step we have  $J_{n_{1_k}}$  which contains both  $\tilde{\Lambda}^{n_{1_k}}$  and the translates of

$\tilde{\Lambda}^{n_{1_1}}, \tilde{\Lambda}^{n_{1_1}}, \dots, \tilde{\Lambda}^{n_{1_{k-1}}}$  by vectors with integer coordinates,

and if  $(n_{1k} \cdot J)\tilde{\Lambda}$  denotes the sum of  $\tilde{\Lambda}^{n_{1k}}$  and these translates then

$$(7) \quad \frac{|(n_{1k} \cdot J)\tilde{\Lambda}|}{|n_{1k} \cdot J|} > \beta + (1 - \varepsilon)(1 - \beta) \cdot \frac{|(n_{1_{k-1}} \cdot J)\tilde{\Lambda}|}{|n_{1_{k-1}} \cdot J|}$$

where  $(n_{1_{k-1}} \cdot J)\tilde{\Lambda}$  is the sum of  $\tilde{\Lambda}^{n_{1_{k-1}}}$  and the translates of  $\tilde{\Lambda}^{n_1}, \tilde{\Lambda}^{n_2}, \dots, \tilde{\Lambda}^{n_{k-2}}$  covering  $J_{n_{1_{k-1}}}$  after  $(k-1)$ -th step.

$$\text{Denote } r_0 \stackrel{\text{df}}{=} \beta, \quad r_1 \stackrel{\text{df}}{=} \frac{|(n_{1_1} \cdot J)\tilde{\Lambda}|}{|n_{1_1} \cdot J|}, \dots$$

$$\dots, r_k \stackrel{\text{df}}{=} \frac{|(n_{1_k} \cdot J)\tilde{\Lambda}|}{|n_{1_k} \cdot J|}$$

By (7)  $1 \geq r_k \geq \beta + (1 - \varepsilon)(1 - \beta)r_{k-1}$  for  $k \in \mathbb{N}$ .

It is easy to prove that the sequence  $(r_k)$  satisfying the above condition tends to  $f(\varepsilon)$  while  $k$  tends to infinity, where  $\lim_{\varepsilon \rightarrow 0} f(\varepsilon) = 1$ . This fact together with (1) ends the proof.

Proof of Theorem 1: Suppose that  $\Lambda_1, \Lambda_2 \subset G$  are cones in  $Z^N$ . Denote  $\eta_1 \stackrel{\text{df}}{=} \liminf_n \frac{1}{\text{card} \Lambda_1^n} H(\mathcal{A}_{\Lambda_1^n})$ ,

$$\eta_2 \stackrel{\text{df}}{=} \limsup_n \frac{1}{\text{card} \Lambda_2^n} H(\mathcal{A}_{\Lambda_2^n}). \text{ Fix } \varepsilon > 0.$$

There exist a sequence  $(n_l)_{l=1}^\infty$  of positive integers such that

$$(8) \quad \frac{1}{\text{card} \Lambda_1^{n_l}} H(\mathcal{A}_{\Lambda_1^{n_l}}) \leq \eta_1 + \varepsilon \quad \text{for } l \in \mathbb{N}.$$

If  $I_w$  is a rectangle from Lemma 1 constructed for  $(n_l)$  and  $\varepsilon$ , then for sufficiently large  $n \in \mathbb{N}$

$$(9) \quad \mathcal{A}_2^n = \bigcup_{i=1}^t (I_{w_i} + \tau_i) \cup (\mathcal{A}_2^n)'$$

where  $\tau_i \in G$ ,  $i=1, \dots, t$ , the sets in the above sum are

pairwise disjoint and 
$$\frac{\text{card}(\mathcal{A}_2^n)'}{\text{card} \mathcal{A}_2^n} < \varepsilon .$$

By (8), (9) and Lemma 1 we have

$$\frac{1}{\text{card} \mathcal{A}_2^n} H(\mathcal{A}_2^n) \leq \eta_1 + \varepsilon + 2\varepsilon H(\mathcal{A}), \text{ so } \eta_2 \leq \eta_1.$$

Definition 1. (a) The topological (measure) entropy of a cover (partition)  $\mathcal{A}$  with respect to an action  $T$  of the semigroup  $G$  is the number

$$h(T, \mathcal{A}) \stackrel{\text{df}}{=} \lim_n \frac{1}{\text{card} \mathcal{A}^n} H(\mathcal{A}^n).$$

(b) The topological (measure) entropy of an action  $T$  of the semigroup  $G$  is the number  $h(T) \stackrel{\text{df}}{=} \sup_{\mathcal{A}} h(T, \mathcal{A})$ .

Example. Let  $H \neq \mathbb{Z}^N$  be a semigroup in  $\mathbb{Z}^N$  containing 0 and a cone in  $\mathbb{Z}^N$ . Equip the set  $\{0, 1\}$  with the discrete topology and put  $X \stackrel{\text{df}}{=} \{0, 1\}^H$  with the product topology. We define an action  $T$  of  $H$  as a shift on  $X$ :  $(T^h(x))_g = x_{h+g}$  for  $x \in X$ ,  $h, g \in H$ . It is easy to prove that  $T$  cannot be extended to an action of a semigroup  $H'$ ,  $H \subsetneq H' \subset \mathbb{Z}^N$ .

This example shows that the above definition is a substantial generalisation of classical one.

It can be easily proved that the above defined notions of entropy possess all the basic properties of entropy which can be found e.g. in [1] and [3].

2. The relation between the entropy of a semigroup and the entropy of its subsemigroup.

For  $A \subset Z^N$ ,  $\langle A \rangle$  will denote the additive group generated by  $A$ .

Let  $P$  be a subsemigroup of  $G$ . We know that for some  $K \in N$  there exists an isomorphism  $\varphi : Z^K \rightarrow \langle P \rangle$ .

$\varphi$  induces a linear mapping  $\tilde{\varphi} : R^K \rightarrow R^N$ . Let

$V \stackrel{\text{df}}{=} \tilde{\varphi} (J_{(1, \dots, 1)}) \cap Z^N$ .  $G$  contains a cone in  $Z^N$ ,

thus there exists  $h \in G$  such that  $V + h \subset G$ .

We set  $\mathcal{A}^V \stackrel{\text{df}}{=} \mathcal{A}_{V+h}$  and  $p \stackrel{\text{df}}{=} \text{card } V$ .

$T_p$  denotes an action of  $P$  on  $X$  defined by  $P \ni g \mapsto T_g^{\mathcal{A}}$ .

Theorem 2 / cf [3] 2.1 /. If  $K = N$  then

$$h(T_p, \mathcal{A}^V) = p \cdot h(T, \mathcal{A}).$$

Proof : I.  $h(T_p, \mathcal{A}^V) \geq p \cdot h(T, \mathcal{A})$ .

By assumption  $\varphi^{-1}(P)$  generates  $Z^N$ , thus there is a cone  $\Lambda_p$  in  $Z^N$ ,  $\varphi(\Lambda_p) \subset P$ .

Fix  $\varepsilon > 0$ . We set  $\eta \stackrel{\text{df}}{=} h(T, \mathcal{A})$ ,  $\eta_p \stackrel{\text{df}}{=} h(T_p, \mathcal{A}^V)$ .

For some  $n_0 \in N$  we have

$$(10) \quad \frac{1}{\text{card } \Lambda_p^n} H(\mathcal{A}_{\varphi(\Lambda_p^n)}) \leq \eta_p + \varepsilon \quad \text{for } n \geq n_0$$

Let  $I_w$  be a rectangle in  $Z^N$  from Lemma 1, constructed for the sequence  $(\Lambda^n)_{n=n_0}^\infty$  and  $\varepsilon$ . For some  $k \in G$ ,  $\varphi(I_w) + V + k \subset G$ , because  $G$  contains a cone in  $Z^N$ . For sufficiently large  $n$  we can find  $s \in N$ ,  $\lambda_j \in G, j=1, \dots, s$  such that

$$(11) \quad \Lambda^n = \bigcup_{j=1}^s (\varphi(I_w) + V + h + k + \lambda_j) \cup (\Lambda^n),$$



where the sets appearing in this sum are pairwise disjoint and  $\frac{\text{card}(\mathcal{A}^n)'}{\text{card } \mathcal{A}^n} < \varepsilon$ .

From (12), (13) and Lemma 1 we get

$$\frac{1}{\text{card } \mathcal{A}^n} H(\mathcal{A}^n) \leq \frac{1}{\text{card } \mathcal{A}^n} \sum_{j=1}^s H(\mathcal{A}_{\varphi(I_w)+V+h+k+\lambda_j}) + \varepsilon \cdot H(\mathcal{A}) \leq \varepsilon \cdot H(\mathcal{A}) + \frac{1}{\text{card}(\varphi(I_w)+V)} \cdot H(\mathcal{A}_{\varphi(I_w)+k}^V) \text{ but}$$

$\text{card}(\varphi(I_w)+V) = p \cdot \text{card } I_w$  and in virtue of (12) and

Lemma 1, and  $\frac{1}{\text{card } I_w} H(\mathcal{A}_{\varphi(I_w)+k}^V) \leq r_p + \varepsilon + \varepsilon \cdot H(\mathcal{A}^V)$ .

Hence  $\frac{1}{\text{card } \mathcal{A}^n} H(\mathcal{A}^n) \leq \frac{1}{p} \cdot r_p + \varepsilon \cdot H(\mathcal{A}) + \frac{1}{p} + \frac{1}{p} H(\mathcal{A}^V)$

which implies  $p \cdot r_p \leq r_p$ .

II.  $p \cdot h(\mathcal{T}, \mathcal{A}) \geq h(\mathcal{T}_p, \mathcal{A}^V)$ .

Fix  $\varepsilon > 0$ . There exists  $n_0 \in \mathbb{N}$  such that

$$(12) \quad \frac{1}{\text{card } \mathcal{A}^n} H(\mathcal{A}^n) \leq r + \varepsilon \quad \text{for } n \geq n_0.$$

Let  $I_w$  be a rectangle in  $Z^N$  from Lemma 1, constructed for  $(\mathcal{A}^n)_{n=n_0}^\infty$  and  $\varepsilon$ . There exists  $t \in \mathbb{N}$ ,

$z_0, z_i \in Z^N, i = 1, \dots, t$ , such that

$$(13) \quad \varphi(I_{z_0}) + V = \bigcup_{i=1}^t (I_w + z_i) \cup (\varphi(I_{z_0}) + V),$$

the sets appearing in this sum are pairwise disjoint and

$$\frac{\text{card}(\varphi(I_{z_0})+V)'}{\text{card}(\varphi(I_{z_0})+V)} < \varepsilon.$$

For  $n \in \mathbb{N}$  sufficiently large we can find  $l \in \mathbb{N}$ ,

$\lambda_i \in \mathcal{A}_p^n, i = 1, \dots, l$ , such that

$$(14) \quad \mathcal{A}_p^n = \bigcup_{i=1}^l (I_{z_0} + \lambda_i) \cup (\mathcal{A}_p^n)',$$

all the sets in the above sum are pairwise disjoint and

$$\frac{\text{card } \Lambda_p^n}{\text{card } \Lambda_p^n} < \varepsilon .$$

By (14) , (15) and (16) we have

$$\begin{aligned} & \frac{1}{\text{card } \varphi(\Lambda_p^n)} H(\mathcal{A}_{\varphi(\Lambda_p^n)}^V) \leq \\ & \leq \varepsilon \cdot H(\mathcal{A}^V) + \frac{1}{\text{card } \Lambda_p^n} \sum_{i=1}^l H(\mathcal{A}_{\varphi(I_{z_0})+V+h+\varphi(\lambda_i)}^V) \leq \varepsilon \cdot H(\mathcal{A}^V) \\ & + \frac{1}{\text{card } \Lambda_p^n} \sum_{i=1}^l \left( \sum_{j=1}^t H(\mathcal{A}_{I_w+z_j+h+\varphi(\lambda_i)}^V) + H(\mathcal{A}_{(\varphi(I_{z_0})+V)+h+\varphi(\lambda_i)}^V) \right) \\ & \leq p \cdot \eta + \varepsilon (p \cdot H(\mathcal{A}) + p + H(\mathcal{A}^V)) \quad \text{which gives the inequality} \end{aligned}$$

$$\eta_p \leq p \cdot \eta .$$

Corollary 1 (cf [3] 2.3) , If  $K = N$  then  $h(T_p) = p \cdot h(T)$  .

Theorem 3 (cf [3] 2.5) . If  $K < N$  and  $h(T) > 0$  then  $h(T_p) = +\infty$  .

Proof: Recall that  $\langle P \rangle \simeq Z^K$  ,  $\varphi : Z^K \rightarrow \langle P \rangle$

is an isomorphism ,  $K < N$  . We extend  $\varphi$  to an isomorphism of  $Z^N$  into  $Z^N$  . In the sequel this extension is denoted also by  $\varphi$  . Let  $p^{\#}$  denotes the index of subsemigroup  $\varphi(Z^N)$  in  $Z^N$  and  $P^{\#} \stackrel{\text{def}}{=} \varphi(Z^N) \cap G$  . By Theorem 1,  $h(T_{P^{\#}}) = p^{\#} \cdot h(T)$  . The extension  $\varphi$  can be chosen in such a way that  $p^{\#}$  is arbitrarily large. Thus it suffices to prove that  $h(T_{P^*}) \leq h(T_p)$  .

$\varphi^{-1}(P)$  contains a cone  $\Lambda_p$  in  $Z^K$ .  $\varphi^{-1}(P^{\#})$  contains a cone  $\Lambda_*$  in  $Z^N$  . Fix  $\varepsilon > 0$  . There exists  $n_0 \in N$  such that for  $n \geq n_0$

$$(15) \quad \frac{1}{\text{card } \Lambda_p^n} H(\mathcal{A}_{\varphi(\Lambda_p^n)}) \leq h(T_p, \mathcal{A}) + \varepsilon .$$

Let  $I_W$  be a rectangle from Lemma 1, constructed for  $(\Lambda_p^n)_{n=n_0}^{\infty}$  and  $\varepsilon$ . For  $n \in \mathbb{N}$  sufficiently large we can cover  $\Lambda_*^n$  by pairwise disjoint translates of  $I_W$  so precisely, that by a standard estimation we obtain the desired inequality.

Corollary 2. /of [3] 2.6./ . If  $K < N$ ,  $h(T_p) < +\infty$ , then  $h(T) = 0$ .

Note that everything that was proved in part 2 is also valid for measure entropy (proofs without modifications).

### 3. Theorem of Dinaburg - Goodwyn - Goodman.

We introduce the following notations :

$\mathcal{M}(X)$  - the space of all Borel, normalised measures on  $X$  with weak  $^*$  - topology.

$\mathcal{M}(X, T)$  - the subspace of all  $T$ -invariant measures in  $\mathcal{M}(X)$ .

$W$  - the set of all neighbourhoods of the diagonal in  $X \times X$  directed by the inclusion.

Let  $\delta \in W$ .  $\delta_C \stackrel{\text{def}}{=} \bigcap_{g \in C} (T^g \times T^g)^{-1} \delta$  for arbitrary  $C \subset G$ .

A finite subset  $e$  of  $X$  is called a/  $(C, \delta)$  - separated, if for all  $x, y \in e$ ,  $x \neq y$  we have  $(x, y) \notin \delta_C$  ;  
 b/  $(C, \delta)$  - spanning, if for all  $x \in X$  there exists  $y \in e$  such that  $(x, y) \in \delta_C$ .

Let  $r(C, \delta) \stackrel{\text{df.}}{=} \min \{ \text{card } e : e \text{ is } (C, \delta)\text{-spanning} \}$ ,  $s(C, \delta) = \max \{ \text{card } e : e \text{ is } (C, \delta)\text{-separated} \}$ . We define

$$\bar{r}_T(\Lambda, \delta) \stackrel{\text{df.}}{=} \limsup_n \frac{1}{\text{card } \Lambda^n} \log r(\Lambda^n, \delta),$$

$$\bar{s}_T(\Lambda, \delta) \stackrel{\text{df.}}{=} \limsup_n \frac{1}{\text{card } \Lambda^n} \log s(\Lambda^n, \delta).$$

By an argument analogous to the one applied in [3] the following definition makes sense,

Definition 3.  $h_T(\Lambda) = \lim_{\delta} \bar{s}_T(\Lambda, \delta) = \lim_{\delta} \bar{r}_T(\Lambda, \delta) = \sup_{\delta} \bar{s}_T(\Lambda, \delta) = \sup_{\delta} \bar{r}_T(\Lambda, \delta)$ .

Theorem 4. For all  $\Lambda \subset G$   $h_T(\Lambda) = h(T)$ .

The proof of this theorem is a translation of the proof [3] 4.8 to the language of the form structure  $W$  on  $X$ .

The following lemma will be used in the proof of Dinaburg-Goodwyn-Goodman theorem.

Lemma 2. Assume that  $\mu \in \mathcal{M}(X, T)$  and  $\mathcal{A}$  is a  $\mu$ -measurable finite partition of  $X$ . Let  $p_i \in \mathbb{Z}_+^{\mathbb{N}}$  for  $i \in \mathbb{N}$  and  $\lim p_i = +\infty$ . Chose  $g_i \in G$  such that  $I_p + g_i \subset G$  for  $i \in \mathbb{N}$ . Then

$$h_{\mu}(T, \mathcal{A}) = \lim_i \frac{1}{\text{card } I_{p_i}} H_{\mu}(\mathcal{A}_{I_p + g_i})$$

Proof :  $\limsup_i \frac{1}{\text{card } I_{p_i}} H_{\mu}(\mathcal{A}_{I_{p_i} + g_i}) \leq h_{\mu}(T, \mathcal{A})$ .

There exists a sequence of positive integers  $(n_l)$  such that  $\frac{1}{\text{card } \Lambda^{n_l}} H_{\mu}(\mathcal{A}_{\Lambda^{n_l}}) \leq h_{\mu}(T, \mathcal{A}) + \epsilon$ .

For  $i$  sufficiently large we cover  $I_{p_i} + g_i$  by pairwise disjoint translates of a rectangle  $I_w$  from Lemma 1, constructed for  $(\Lambda^{n_i})$  and  $\varepsilon$ .

A standard estimation yields the desired inequality.

$$\text{II. } h_\mu(\mathbb{T}, \mathcal{A}) \leq \liminf_i \frac{1}{\text{card } I_{p_i}} H_\mu(\mathcal{A}_{I_{p_i} + g_i}).$$

If  $i \in \mathbb{N}$  then for sufficiently large  $n \in \mathbb{N}$  we can find  $k \in \mathbb{N}$ ,  $\lambda_l \in \Lambda^n$ ,  $l = 1, \dots, k$ , such that  $\Lambda^n = \bigcup_{l=1}^k (I_{p_i} + \lambda_l) \cup (\Lambda^n)'$ , where the sets appearing in this sum are pairwise disjoint and  $\frac{\text{card } (\Lambda^n)'}{\text{card } \Lambda^n} < \varepsilon$ . Since

$$\begin{aligned} \text{for } l = 1, \dots, k, \quad H_\mu(\mathcal{A}_{I_{p_i} + \lambda_l}) &= H_\mu(\mathcal{A}_{I_{p_i} + \lambda_l + g_i}) = \\ &= H_\mu(\mathcal{A}_{I_{p_i} + g_i}), \quad \text{the following inequality holds:} \\ \frac{1}{\text{card } \Lambda^n} H_\mu(\mathcal{A}_{\Lambda^n}) &\leq \varepsilon \cdot H_\mu(\mathcal{A}) + \frac{1}{\text{card } I_{p_i}} H_\mu(\mathcal{A}_{I_{p_i} + g_i}). \end{aligned}$$

This inequality implies II.

Theorem 5. /Dinaburg-Goodwyn-Goodman/.

$$h(\mathbb{T}) = \sup_{\mu \in \mathcal{M}(X, \mathbb{T})} h_\mu(\mathbb{T}).$$

Proof: I.  $\sup_{\mu \in \mathcal{M}(X, \mathbb{T})} h_\mu(\mathbb{T}) \leq h(\mathbb{T})$  /Goodwyn/.

The proof is analogous to the proof of Theorem 4.1 in [4].

$$\text{II. } h(\mathbb{T}) \leq \sup_{\mu \in \mathcal{M}(X, \mathbb{T})} h_\mu(\mathbb{T}) \quad \text{/cf [5] /.$$

Fix  $\delta > 0$  and  $\delta \in W$ . Let for all  $n \in \mathbb{N}$   $e_n$  be a set  $(\Lambda^n, \delta)$ -separated of maximal cardinality.

For some sequence  $(n_k)$  of positive integers there exists

$$\lim_k \frac{1}{\text{card } \Lambda^{n_k}} \log \text{card } e_{n_k} = h_T(\Lambda, \delta) .$$

We construct a measure  $\mu \in \mathcal{M}(X, T)$  in the way indicated in [5] :  $\sigma_n(\{y\}) = \frac{1}{\text{card } e_n}$  for  $y \in e_n$ ,

$\mu_n \stackrel{\text{df}}{=} \frac{1}{\text{card } \Lambda^n} \sum_{g \in \Lambda^n} T^{ng} g_{\sigma_n}$  /definition of  $T^{ng}$  is given in [5] /. In virtue of the theorem of Alaoglu there exists a cluster point  $\mu \in \mathcal{M}(X)$  of the sequence  $(\mu_{n_k})$ . As in [5] one proves that  $\mu \in \mathcal{M}(X, T)$ .

Let  $\mathcal{A}$  be a finite Borel partition of  $X$  such that  $a \times a \subset \delta$  for  $a \in \mathcal{A}$ . Then for  $a \in \mathcal{A}_{\Lambda^n}$   $a \times a \subset \delta_{\Lambda^n}$  thus  $\forall a \in \mathcal{A}_{\Lambda^n}$   $\text{card}(e_n \cap a) \leq 1$ , so

$$H_{\sigma_n}(\mathcal{A}_{\Lambda^n}) = - \sum_{y \in e_n} \sigma_n(\{y\}) \log \sigma_n(\{y\}) = \log \text{card } e_n .$$

Let  $(I_{p_i} + g_i)$  be a sequence from Lemma 2.

We can assume that  $g_i \in \mathbb{Z}_+^N$  for  $i \in \mathbb{N}$ .

Fix  $m \in \mathbb{N}$  and  $\varepsilon$ ,  $0 < \varepsilon < \frac{\sigma}{2 \log \text{card } \mathcal{A}}$ . There exists  $l_0 \in \mathbb{N}$  such that for  $l \geq l_0$   $p_l - g_m - p_m \in \mathbb{Z}_+^N$

and

$$(16) \quad \frac{\text{card } I_{p_l - g_m - p_m}}{\text{card } I_{p_l}} \geq 1 - \varepsilon .$$

If  $l \geq l_0$ ,  $l \in \mathbb{N}$ , then for  $n$  sufficiently large we can find  $t \in \mathbb{N}$ ,  $\lambda_i \in \Lambda^n$ ,  $i = 1, \dots, t$ , such that  $\Lambda^n = \bigcup_{i=1}^t (I_{p_l} + \lambda_i) \cup (\Lambda^n)^c$ ; the sets appearing

in this sum are pairwise disjoint and  $\frac{\text{card}(\Lambda^n)'}{\text{card} \Lambda^n} \leq \varepsilon$ .

Now, let  $q \in I_{p_m}$ . We define

$$s(q) = \left( \left[ \frac{p_1^1 - g_m^1 - q^1}{p_m^1} \right], \dots, \left[ \frac{p_1^N - g_m^N - q^N}{p_m^N} \right] \right).$$

Observe that  $I_{p_l} = \bigcup_{r \in I_{s(q)}} (I_p + g_m + q + r \cdot p_m) \cup (I_{p_l})'$ ,

where the sets appearing in this sum are pairwise disjoint and  $\text{card} (I_{p_l})' \leq \text{card} I_{p_l} - \text{card} I_{p_l - g_m - p_m} \leq \varepsilon \cdot \text{card} I_{p_l}$

/by (16) /. So, finally we can represent  $\Lambda^n$  as a sum of pairwise disjoint sets as follows  $\Lambda^n = \bigcup_{i=1}^t \left( \bigcup_{r \in I_{s(q)}} (I_{p_m} + \lambda_i + g_m + q + r \cdot p_m) \cup (I_{p_l}' + \lambda_i) \right) \cup (\Lambda^n)'$ . Thus, for all  $q \in I_{p_m}$

$$(17_q) \quad H_{\sigma_n}(\mathcal{A}_{\Lambda^n}) \leq \text{card}(\Lambda^n)' \cdot \log \text{card} \mathcal{A} + \sum_{i=1}^t \text{card} I_{p_l}' \log \text{card} \mathcal{A} + \sum_{i=1}^t \sum_{r \in I_{s(q)}} H_{\sigma_n} \left( \left( \mathbb{T}^{\lambda_i + q + r \cdot p_m} \right)^{-1} \mathcal{A}_{I_{p_m} + g_m} \right).$$

Adding the inequalities (17q),  $q \in I_p$ , by sides we obtain

$$(18) \quad \text{card} I_{p_m} \cdot \log \text{card} e_n \leq \text{card} I_p \cdot \log \text{card} \mathcal{A} \cdot \left( \text{card}(\Lambda^n)' + t \cdot \text{card} I_{p_l}' \right) + \sum_{i=1}^t \left( \sum_{q \in I_{p_m}} \sum_{r \in I_{s(q)}} H_{\sigma_n} \left( \left( \mathbb{T}^{\lambda_i + q + r \cdot p_m} \right)^{-1} \mathcal{A}_{I_{p_m} + g_m} \right) \right) \leq \text{card} I_{p_m} \cdot \log \text{card} \mathcal{A} \left( \text{card}(\Lambda^n)' + t \cdot \text{card} I_{p_l}' \right) + \sum_{g \in \Lambda^n} H_{\sigma_n} \left( \left( \mathbb{T}^g \right)^{-1} \mathcal{A}_{I_{p_m} + g_m} \right).$$

Dividing the inequality (18) by  $\text{card} I_p \cdot \text{card} \Lambda^n$  and applying the inequalities

$$\frac{1}{\text{card } \Lambda^n} \sum_{g \in \Lambda^n} H_{\sigma_n} \left( (T^g)^{-1} \mathcal{A}_{I_{p_m} + g_m} \right) \leq H_{\mu}(\mathcal{A}_{I_{p_m} + g_m})$$

and  $\frac{t \cdot \text{card } I_{p_t}}{\text{card } \Lambda^n} \leq \frac{t \cdot \text{card } I_{p_t} \cdot \varepsilon}{\text{card } \Lambda^n} \leq \varepsilon$  , we obtain

$$(19) \quad \frac{1}{\text{card } \Lambda^n} \log \text{card } e_n \leq 2 \cdot \varepsilon \log \text{card } \mathcal{A} +$$

$$+ \frac{1}{\text{card } I_{p_m}} \cdot H_{\mu_n} \left( \mathcal{A}_{I_{p_m} + g_m} \right).$$

Inequality (19) is true for all  $n \in \mathbb{N}$  sufficiently large and  $\mathcal{A}$  can be chosen in such a way that the boundaries of the elements of  $\mathcal{A}$  have measure  $\mu$  zero, hence taking the limit with respect to  $n$ /or with respect to a subsequence  $(n_k)$  if necessary / we get  $h_T(\Lambda, \delta) \leq 2 \cdot \varepsilon \log \text{card } \mathcal{A} +$

$$+ \frac{1}{\text{card } I_{p_m}} H_{\mu} \left( \mathcal{A}_{I_{p_m} + g_m} \right) \leq \sigma + \frac{1}{\text{card } I_{p_m}} \cdot H_{\mu} \left( \mathcal{A}_{I_{p_m} + g_m} \right)$$

for all  $\delta \in W$  and  $m \in \mathbb{N}$  . Passing to the limit with  $\delta$  and  $m$  , owing to the arbitraryness of  $\sigma$  , we obtain finally  $h(T) \leq h_{\mu}(T)$  .

Corollary 3. If  $T_{\Omega}$  denotes on action of  $G$  on the set of nonwandering points  $\Omega$  defined by  $T_{\Omega}^g(x) = T^g(x)$  for  $x \in \Omega$  , then  $h(T_{\Omega}) = h(T)$  .



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