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On topological and measure entropies of semigroups

by

Krystyna Małgorzata Ziemiańska

The present paper contains a generalization of the theory of topological and measure entropies to the case of an action of an arbitrary subsemigroup of Z^N . Some ideas were suggested to the author by M. Misiurewicz.

1. Definitions of the topological and measure entropies.

A subset $\tilde{\Lambda} \subset \mathbb{R}^N$ will be called a cone in \mathbb{R}^N if $\forall x \in \tilde{\Lambda} \forall t > 0 \quad t \cdot x \in \tilde{\Lambda}$ and $\tilde{\Lambda} \cap B(0,1)$ is of positive Jordan measure, where $B(0,1)$ is the unit-ball in \mathbb{R}^N . The set Λ of the form $\Lambda = \tilde{\Lambda} \cap Z^N$, where $\tilde{\Lambda}$ is a cone in \mathbb{R}^N , will be called a cone in Z^N .

If G is a semigroup in Z^N then G generates a subgroup of Z^N isomorphic to $Z^{N'}$ for some $N' \in \mathbb{N} / \mathbb{N}$ as usually denotes the set of positive integers /. Thus without loss of generality, we can restrict ourselves to the study of these semigroups in Z^N which generate Z^N . It is easy to prove the following.

Proposition 1. A semigroup $G \subset Z^N$ generates Z^N iff G contains a cone in Z^N .

Commencing from now G is a fixed semigroup in Z^N containing a cone Λ in Z^N .

We introduce the following notations:

For $r_1 = (r_1^1, \dots, r_1^N)$, $r_2 = (r_2^1, \dots, r_2^N) \in \mathbb{R}^N$

the relation $r_1 < r_2 / r_1 < r_2 /$ means that $r_1^i < r_2^i / r_1^i < r_2^i /$
for $i = 1, \dots, N$.

$\mathbb{R}_+^N \stackrel{\text{df}}{=} \{x \in \mathbb{R}^N : x \geq 0\}$.

For $\varrho \in \mathbb{R}_+^N$ $J_\varrho \stackrel{\text{df}}{=} \{x \in \mathbb{R}_+^N : x < \varrho\} \cdot J_\varrho + s$, where
 $s \in \mathbb{R}^N$, will be called a rectangle in \mathbb{R}^N .

$\mathbb{Z}_+^N \stackrel{\text{df}}{=} \{z \in \mathbb{Z}^N : z \geq 0\}$.

For $w \in \mathbb{Z}_+^N$ $I_w \stackrel{\text{df}}{=} \{x \in \mathbb{Z}_+^N : x < w\} \cdot I_w + s$, where
 $s \in \mathbb{Z}^N$, will be called a rectangle in \mathbb{Z}^N .

X is a non-empty, compact Hausdorff (probability) space.

T is an action of G in X (it is not assumed that $T^0 = \text{id}_X$).

\mathcal{A} denotes an open cover (a finite measurable partition) of X .

For every subset B of G we set $\mathcal{A}_B \stackrel{\text{df}}{=} \bigvee_{g \in B} (Tg)^{-1} \mathcal{A}$.

$H(\mathcal{A}_B)$ stands for the topological (measure) entropy of the
cover (partition) \mathcal{A}_B .

For $n \in \mathbb{N}$ we set $\Lambda^n \stackrel{\text{df}}{=} \Lambda \cap B(0, n)$, where $B(0, n)$ is
the ball with center 0 and radius n .

Theorem 1. $\lim_n \frac{1}{\text{card } \Lambda^n} H(\mathcal{A}_{\Lambda^n})$ exists and does not

depend on the choice of $\Lambda \subset G$.

Lemma 1. Let δ be an arbitrary positive number. If
 Λ is a cone in \mathbb{Z}^N and (n_1) is a sequence of positive

integers such that $\lim_l n_l = +\infty$ then there exist

(i) positive integers $l_1, \dots, l_k, t_1, \dots, t_k$

(ii) $w \in \mathbb{Z}_+^N$

(iii) $z_{i,j} \in I_w, j=1, \dots, t_i, i=1, \dots, k$

such that $I_w = \bigcup_{j=1}^{t_1} (\Lambda^{n_{l_1}} + z_{1,j}) \cup \dots \cup$

$\dots \cup \bigcup_{j=1}^{t_k} (\Lambda^{n_{l_k}} + z_{k,j}) \cup I'_w$ where all the sets in the

above sum are pairwise disjoint and $\frac{\text{card } I'_w}{\text{card } I_w} < \delta$.

Proof: By assumption, $\Lambda = \tilde{\Lambda} \cap \mathbb{Z}^N, \Lambda^{n_l} = \Lambda \cap B(0, n_l) =$
 $= \tilde{\Lambda} \cap B(0, n_l) \cap \mathbb{Z}^N$ for $l \in \mathbb{N}$. Let $\tilde{\Lambda}^{n_l} \stackrel{\text{df}}{=} \tilde{\Lambda} \cap B(0, n_l) \subset \mathbb{R}^N$.

Fix $\varepsilon > 0$. If $|\cdot|$ denotes the Jordan measure on \mathbb{R}^N then

$$(1) \quad \lim_l \frac{\text{card}(\tilde{\Lambda}^{n_l} \cap \mathbb{Z}^N)}{|\tilde{\Lambda}^{n_l}|} = 1,$$

by definition of Jordan measure.

Let $J \subset \mathbb{R}^N$ be a rectangle with vertices belonging to \mathbb{Z}^N such that $\tilde{\Lambda}^1 \subset J$. Denote

$$(2) \quad \beta \stackrel{\text{df}}{=} \frac{|\tilde{\Lambda}^1|}{|J|}$$

I_w can be constructed inductively. The idea is the following. We chose $l_1 \in \mathbb{N}$ such that $n_{l_1} \cdot J \setminus \tilde{\Lambda}^{n_{l_1}}$ can be covered by pairwise disjoint translates of $n_{l_1} \cdot J$ by vectors with integer coordinates so precisely that if we denote the covered part of $n_{l_1} \cdot J$ by $(n_{l_1} \cdot J)_c$ then

$$(3) \quad \frac{|(n_{1_1} \cdot J)_c|}{|n_{1_1} \cdot J \setminus \tilde{\Lambda}^{n_{1_1}}|} > 1 - \varepsilon .$$

Then, $n_{1_1} \cdot J$ contains both $\tilde{\Lambda}^{n_{1_1}}$ and the translates of $\tilde{\Lambda}^{n_{1_1}}$. Now, if $(n_{1_1} \cdot J)_{\tilde{\Lambda}}$ denotes the sum of and these translates then, in virtue of (2) and (3),

$$(4) \quad \frac{|(n_{1_1} \cdot J)_{\tilde{\Lambda}}|}{|n_{1_1} \cdot J|} > \beta + (1 - \varepsilon)(1 - \beta) \cdot \beta .$$

Now, we chose $l_2 \in \mathbb{N}$ such that $n_{1_2} \cdot J \setminus \tilde{\Lambda}^{n_{1_2}}$ can be covered pairwise disjoint translates of $n_{1_1} \cdot J$ by vectors with integer coordinates, so precisely that if we denote the covered part of $n_{1_2} \cdot J$ by $(n_{1_2} \cdot J)_c$ then

$$(5) \quad \frac{|(n_{1_2} \cdot J)_c|}{|n_{1_2} \cdot J \setminus \tilde{\Lambda}^{n_{1_2}}|} > 1 - \varepsilon .$$

Then, $n_{1_2} \cdot J$ contains both $\tilde{\Lambda}^{n_{1_2}}$ and the translates of $\tilde{\Lambda}^{n_{1_1}}$ and $\tilde{\Lambda}^{n_{1_1}}$. Now, if $(n_{1_2} \cdot J)_{\tilde{\Lambda}}$ denotes the sum of $\tilde{\Lambda}^{n_{1_2}}$ and these translates then by (2), (4) and (5) we have

$$(6) \quad \frac{|(n_{1_2} \cdot J)_{\tilde{\Lambda}}|}{|n_{1_2} \cdot J|} > \beta + (1 - \varepsilon)(1 - \beta) \cdot \frac{|(n_{1_1} \cdot J)_{\tilde{\Lambda}}|}{|n_{1_1} \cdot J|}$$

Continuing this procedure, after the k -th step we have $J_{n_{1_k}}$ which contains both $\tilde{\Lambda}^{n_{1_k}}$ and the translates of

$\tilde{\Lambda}^{n_{1_1}}, \tilde{\Lambda}^{n_{1_1}}, \dots, \tilde{\Lambda}^{n_{1_{k-1}}}$ by vectors with integer coordinates,

and if $(n_{1_k} \cdot J)\tilde{\Lambda}$ denotes the sum of $\tilde{\Lambda}^{n_{1_k}}$ and these translates then

$$(7) \quad \frac{|(n_{1_k} \cdot J)\tilde{\Lambda}|}{|n_{1_k} \cdot J|} > \beta + (1 - \varepsilon)(1 - \beta) \cdot \frac{|(n_{1_{k-1}} \cdot J)\tilde{\Lambda}|}{|n_{1_{k-1}} \cdot J|}$$

where $(n_{1_{k-1}} \cdot J)\tilde{\Lambda}$ is the sum of $\tilde{\Lambda}^{n_{1_{k-1}}}$ and the translates of $\tilde{\Lambda}^{n_1}, \tilde{\Lambda}^{n_2}, \dots, \tilde{\Lambda}^{n_{k-2}}$ covering $J_{n_{1_{k-1}}}$ after $(k-1)$ -th step.

$$\text{Denote } r_k \stackrel{\text{df}}{=} \beta, \quad r_1 \stackrel{\text{df}}{=} \frac{|(n_{1_1} \cdot J)\tilde{\Lambda}|}{|n_{1_1} \cdot J|}, \dots$$

$$\dots, r_k \stackrel{\text{df}}{=} \frac{|(n_{1_k} \cdot J)\tilde{\Lambda}|}{|n_{1_k} \cdot J|}$$

By (7) $1 \geq r_k \geq \beta + (1 - \varepsilon)(1 - \beta)r_{k-1}$ for $k \in \mathbb{N}$.

It is easy to prove that the sequence (r_k) satisfying the above condition tends to $f(\varepsilon)$ while k tends to infinity, where $\lim_{\varepsilon \rightarrow 0} f(\varepsilon) = 1$. This fact together with (1) ends the proof.

Proof of Theorem 1: Suppose that $\Lambda_1, \Lambda_2 \subset G$ are cones in Z^N . Denote $\eta_1 \stackrel{\text{df}}{=} \liminf_n \frac{1}{\text{card } \Lambda_1^n} H(\mathcal{A}_{\Lambda_1}^n)$,

$$\eta_2 \stackrel{\text{df}}{=} \limsup_n \frac{1}{\text{card } \Lambda_2^n} H(\mathcal{A}_{\Lambda_2}^n). \text{ Fix } \varepsilon > 0.$$

There exist a sequence $(n_l)_{l=1}$ of positive integers such that

$$(8) \quad \frac{1}{\text{card } \Lambda_1^{n_l}} H(\mathcal{A}_{\Lambda_1}^{n_l}) \leq \eta_1 + \varepsilon \quad \text{for } l \in \mathbb{N}.$$

If I_w is a rectangle from Lemma 1 constructed for (n_1) and ε , then for sufficiently large $n \in \mathbb{N}$

$$(9) \quad \mathcal{A}_2^n = \bigcup_{i=1}^t (I_{w_i} + \mathcal{F}_i) \cup (\mathcal{A}_2^n)'$$

where $\mathcal{F}_i \in \mathcal{G}$, $i=1, \dots, t$, the sets in the above sum are pairwise disjoint and $\frac{\text{card}(\mathcal{A}_2^n)'}{\text{card} \mathcal{A}_2^n} < \epsilon$.

By (8), (9) and Lemma 1 we have

$$\frac{1}{\text{card} \mathcal{A}_2^n} H(\mathcal{A}_2^n) \leq \eta_1 + \epsilon + 2 \epsilon H(\mathcal{A}), \text{ so } \eta_2 \leq \eta_1.$$

Definition 1. (a) The topological (measure) entropy of a cover (partition) \mathcal{A} with respect to an action T of the semigroup G is the number

$$h(T, \mathcal{A}) \stackrel{\text{df}}{=} \lim_n \frac{1}{\text{card} \mathcal{A}^n} H(\mathcal{A}^n).$$

(b) The topological (measure) entropy of an action T of the semigroup G is the number $h(T) \stackrel{\text{df}}{=} \sup_{\mathcal{A}} h(T, \mathcal{A})$.

Example. Let $H \neq \mathbb{Z}^N$ be a semigroup in \mathbb{Z}^N containing 0 and a cone in \mathbb{Z}^N . Equip the set $\{0, 1\}$ with the discrete topology and put $X \stackrel{\text{df}}{=} \{0, 1\}^H$ with the product topology. We define an action T of H as a shift on $X : (T^h(x))_g = x_{h+g}$ for $x \in X$, $h, g \in H$. It is easy to prove that T cannot be extended to an action of a semigroup H' , $H \subsetneq H' \subset \mathbb{Z}^N$.

This example shows that the above definition is a substantial generalisation of classical one.

It can be easily proved that the above defined notions of entropy possess all the basic properties of entropy which can be found e.g. in [1] and [3].

2. The relation between the entropy of a semigroup and the entropy of its subsemigroup.

For $A \subset Z^N$, $\langle A \rangle$ will denote the additive group generated by A .

Let P be a subsemigroup of G . We know that for some $K \in N$ there exists an isomorphism $\varphi : Z^K \rightarrow \langle P \rangle$.

φ induces a linear mapping $\tilde{\varphi} : R^K \rightarrow R^N$. Let

$V \stackrel{\text{df}}{=} \tilde{\varphi} (J_{(1, \dots, 1)}) \cap Z^N$. G contains a cone in Z^N ,

thus there exists $h \in G$ such that $V + h \subset G$.

We set $\mathcal{A}^V \stackrel{\text{df}}{=} \mathcal{A}_{V+h}$ and $p \stackrel{\text{df}}{=} \text{card } V$.

T_p denotes an action of P on X defined by $P \ni g \mapsto T_g^{\mathcal{A}}$.

Theorem 2 / cf [3] 2.1 /. If $K = N$ then

$$h(T_p, \mathcal{A}^V) = p \cdot h(T, \mathcal{A}).$$

Proof : I. $h(T_p, \mathcal{A}^V) \geq p \cdot h(T, \mathcal{A})$.

By assumption $\varphi^{-1}(P)$ generates Z^N , thus there is a cone Λ_p in Z^N , $\varphi(\Lambda_p) \subset P$.

Fix $\varepsilon > 0$. We set $\eta \stackrel{\text{df}}{=} h(T, \mathcal{A})$, $\eta_p \stackrel{\text{df}}{=} h(T_p, \mathcal{A}^V)$.

For some $n_0 \in N$ we have

$$(10) \quad \frac{1}{\text{card } \Lambda_p^n} H(\mathcal{A}_{\varphi(\Lambda_p^n)}) \leq \eta_p + \varepsilon \quad \text{for } n \geq n_0$$

Let I_w be a rectangle in Z^N from Lemma 1, constructed for the sequence $(\Lambda^n)_{n=n_0}^\infty$ and ε . For some $k \in G$, $\varphi(I_w) + V + k \subset G$, because G contains a cone in Z^N . For sufficiently large n we can find $s \in N$, $\lambda_j \in G, j=1, \dots, s$ such that

$$(11) \quad \Lambda^n = \bigcup_{j=1}^s (\varphi(I_w) + V + h + k + \lambda_j) \cup (\Lambda^n),$$

where the sets appearing in this sum are pairwise disjoint and $\frac{\text{card}(\Lambda^n)'}{\text{card } \Lambda^n} < \varepsilon$.

From (12), (13) and Lemma 1 we get

$$\frac{1}{\text{card } \Lambda^n} H(\mathcal{A}_{\Lambda^n}) \leq \frac{1}{\text{card } \Lambda^n} \sum_{j=1}^5 H(\mathcal{A}_{\varphi(I_w)+V+h+k+\lambda_j}) + \varepsilon \cdot H(\mathcal{A}) \leq \varepsilon \cdot H(\mathcal{A}) + \frac{1}{\text{card}(\varphi(I_w)+V)} \cdot H(\mathcal{A}_{\varphi(I_w)+k}^V) \text{ but}$$

$\text{card}(\varphi(I_w)+V) = p \text{ card } I_w$ and in virtue of (12) and

Lemma 1, and $\frac{1}{\text{card } I_w} H(\mathcal{A}_{\varphi(I_w)+k}^V) \leq r_p + \varepsilon + \varepsilon \cdot H(\mathcal{A}^V)$.

Hence $\frac{1}{\text{card } \Lambda^n} H(\mathcal{A}_{\Lambda^n}) \leq \frac{1}{p} \cdot r_p + \varepsilon \cdot H(\mathcal{A}) + \frac{1}{p} + \frac{1}{p} H(\mathcal{A}^V)$

which implies $p \cdot r \leq r_p$.

$$\text{II. } p \cdot h(\mathbb{T}, \mathcal{A}) \geq h(\mathbb{T}_p, \mathcal{A}^V) \quad .$$

Fix $\varepsilon > 0$. There exists $n_0 \in \mathbb{N}$ such that

$$(12) \quad \frac{1}{\text{card } \Lambda^n} H(\mathcal{A}_{\Lambda^n}) \leq r + \varepsilon \quad \text{for } n \geq n_0 .$$

Let I_w be a rectangle in Z^N from Lemma 1, constructed for $(\Lambda^n)_{n=n_0}^{\infty}$ and ε . There exists $t \in \mathbb{N}$,

$z_0, z_i \in Z^N, i = 1, \dots, t$, such that

$$(13) \quad \varphi(I_{z_0}) + V = \bigcup_{i=1}^t (I_w + z_i) \cup (\varphi(I_{z_0}) + V)'$$

the sets appearing in this sum are pairwise disjoint and

$$\frac{\text{card}(\varphi(I_{z_0})+V)'}{\text{card}(\varphi(I_{z_0})+V)} < \varepsilon .$$

For $n \in \mathbb{N}$ sufficiently large we can find $l \in \mathbb{N}$,

$\lambda_i \in \Lambda_p^n, i = 1, \dots, l$, such that

$$(14) \quad \Lambda_p^n = \bigcup_{i=1}^l (I_{z_0} + \lambda_i) \cup (\Lambda_p^n)'$$

all the sets in the above sum are pairwise disjoint and

$$\frac{\text{card } \Lambda_p^n}{\text{card } \Lambda_p^n} < \varepsilon .$$

By (14) , (15) and (16) we have $\frac{1}{\text{card } \varphi(\Lambda_p^n)} H(\mathcal{A}_{\varphi(\Lambda_p^n)}^V) \leq$

$$\begin{aligned} &\leq \varepsilon \cdot H(\mathcal{A}^V) + \frac{1}{\text{card } \Lambda_p^n} \sum_{i=1}^l H(\mathcal{A}_{\varphi(I_{z_0})+V+h+\varphi(\lambda_i)}^V) \leq \varepsilon \cdot H(\mathcal{A}^V) \\ &+ \frac{1}{\text{card } \Lambda_p^n} \sum_{i=1}^l \left(\sum_{j=1}^t H(\mathcal{A}_{I_w+z_j+h+\varphi(\lambda_i)}^V) + H(\mathcal{A}_{(\varphi(I_{z_0})+V)+h+\varphi(\lambda_i)}^V) \right) \\ &\leq p \cdot \eta + \varepsilon (p \cdot H(\mathcal{A}) + p + H(\mathcal{A}^V)) \quad \text{which gives the inequality} \end{aligned}$$

$$\eta_p \leq p \cdot \eta .$$

Corollary 1 (cf [3] 2.3) , If $K = N$ then $h(T_p) = p \cdot h(T)$.

Theorem 3 (cf [3] 2.5) . If $K < N$ and $h(T) > 0$ then $h(T_p) = +\infty$.

Proof: Recall that $\langle P \rangle \simeq Z^K$, $\varphi : Z^K \rightarrow \langle P \rangle$

is an isomorphism , $K < N$. We extend φ to an isomorphism of Z^N into Z^N . In the sequel this extension is denoted also by φ . Let $p^{\#}$ denotes the index of subsemigroup $\varphi(Z^N)$ in Z^N and $P^{\#} \stackrel{\text{def}}{=} \varphi(Z^N) \cap G$. By Theorem 1, $h(T_{P^{\#}}) = p^{\#} \cdot h(T)$. The extension φ can be chosen in such a way that $p^{\#}$ is arbitrarily large. Thus it suffices to prove that $h(T_{P^*}) \leq h(T_p)$.

$\varphi^{-1}(P)$ contains a cone Λ_p in Z^K . $\varphi^{-1}(P^{\#})$ contains a cone Λ_* in Z^N . Fix $\varepsilon > 0$. There exists $n_0 \in N$ such that for $n \geq n_0$

$$(15) \quad \frac{1}{\text{card } \Lambda_p^n} H(\mathcal{A}(\Lambda_p^n)) \leq h(T_p, \mathcal{A}) + \varepsilon .$$

Let I_W be a rectangle from Lemma 1, constructed for $(\Lambda_p^n)_{n=n_0}^\infty$ and ε . For $n \in \mathbb{N}$ sufficiently large we can cover Λ_p^n by pairwise disjoint translates of I_W so precisely, that by a standard estimation we obtain the desired inequality.

Corollary 2. /of [3] 2.6./ . If $K < N$, $h(T_p) < +\infty$, then $h(T) = 0$.

Note that everything that was proved in part 2 is also valid for measure entropy (proofs without modifications).

3. Theorem of Dinaburg - Goodwyn - Goodman.

We introduce the following notations :

$\mathcal{M}(X)$ - the space of all Borel, normalised measures on X with weak σ - topology.

$\mathcal{M}(X, T)$ - the subspace of all T -invariant measures in $\mathcal{M}(X)$.

W - the set of all neighbourhoods of the diagonal in $X \times X$ directed by the inclusion.

Let $\delta \in W$. $\delta_C \stackrel{\text{def}}{=} \bigcap_{g \in C} (T^g \times T^g)^{-1} \delta$ for arbitrary $C \subset G$.

A finite subset e of X is called a/ (C, δ) - separated, if for all $x, y \in e$, $x \neq y$ we have $(x, y) \notin \delta_C$;
 b/ (C, δ) - spanning, if for all $x \in X$ there exists $y \in e$ such that $(x, y) \in \delta_C$.

Let $\tau(C, \delta) \stackrel{\text{df.}}{=} \min \{ \text{card } e : e \text{ is } (C, \delta)\text{-spanning} \}$, $s(C, \delta) = \max \{ \text{card } e : e \text{ is } (C, \delta)\text{-separated} \}$. We define

$$\bar{r}_T(\Lambda, \delta) \stackrel{\text{df.}}{=} \limsup_n \frac{1}{\text{card } \Lambda^n} \log \tau(\Lambda^n, \delta) ,$$

$$\bar{s}_T(\Lambda, \delta) \stackrel{\text{df.}}{=} \limsup_n \frac{1}{\text{card } \Lambda^n} \log s(\Lambda^n, \delta) .$$

By an argument analogous to the one applied in [3] the following definition makes sense,

Definition 3. $h_T(\Lambda) = \lim_{\delta} \bar{s}_T(\Lambda, \delta) = \lim_{\delta} \bar{r}_T(\Lambda, \delta) = \sup_{\delta} \bar{s}_T(\Lambda, \delta) = \sup_{\delta} \bar{r}_T(\Lambda, \delta)$.

Theorem 4. For all $\Lambda \subset G$ $h_T(\Lambda) = h(T)$.

The proof of this theorem is a translation of the proof [3] 4.8 to the language of the form structure W on X .

The following lemma will be used in the proof of Dinaburg-Goodwyn-Goodman theorem.

Lemma 2. Assume that $\mu \in \mathcal{M}(X, T)$ and \mathcal{A} is a μ -measurable finite partition of X . Let $p_i \in \mathbb{Z}_+^N$ for $i \in \mathbb{N}$ and $\lim p_i = +\infty$. Chose $g_i \in G$ such that $I_{p_i} + g_i \subset G$ for $i \in \mathbb{N}$. Then

$$h_{\mu}(T, \mathcal{A}) = \lim_i \frac{1}{\text{card } I_{p_i}} H_{\mu}(\mathcal{A}_{I_{p_i} + g_i})$$

Proof : $I \limsup_i \frac{1}{\text{card } I_{p_i}} H_{\mu}(\mathcal{A}_{I_{p_i} + g_i}) \leq h_{\mu}(T, \mathcal{A})$.

There exists a sequence of positive integers (n_l) such that $\frac{1}{\text{card } \Lambda^{n_l}} H_{\mu}(\mathcal{A}_{\Lambda^{n_l}}) \leq h_{\mu}(T, \mathcal{A}) + \epsilon$.

For i sufficiently large we cover $I_{p_i} + g_i$ by pairwise disjoint translates of a rectangle I_w from Lemma 1, constructed for (Λ^{n_i}) and ε .

A standard estimation yields the desired inequality.

$$\text{II. } h_\mu(\mathbb{T}, \mathcal{A}) \leq \liminf_i \frac{1}{\text{card } I_{p_i}} H_\mu(\mathcal{A}_{I_{p_i} + g_i}).$$

If $i \in \mathbb{N}$ then for sufficiently large $n \in \mathbb{N}$ we can find $k \in \mathbb{N}$, $\lambda_l \in \Lambda^n$, $l = 1, \dots, k$, such that $\Lambda^n = \bigcup_{l=1}^k (I_{p_i} + \lambda_l) \cup (\Lambda^n)'$, where the sets appearing in this sum are pairwise disjoint and $\frac{\text{card } (\Lambda^n)'}{\text{card } \Lambda^n} < \varepsilon$. Since

$$\begin{aligned} \text{for } l = 1, \dots, k, \quad & H_\mu(\mathcal{A}_{I_{p_i} + \lambda_l}) = H_\mu(\mathcal{A}_{I_{p_i} + \lambda_l + g_i}) = \\ & = H_\mu(\mathcal{A}_{I_{p_i} + g_i}), \quad \text{the following inequality holds:} \\ \frac{1}{\text{card } \Lambda^n} H_\mu(\mathcal{A}_{\Lambda^n}) & \leq \varepsilon \cdot H_\mu(\mathcal{A}) + \frac{1}{\text{card } I_{p_i}} H_\mu(\mathcal{A}_{I_{p_i} + g_i}). \end{aligned}$$

This inequality implies II.

Theorem 5. /Dinaburg-Goodwyn-Goodman/.

$$h(\mathbb{T}) = \sup_{\mu \in \mathcal{M}(X, \mathbb{T})} h_\mu(\mathbb{T}).$$

Proof: I. $\sup_{\mu \in \mathcal{M}(X, \mathbb{T})} h_\mu(\mathbb{T}) \leq h(\mathbb{T})$ /Goodwyn/.

The proof is analogous to the proof of Theorem 4.1 in [4].

$$\text{II. } h(\mathbb{T}) \leq \sup_{\mu \in \mathcal{M}(X, \mathbb{T})} h_\mu(\mathbb{T}) \quad \text{/cf [5] /}.$$

Fix $\delta > 0$ and $\delta \in W$. Let for all $n \in \mathbb{N}$ e_n be a set (Λ^n, δ) -separated of maximal cardinality.

For some sequence (n_k) of positive integers there exists

$$\lim_k \frac{1}{\text{card } \Lambda^{n_k}} \log \text{card } e_{n_k} = h_T(\Lambda, \delta) .$$

We construct a measure $\mu \in \mathcal{M}(X, T)$ in the way indicated in [5] : $\sigma_n(\{y\}) = \frac{1}{\text{card } e_n}$ for $y \in e_n$,

$$\mu_n \stackrel{\text{df}}{=} \frac{1}{\text{card } \Lambda^n} \sum_{g \in \Lambda^n} T^{ng} \sigma_n \text{ /definition of } T^{ng} \text{ is given}$$

in [5] /. In virtue of the theorem of Alaoglu there exists a cluster point $\mu \in \mathcal{M}(X)$ of the sequence (μ_{n_k}) . As in [5] one proves that $\mu \in \mathcal{M}(X, T)$.

Let \mathcal{A} be a finite Borel partition of X such that $a \cap a \subset \delta$ for $a \in \mathcal{A}$. Then for $a \in \mathcal{A}_{\Lambda^n}$ $a \cap a \subset \delta_{\Lambda^n}$ thus $\forall a \in \mathcal{A}_{\Lambda^n}$ $\text{card}(e_n \cap a) \leq 1$, so

$$H_{\sigma_n}(\mathcal{A}_{\Lambda^n}) = - \sum_{y \in e_n} \sigma_n(\{y\}) \log \sigma_n(\{y\}) = \log \text{card } e_n .$$

Let $(I_{p_i} + g_i)$ be a sequence from Lemma 2.

We can assume that $g_i \in Z_+^N$ for $i \in \mathbb{N}$.

Fix $m \in \mathbb{N}$ and ε , $0 < \varepsilon < \frac{\sigma}{2 \log \text{card } \mathcal{A}}$. There

exists $l_0 \in \mathbb{N}$ such that for $l \geq l_0$ $p_l - g_m - p_m \in Z_+^N$

and

$$(16) \quad \frac{\text{card } I_{p_l - g_m - p_m}}{\text{card } I_{p_l}} \geq 1 - \varepsilon .$$

If $l \geq l_0$, $l \in \mathbb{N}$, then for n sufficiently large we can find $t \in \mathbb{N}$, $\lambda_i \in \Lambda^n$, $i = 1, \dots, t$, such that $\Lambda^n = \bigcup_{i=1}^t (I_{p_l} + \lambda_i) \cup (\Lambda^n)^c$; the sets appearing

in this sum are pairwise disjoint and $\frac{\text{card}(\Lambda^n)'}{\text{card} \Lambda^n} \leq \varepsilon$.

Now, let $q \in I_{p_m}$. We define

$$s(q) = \left(\left[\frac{p_1^1 - g_m^1 - q^1}{p_m} \right], \dots, \left[\frac{p_1^N - g_m^N - q^N}{p_m} \right] \right).$$

Observe that $I_{p_l} = \bigcup_{r \in I_{s(q)}} (I_p + g_m + q + r \cdot p_m) \cup (I_{p_l})'$,

where the sets appearing in this sum are pairwise disjoint and $\text{card} (I_{p_l})' \leq \text{card} I_{p_l} - \text{card} I_{p_l - q_m - p_m} \leq \varepsilon \cdot \text{card} I_{p_l}$

/by (16) /. So, finally we can represent Λ^n as a sum of pairwise disjoint sets as follows $\Lambda^n = \bigcup_{i=1}^t (\bigcup_{r \in I_{s(q)}} (I_{p_m} + \lambda_i + g_m + q + r \cdot p_m) \cup (I_{p_l} + \lambda_i)) \cup (\Lambda^n)'$. Thus, for all $q \in I_{p_m}$

$$(17q) \quad H_{\sigma_n}(\mathcal{A}_{\Lambda^n}) \leq \text{card}(\Lambda^n)' \cdot \log \text{card} \mathcal{A} + \sum_{i=1}^t \text{card} I_{p_l}' \log \text{card} \mathcal{A} + \sum_{i=1}^t \sum_{r \in I_{s(q)}} H_{\sigma_n}((T^{\lambda_i + q + r \cdot p_m})^{-1} \mathcal{A}_{I_{p_m} + g_m}^{-1}).$$

Adding the inequalities (17q), $q \in I_{p_m}$, by sides we obtain

$$(18) \quad \text{card} I_{p_m} \cdot \log \text{card} e_n \leq \text{card} I_p \cdot \log \text{card} \mathcal{A} \cdot (\text{card}(\Lambda^n)' + t \cdot \text{card} I_{p_l}') + \sum_{i=1}^t \left(\sum_{q \in I_{p_m}} \sum_{r \in I_{s(q)}} H_{\sigma_n}((T^{\lambda_i + q + r \cdot p_m})^{-1} \mathcal{A}_{I_{p_m} + g_m}) \right) \leq \text{card} I_{p_m} \cdot \log \text{card} \mathcal{A} (\text{card}(\Lambda^n)' + t \cdot \text{card} I_{p_l}') + \sum_{g \in \Lambda^n} H_{\sigma_n}((T^g)^{-1} \mathcal{A}_{I_{p_m} + g_m}).$$

Dividing the inequality (18) by $\text{card} I_p \cdot \text{card} \Lambda^n$ and applying the inequalities

$$\frac{1}{\text{card } \Lambda^n} \sum_{g \in \Lambda^n} H_{\sigma_n}((T^g)^{-1} \mathcal{A}_{I_{p_m} + g_m}) \leq H_{\mu}(\mathcal{A}_{I_{p_m} + g_m})$$

and $\frac{t \cdot \text{card } I_{p_l}}{\text{card } \Lambda^n} \leq \frac{t \cdot \text{card } I_{p_l} \cdot \varepsilon}{\text{card } \Lambda^n} \leq \varepsilon$, we obtain

$$(19) \quad \frac{1}{\text{card } \Lambda^n} \log \text{card } e_n \leq 2 \cdot \varepsilon \log \text{card } \mathcal{A} +$$

$$+ \frac{1}{\text{card } I_{p_m}} \cdot H_{\mu_n}(\mathcal{A}_{I_{p_m} + g_m}).$$

Inequality (19) is true for all $n \in \mathbb{N}$ sufficiently large and \mathcal{A} can be chosen in such a way that the boundaries of the elements of \mathcal{A} have measure μ zero, hence taking the limit with respect to n /or with respect to a subsequence (n_k) if necessary / we get $h_T(\Lambda, \delta) \leq 2 \cdot \varepsilon \log \text{card } \mathcal{A} +$

$$+ \frac{1}{\text{card } I_{p_m}} H_{\mu}(\mathcal{A}_{I_{p_m} + g_m}) \leq \varepsilon + \frac{1}{\text{card } I_{p_m}} \cdot H_{\mu}(\mathcal{A}_{I_{p_m} + g_m})$$

for all $\delta \in W$ and $m \in \mathbb{N}$. Passing to the limit with δ and m , owing to the arbitraryness of ε , we obtain finally $h(T) \leq h_{\mu}(T)$.

Corollary 3. If T_{Ω} denotes on action of G on the set of nonwandering points Ω defined by $T_{\Omega}^g(x) = T^g(x)$ for $x \in \Omega$, then $h(T_{\Omega}) = h(T)$.

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