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## OSCILLATING GEODESICS ON 2-DIMENSIONAL MANIFOLDS

Maciej Wojtkowski

Introduction. The aim of this paper is to study the phenomenon of oscillating motions in the case of a geodesic flow on 2-dimensional open manifolds.

In 1. we classify trajectories of an abstract flow into bounded, escaping and oscillating, in future and in the past. We prove that the sets of trajectories oscillating in future and in the past differ merely by a set of an invariant measure zero. This theorem completes the theorem of E.Hopf /3/ in which the same is stated for escaping trajectories.

In 2. we tackle the problem of existence of oscillating geodesics on a 2-dimensional, open, finitely connected manifold. It turns out that their existence is closely connected with the presence of a horn on the manifold, in the sense of Cohn-Vossen /6/. We prove the existence of continuum of oscillating geodesics by purely topological methods for manifolds with a free fundamental group with more than one generator. The idea is to characterize a geodesic by an infinite word of generators of a fundamental group and it goes back to Hadamard /4/ and Morse /5/. Topological methods do not work when the fundamental group is commutative or trivial.

The rest of the paper is aimed at the construction of differential symbolic dynamics for a geodesic flow. It allows under weak conditions to find continuum of oscillating

geodesics, also in the case, when the fundamental group is trivial or commutative.

Also a large family of geodesics with different geometrical properties is obtained. We use the results of Alekseev /1/ and simplifications made by Moser /2/.

Special thanks go to Professor V.M.Alekseev who inspired me to do this work.

I. Let  $X$  be a locally compact, separable topological space and  $\{\varphi_t\}_{t \in \mathbb{R}}$  a continuous flow on  $X$ .

Definition. We call  $x \in X$

- a) bounded for  $t \rightarrow +\infty (t \rightarrow -\infty)$  iff there is a compact set  $K$  such that  $\varphi_t x \in K$  for  $t \geq 0 (t \leq 0)$ ;
- b) escaping for  $t \rightarrow +\infty (t \rightarrow -\infty)$  iff for every compact set  $K$  there is  $T \in \mathbb{R}$  such that  $\varphi_t x \notin K$  for  $t \geq T (t \leq -T)$ ;
- c) oscillating for  $t \rightarrow +\infty (t \rightarrow -\infty)$  iff a) and b) are not satisfied.

By  $B^+(B^-)$  we denote the set of all bounded elements for  $t \rightarrow +\infty (t \rightarrow -\infty)$ , by  $H^+(H^-)$  the set of escaping elements and by  $OS^+(OS^-)$  the set of oscillating ones. These sets are invariant under  $\{\varphi_t\}_{t \in \mathbb{R}}$ .

Suppose  $\nu$  is a Borel measure on  $X$ , invariant under  $\{\varphi_t\}_{t \in \mathbb{R}}$  and suppose  $\nu$  is finite for every compact subset of  $X$ . By a well known theorem of E.Hopf /3/  $\nu(H^+ \Delta H^-) = 0$ . We prove

Theorem I.

$$\nu(OS^+ \Delta OS^-) = 0$$

Proof. We shall prove that  $\nu(B^+ \Delta B^-) = 0$ . It is easy to see that the set of bounded elements for  $\{\varphi_n\}_{n \in \mathbb{Z}}$   $B_1^+(B_1^-)$  coincides with  $B^+(B^-)$ . Suppose  $K_i$  are compact subsets of  $X$  such that  $K_i \subset K_{i+1}$  for  $i = 1, 2, \dots$  and  $\bigcup_{i=1}^{\infty} K_i = X$ .

Let  $B_i^\pm(K_i) = \{x \in B_i^\pm \mid \varphi_n x \in K_i \text{ for } n \in \mathbb{Z}, n \geq 0 (n \leq 0)\}$

We see that  $\nu(B_i^\pm(K_i)) < +\infty$ ,  $B_i^\pm(K_i) \subset B_i^\pm(K_{i+1})$ ,  $i = 1, 2, \dots$

and  $B_i^\pm = \bigcup_{i=1}^{\infty} B_i^\pm(K_i)$ . Now  $\varphi_n B_i^\pm(K_i) \subset B_i^\pm(K_i)$  for  $n \in \mathbb{Z}^+$

and so  $\nu(\varphi_n B_i^\pm(K_i) \Delta B_i^\pm(K_i)) = 0$ . It gives

$\nu(\bigcap_{n=0}^{\infty} \varphi_n B_i^\pm(K_i) \Delta B_i^\pm(K_i)) = 0$ . But  $\bigcap_{n=0}^{\infty} \varphi_n B_i^\pm(K_i) \subset B_i^\pm(K_i)$ , so we

have  $\nu(B_i^\pm(K_i) \setminus B_i^\pm(K_i)) = 0$ . By symmetry  $\nu(B_i^+(K_i) \Delta B_i^-(K_i)) = 0$ .

Because of  $B_i^+ \Delta B_i^- \subset \bigcup_{i=1}^{\infty} B_i^+(K_i) \Delta B_i^-(K_i)$  the theorem is

proved.

2. Let  $M$  be a 2-dimensional manifold with a Riemannian metric of class  $C^2$ . We suppose  $M$  to be complete in this metric, open and finitely connected. There is a homeomorphism taking  $M$  onto  $X \setminus \{x_1, \dots, x_n\}$ , where  $X$  is a compact 2-dimensional manifold and  $x_i \in X$ , for  $i = 1, 2, \dots, n$ . An open subset  $U \subset M$ , homeomorphic to an open ring, will be called an escaping domain, if it is taken onto a deleted neighbourhood of  $x_i$  for some  $i = 1, 2, \dots, n$  by the homeomorphism above and if the closure of  $U$  in  $X$  contains only one of the points  $x_i$ ,  $i = 1, \dots, n$ .

Consider closed, rectifiable, Jordan curves in  $U$  which cannot be contracted to a point in  $U$ . By  $g(U)$  we denote the infimum of their lengths.

$U$  is called a horn if any sequence of curves which realises the infimum  $g(U)$  is not contained in a compact subset of  $M$ .  $U$  is called a cup if it is not a horn [6]. For a cup on a surface of rotation we see that geodesics that go far enough in the cup, escape to infinity. So as we are interested in oscillating geodesics we shall deal with surfaces containing horns.

A horn  $U$  is called regular if every escaping domain  $U_1$  such that  $U_1 \cap U$  is unbounded, is also a horn. A horn  $U$  is

called sharp if  $g(U) = 0$ . It is easy to see that if for an escaping domain  $U$  on  $M \neq \mathbb{R}^2$ ,  $g(U) = 0$ , then  $U$  is a regular horn.

Theorem 2. If  $M$  contains a regular horn and  $M$  is not homeomorphic to a plane, a cylinder or a projective plane without a point, then for every point  $m \in M$  there is continuum of oscillating geodesics beginning in  $m$ .

The sketch of the proof. For simplicity we consider the case of  $M$  homeomorphic to  $S^2$  without three points. We represent  $M$  as the half cylinder  $S^1 \times \mathbb{R}^+$  with two points of  $S^1 \times \{0\}$  taken away and appropriate identifications. We assume that  $S^1 \times (\mathbb{R}^+ \setminus \{0\})$  is a horn (this assumption is possible because  $M$  contains a regular horn). Consider the fundamental group  $\pi_1(M, m)$ . Let  $a$  be an element of  $\pi_1(M, m)$  represented by a loop homotopic to  $S^1 \times \{1\}$ . We choose  $b$ , another generator of  $\pi_1(M, m)$ , represented by a loop connecting  $m$  and  $S^1 \times \{0\}$ . For a sequence of natural numbers  $n_1, n_2, \dots$  we consider  $c_1, c_2, \dots$  elements of  $\pi_1(M, m)$  :

$$\begin{aligned} c_1 &= a^{n_1} \\ c_2 &= a^{n_1} b a^{n_2} \\ &\vdots \\ c_i &= a^{n_1} b a^{n_2} \dots b a^{n_i} \\ &\vdots \end{aligned}$$

We choose  $\gamma_i$  to be the shortest geodesic with  $m$  as a beginning and an end such that the homotopy class of  $\gamma_i$  is equal to  $c_i$ , for  $i = 1, 2, \dots$ . Consider one of the limitary geodesic rays -  $\gamma$ . It begins in  $m$  and has the property to be the shortest connection between two points in its homotopy class. Roughly speaking it loops  $n_1$  times around the cylinder  $S^1 \times \mathbb{R}^+$ , then it intersects  $S^1 \times \{0\}$ , then it loops  $n_2$  times around

$S^1 \times \mathbb{R}^+$  and so on. So for different sequences  $n_1, n_2, \dots$  we get different geodesic rays  $\gamma$ .

For an arbitrary compact subset  $K \subset M$  a curve connecting two points of  $S^1 \times \{0\}$ , looping  $n$  times around  $S^1 \times \mathbb{R}^+$  and the shortest in its homotopy class goes out of  $K$  if only  $n$  is sufficiently large. It is so because if such a curve lies in  $K$ , its length is not less than  $(n-1)g_K$  for some  $g_K$ , such that  $g_K > g + \varepsilon$  for some  $\varepsilon > 0$ . But if we do not restrict ourselves to  $K$ , we can easily find a curve, looping  $n$  times, the length of which does not exceed  $n(g + \varepsilon) + C$ , where  $C$  is independent of  $n$ . Now if we take  $n > \frac{C + g_K}{g_K - g - \varepsilon}$  the corresponding geodesic cannot lie in  $K$ . So we get that if the sequence  $n_1, n_2, \dots$  is unlimited, then  $\gamma$  is oscillating. The theorem is proved.

The examples of surfaces of rotation (Fig. I) show that the theorem 2 is not true for the excluded cases of surfaces with a trivial or commutative fundamental group. Nevertheless it will be our aim to show that these are exceptional, degenerated cases and under small perturbation we obtain the same picture as for surfaces with a free fundamental group.

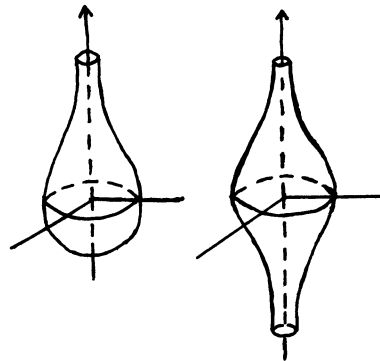


Fig. I

3. A horn  $U$  is called hyperbolic if Gaussian curvature in it is nonpositive. From now on we shall consider only hyperbolic horns. Cohn-Vossen /6/ proved that if Gaussian curvature in  $U$  is non-negative then  $U$  is a cup, so our presumption is not so

restrictive as it seems.

In a hyperbolic horn  $U$  we can introduce convenient coordinates:

Theorem 3. If  $U$  is a hyperbolic horn, there is an escaping domain  $U_1$ ,  $\bar{U}_1 \subset U$ , such that in  $\bar{U}_1$  we can introduce coordinates  $(r, s)$  of class  $C^1$ ,  $r \geq 0$ ,  $s \bmod d$ , so that

(i) the curves  $s = \text{const}$  are the only escaping geodesics in  $U_1$  with the property to be the shortest connection between any two of their points

(ii) the curve  $r=0$ , with  $s$  as a parameter on it is of class  $C^2$

(iii) the Riemannian metric in coordinates  $(r, s)$  is of the form  $d^2r + Y^2(r, s)d^2s$  where  $Y$  is a solution of the equation

$Y_r'' = -k(r, s)Y$  with  $Y(0, s) = 1$  and  $Y_r'(r, s) \leq 0$  (so  $Y$  is a principal solution),  $k(r, s)$  denotes Gaussian curvature.

(iv) for an arbitrary geodesic in  $U_1$ ,  $\alpha(s)$  denotes the angle between the vector of velocity of the geodesic and the curve  $s = \text{const}$ , then  $\alpha(s)$  is of class  $C^1$  and

$$\frac{d\alpha}{ds} = -Y_r'$$

The proof of this theorem resembles the construction of stable manifolds (oricycles) for geodesic flows on surfaces of negative curvature.

Proposition.  $U_1$  is a sharp horn if and only if  $\int_0^\infty k(r, s)r dr = \infty$  for almost all  $s$ .

Proof. It is easy to see that  $U_1$  is a sharp horn if and only if  $Y(r, s) \rightarrow 0$  when  $r \rightarrow +\infty$ , for almost all  $s$ , and so the proposition follows from the properties of principal solutions, Hartman /7/, chapter XI.

The behaviour of geodesics in  $U_1$  depends much on  $U_1$  being a sharp horn or not.

Theorem 4.  $U_1$  is a sharp horn if and only if the curves  $s = \text{const}$  are the only escaping geodesics in  $U_1$ .

Proof. Suppose  $(r(\tau), s(\tau))$ ,  $\tau \geq 0$  is an escaping geodesic in  $U_1$  and  $s(\tau) \neq \text{const}$ , for instance  $\frac{ds}{d\tau} > 0$ . Then  $\alpha(s(\tau))$  increases and  $\alpha(s(\tau)) < \frac{\pi}{2}$  for  $\tau \geq 0$ . Consider a real function  $f(r) = \int_0^r Y(r,s) ds$ .  $f$  is positive, decreasing and convex ( $f'' \geq 0$ ). It is easy to see, that  $U_1$  is a sharp horn if and only if  $\lim_{r \rightarrow \infty} f(r) = 0$ . Let  $r_n < r_{n+1}$ ,  $n = 0, 1, \dots$ , denote the coordinates of the points of intersection of our escaping geodesic and the curve  $s = 0$ , and  $\alpha_n$  - the angles in these points.

We have

$$\frac{dr}{d\tau} = \cos \alpha \quad \text{and} \quad \frac{ds}{d\tau} = \frac{\sin \alpha}{Y}$$

so  $\frac{dr}{ds} = Y \operatorname{ctg} \alpha$ .

Further

$$\Delta r_n = r_{n+1} - r_n = \int_0^{\alpha_n} \frac{dr}{d\alpha} d\alpha = \int_0^{\alpha_n} Y \operatorname{ctg} \alpha d\alpha \leq \operatorname{ctg} \alpha_n \int_0^{\alpha_n} Y d\alpha \leq \operatorname{ctg} \alpha_n f(r_n)$$

$$\Delta \alpha_n = \alpha_{n+1} - \alpha_n = \int_0^{\alpha_n} \frac{d\alpha}{ds} ds = \int_0^{\alpha_n} -Y'_r ds \geq -f'(r_{n+1})$$

So we have

$$\sum_{i=1}^{\infty} -f'(r_i) \leq \sum_{i=0}^{\infty} \Delta \alpha_i = \alpha_{\infty} - \alpha_0 < \frac{\pi}{2}$$

On the other hand

$$f(r_{n+1}) \geq f(r_n) + f'(r_n) \Delta r_n$$

and

$$\frac{f(r_{n+1})}{f(r_n)} \geq 1 + f'(r_n) \frac{\Delta r_n}{f(r_n)} \geq 1 + f'(r_n) \operatorname{ctg} \alpha_n \geq 1 + f'(r_n) \operatorname{ctg} \alpha_0, \text{ where } 1 + f'(r_n) \operatorname{ctg} \alpha_0 > 0$$

for sufficiently large  $n$ . Now we get



$$\lim_{r \rightarrow \infty} \frac{f(r)}{f(r_k)} = \prod_{i=k}^{\infty} \frac{f(r_{i+1})}{f(r_i)} \geq \prod_{i=k}^{\infty} (1 + f'(r_i) \alpha g_{d_0})$$

and so as the infinite product converges,  $\lim_{r \rightarrow \infty} f(r) > 0$  and hence  $U_1$  is not a sharp horn. We omit the proof of the reverse part of the theorem.

4. Now we restrict our attention to a sharp hyperbolic horn.\*

Consider the set  $R$  of unit tangent vectors whose carriers belong to the curve  $r=0$ . In this set we introduce coordinates  $(s, \varphi)$ ,  $s \pmod{\alpha}$  along the curve  $r=0$  and  $\varphi \pmod{2\pi}$  - the angle between the tangent vector and the curve  $s = \text{const}$ . Because of the Theorem 3 (ii)  $R$  with these coordinates is a closed submanifold of  $T_1 R$  of class  $C^1$ . Let

$$R^+ = \{(s, \varphi) \mid -\frac{\pi}{2} < \varphi < \frac{\pi}{2}, \varphi \neq 0\}$$

$$R^- = \{(s, \varphi) \mid \frac{\pi}{2} < \varphi < \frac{3\pi}{2}, \varphi \neq \pi\}$$

The geodesic flow in  $T_1 R$  defines the Poincare map  $\Phi$  taking, in view of Theorem 4,  $R^+$  onto  $R^-$ .  $\Phi$  is a diffeomorphism. It is easy to prove the following lemma.

Lemma I. Let  $(s_1, \varphi_1) = \Phi(s_0, \varphi_0)$ , then for a fixed  $s_0$ , if  $\varphi_0 \rightarrow 0$  then  $\varphi_1 \rightarrow \pi$  and  $s_1 \rightarrow \infty$ .

Now our aim is to investigate the behaviour of  $d\Phi$  for  $(s_0, \varphi_0)$  with  $\varphi_0$  close to zero.

By  $(\xi^s, \xi^\varphi)$  we denote coordinates in  $T R$  defined by coordinates  $(s, \varphi)$  in  $R$ .

The main tool in the construction of symbolic dynamics is the following theorem.

\* Added in proof : the case of a hyperbolic horn that is not sharp can be treated in an analogous way.

Theorem 5. Let  $\bar{s}_0$  and  $\bar{s}_1$  be fixed and let  $\Phi(s_0, \varphi_0) = (s_1, \varphi_1)$ ,  $d\Phi_{(s_0, \varphi_0)}(\xi^s, \xi^\varphi) = (\xi_1^s, \xi_1^\varphi)$  then

(i) for every  $\delta > 0$  there is  $\eta > 0$ , such that if  $|\varphi_0| < \eta$ ,  $|s_0 - \bar{s}_0| < \eta$ ,  $|\varphi_1 - \pi| < \eta$  and  $|s_1 - \bar{s}_1| < \eta$  then from  $|\xi_1^\varphi| \geq \delta |\xi_1^s|$  follows that  $|\xi_1^\varphi| \leq \delta |\xi_1^s|$  ;

(ii) if  $Y(\tau, \bar{s}_0) \rightarrow 0$  when  $\tau \rightarrow +\infty$  then for every  $\delta > 0$  and  $\lambda > 1$  there is  $\eta > 0$  such that if  $|\varphi_0| < \eta$  and  $|s_0 - \bar{s}_0| < \eta$  then from  $|\xi_1^\varphi| \geq \delta |\xi_1^s|$  follows that  $|\xi_1^s| \geq \lambda |\xi_1^\varphi|$ .

The proof of this theorem is obtained by the detailed study of the Jacobi equation with the help of the following lemma.

Lemma 2. Let  $k(t)$  be a continuous, nonpositive function for  $t \geq 0$  and let  $y_0(t)$  be a principal solution of the equation

$$y'' = -k(t)y$$

then for every  $\varepsilon > 0$  there are  $\tau > 0$  and  $T > 0$  such that if

$k_1(t)$  is a continuous, nonpositive function for  $0 \leq t \leq T$  and

$$|k(t) - k_1(t)| < \tau \quad \text{for } 0 \leq t \leq T \quad \text{then}$$

from  $\left| \frac{y_0'(0)}{y_0(0)} - \frac{y_1'(0)}{y_1(0)} \right| > \varepsilon$  follows that  $\frac{y_1'(T)}{y_1(T)} > 0$

where  $y_1(t)$  is a solution of the equation

$$y'' = -k_1(t)y.$$

5. Let us now suppose that there is a geodesic on  $M$  that escapes to infinity in  $\mathcal{U}_1$  both in future and in the past, that is some  $(\bar{s}_1, \pi) \in \mathcal{R}^-$  is taken by the geodesic flow into  $(\bar{s}_0, 0) \in \mathcal{R}^+$ . The geodesic intersects  $\mathcal{R}^-$  and  $\mathcal{R}^+$  transversally in  $\mathbb{T}_1 M$ .

Let  $\Psi$  be the Poincare map taking a neighbourhood of  $(\bar{s}_1, \pi)$  in  $\mathcal{R}^-$  onto a neighbourhood of  $(\bar{s}_0, 0)$  in  $\mathcal{R}^+$ . We assume that the following conditions are satisfied:

(A) (transversality condition) the image of the curve  $(s_1, \pi)$ ,  $|s_1 - \bar{s}_1| < \varepsilon$  under  $\Psi$  transversally intersects the curve  $(s_0, 0)$ ,  $|s_0 - \bar{s}_0| < \varepsilon$ .

(B)  $Y(r, \bar{s}_0) \rightarrow 0$  and  $Y(r, \bar{s}_1) \rightarrow 0$  when  $r \rightarrow \infty$ .

By  $S(N)$  we denote the space of sequences  $\{a_n; n_1 \leq n \leq n_2\}$ ,  $-\infty \leq n_1 \leq -1 < 0 \leq n_2 \leq +\infty$  where  $a_n \in \mathbb{Z}$ ,  $|a_n| > N$  for  $n_1 < n < n_2$  and  $a_{n_1} = a_{n_2} = \infty$ . Defining a topology in  $S(N)$  as in /2/ we obtain a compact space. The left shift  $\tau$  takes  $\Delta^+ \subset S(N)$  onto  $\Delta^- \subset S(N)$  where  $\Delta^+$  contains the sequences with  $n_2 \geq 1$  and  $\Delta^-$  the sequences with  $n_1 \leq -2$ .

The construction of symbolic dynamics is contained in the following theorem:

Theorem 6. If (A) and (B) hold then there are  $N > 0, \eta > 0$  and homeomorphism  $h: S(N) \rightarrow \mathcal{U}_\eta$ , where  $\mathcal{U}_\eta = \{(s_0, \varphi_0) \mid |s_0 - \bar{s}_0| \leq \eta, |\varphi_0| \leq \eta\}$ , such that the diagram

$$\begin{array}{ccc} \Delta^+ & \xrightarrow{\tau} & \Delta^- \\ h \downarrow & & \downarrow h \\ \mathcal{U}_\eta & \xrightarrow{\Psi \circ \Phi} & \mathcal{U}_\eta \end{array}$$

is commutative. (More exactly  $\Psi \circ \Phi$  takes  $h(\Delta^+)$  onto  $h(\Delta^-)$  and  $\Psi \circ \Phi \circ h = h \circ \tau$  on  $\Delta^+$ ).

Moreover for  $(s_0, \varphi_0) = h(\{a_n; n_1 \leq n \leq n_2\})$  if  $n_2 = 0$  then  $\varphi_0 = 0$ , and if  $n_2 \geq 1$  then for  $(s_1, \varphi_1) = \Phi(s_0, \varphi_0)$  we have

$$|s_1 - s_0 - a_0 d| \leq |\bar{s}_1 - \bar{s}_0| + 2\eta.$$

Analogously for  $(s_1, \varphi_1) = (\Psi^{-1} \circ h)(\{a_n; n_1 \leq n \leq n_2\})$  if  $n_1 = -1$  then  $\varphi_1 = \pi$ .

The sketch of the proof. By Lemma I  $\Phi(\mathcal{U}_\eta)$  and  $\Psi^{-1}(\mathcal{U}_\eta)$  intersect in infinitely many components, see Fig.2.

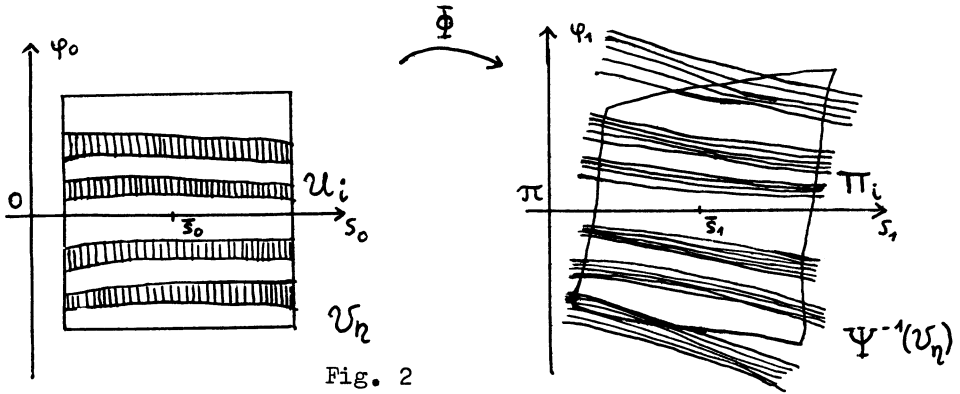


Fig. 2

Each component corresponds to geodesics that loop one and the same number of times around  $\mathcal{U}_1$  and we index the components by these numbers  $\pm N_1, \pm(N_1+1), \dots$  - a sign showing the direction in which the geodesic loops. Dropping finitely many of the components we get a sequence  $\Pi_i$ ,  $i = \pm N_2, \pm(N_2+1), \dots$  of curvilinear quadrilaterals connecting the opposite sides of  $\Psi^{-1}(\mathcal{U}_\eta)$  with horizontal sides  $\mathcal{C}^1$  convergent to the curve  $(s_1, \pi)$ , in view of Theorem 5.

The pre-images  $\Phi^{-1}(\Pi_i)$  form a sequence  $\mathcal{U}_i$  of curvilinear quadrilaterals connecting the opposite sides of  $\mathcal{U}_\eta$  with horizontal sides  $\mathcal{C}^1$  convergent to the curve  $(s_0, 0)$ ,  $|s_0 - \bar{s}_0| \leq \eta$ . The image of the curve  $(s_1, \pi)$ , under  $\Psi$  is by transversality condition connecting the opposite sides of  $\mathcal{U}_\eta$  if  $\eta$  is sufficiently small. The same is true of  $\mathcal{V}_i = \Psi(\Pi_i)$ : they connect the opposite sides of  $\mathcal{U}_\eta$  and their "vertical" sides converge to the  $\Psi$ -image of the curve  $(s_1, \pi)$ , (Fig.3).

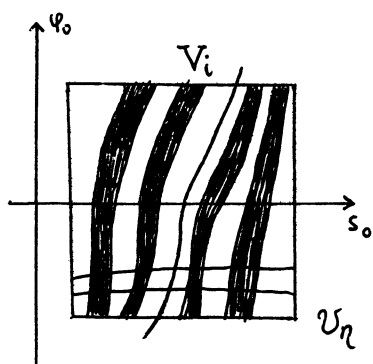


Fig. 3

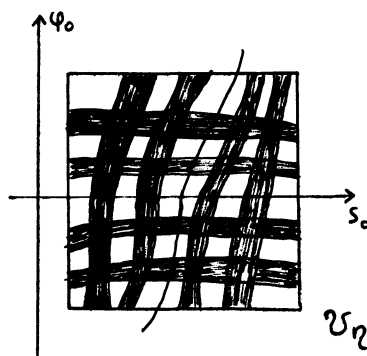


Fig. 4

So we get the family of "horizontal" strips  $\mathcal{U}_i$  that are taken by  $\Psi \circ \Phi$  onto "vertical" strips  $V_i$ ,  $i = \pm N, \pm(N+1), \dots$  in such a way that vertical sides of  $\mathcal{U}_i$  are transformed onto vertical sides of  $V_i$  and horizontal sides onto horizontal (Fig. 4). It is very important that when we take lesser  $\eta$  the picture is not destroyed, merely some of the strips are dropped and the rest of them are narrowed. This is the situation studied by Alekseev /1/ (for a simple exposition see Moser /2/) and we use the construction of symbolic dynamics described there. By Theorem 5, for sufficiently small  $\eta$ , we obtain that the necessary conditions for  $d(\Psi \circ \Phi)$  on  $\cup \mathcal{U}_i$  and  $d(\Phi^{-1} \circ \Psi^{-1})$  on  $\cup V_i$  are satisfied and by corresponding theorems in /1/ or /2/ we get homeomorphism  $h$  that takes infinite sequence  $\{a_n; -\infty \leq n \leq +\infty\}$  into such a point  $x = h(\{a_n; -\infty \leq n \leq +\infty\})$  that  $(\Psi \circ \Phi)^n x \in \mathcal{U}_{a_n}$ . It is easily extendable to the whole of  $S(N)$  and for this homeomorphism the theorem holds.

This theorem provides a large family of geodesics with different geometrical properties. If a sequence  $\{a_n; -\infty \leq n \leq +\infty\}$

is bounded on the right the corresponding geodesic is bounded in future and when it is unbounded the geodesic is oscillating. Finite sequences correspond to escaping geodesics.

And thus we get continuum of oscillating geodesics.

6. We end with a discussion of situations when the assumptions of Theorem 6 are satisfied.

If there are two hyperbolic sharp horns on  $M$  then obviously there is a geodesic escaping in one horn in the past and in another in future. If analogs of conditions (A) and (B) for this geodesic hold then we can construct symbolic dynamics analogous to one described in Theorem 6. And a posteriori we find a geodesic escaping in future and in the past in one of the horns. Moreover the transversality condition is automatically satisfied. If condition (B) is also satisfied Theorem 6 applies.

Suppose now there is a geodesic on  $M$  escaping in a sharp hyperbolic horn both in future and in the past but transversality condition is not satisfied. Then we can take a neighbourhood of a point on the geodesic and change there the metric in such a way that the geodesic persists to be a geodesic of the new metric and Gaussian curvature on it increases (or decreases). It is enough for the transversality condition to hold if the change of Gaussian curvature is not too big. The change of metric can be made  $C^\infty$  arbitrary small.

Now if (A) holds but not (B) we can take a  $C^\infty$  arbitrary close metric and get both (A) and (B) for a close geodesic. It should be pointed out that in view of Proposition condition (B) holds automatically if Gaussian curvature satisfies some uniformity condition in  $U_\lambda$ , for instance  $k(r,s) \leq k_0(r) \leq 0$

for all  $(r,s)$  and  $\int_0^{\infty} k_0(r) r dr = \infty$ .

In conclusion we see that Theorem 6 applies to small perturbations of surfaces of rotation from Fig. I.

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REFERENCES

- /1/ ALEKSEEV, V.M., Quasirandom dynamical systems I, Math. USSR Sbornik 5, 73-128 (1968) .
- /2/ MOSER, J., Stable and Random Motions in Dynamical Systems, Princeton Univ. Press, 1973 .
- /3/ HOPF, E., Ergodentheorie, Springer-Verlag, 1937, in particular, § 13.
- /4/ HADAMARD, J., Les surfaces a courbures opposés et leurs lignes géodésiques, Journ. de Math. (5) 4, 27-73 (1898).
- /5/ MORSE, H.M., A one-to-one representation of geodesics on a surface of negative curvature, Am. J. Math., 43, 33-51 (1921).
- /6/ COHN-VOSSEN, S., Kürzeste Wege und Totalkrümmung auf Flächen, Compos. Math. 2, 69-133 (1935).
- /7/ HARTMAN, P., Ordinary Differential Equations, J. Wiley and Sons, Inc., 1964.

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