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Topological entropy and Hausdorff dimension
 for area preserving diffeomorphisms of surfaces

by

S. E. Newhouse

In this paper we present two new generic properties of C^1 area preserving diffeomorphisms of a compact oriented surface. We obtain a lower bound for the topological entropy of a generic diffeomorphism, and we show that such a diffeomorphism always has closed invariant sets with dense orbits and Hausdorff dimension two.

Before stating our results precisely, let us fix some notation and recall some definitions. Let M be a C^∞ compact, connected, orientable 2-manifold, and let ω be a C^∞ area form on M . That is, ω is a nowhere vanishing differential 2-form of class C^∞ . Let $\text{Diff}_\omega^1 M$ denote the space of C^1 diffeomorphisms of M which preserve ω , and give $\text{Diff}_\omega^1 M$ the uniform C^1 topology.

For f in $\text{Diff}_\omega^1 M$, a point $p \in M$ is periodic if $f^n p = p$ for some $n > 0$. Let $\tau(p) = \inf\{n > 0 : f^n p = p\}$. This is the period of p . The periodic point p is hyperbolic if all eigenvalues of $T_p f^{\tau(p)}$ have norm different from one. In our case this means that $T_p f^{\tau(p)}$ has a single eigenvalue of norm bigger than one. Call this eigenvalue $\lambda(p)$. Let $n > 0$ be a positive integer, and let $\text{Hyp}_n f$ denote the set of hyperbolic periodic points of f with period less than or equal to n . Define $s_n(f) = \max\{\frac{1}{\tau(p)} \log |\lambda(p)| : p \in \text{Hyp}_n f\}$, and set $s(f) = \sup_{n \geq 1} s_n(f)$.

Let d be a topological metric on M . For $\epsilon > 0$, $n > 0$, a set $E \subset M$ is (n, ϵ) -separated if for any $x \neq y$ in E , there is a $0 \leq j < n$ such that $d(f^j x, f^j y) > \epsilon$. Let $r(n, \epsilon, f)$ be the maximal cardinality of an (n, ϵ) -separated set. The number $h(f) = \lim_{\epsilon \rightarrow 0} \limsup_{n \rightarrow \infty} r(n, \epsilon, f)$ is the topological entropy of f . It is a rough asymptotic measure of how much f mixes up the points in M . For any C^1 diffeomorphism, $0 \leq h(f) < \infty$.

If $\Lambda \subset M$ is a closed f -invariant set, then $h(f|\Lambda)$ is defined similarly, and it is easy to see that $h(f|\Lambda) \leq h(f)$. Also, for any integer n ,

$$h(f^n|\Lambda) = |n|h(f|\Lambda), \text{ and if } \phi : \Lambda \rightarrow \Lambda_1$$

is a homeomorphism, then $h(\phi f \phi^{-1}|\Lambda_1) = h(f|\Lambda)$. For more properties of h we refer to [2]. If p is a hyperbolic periodic point of the diffeomorphism f with orbit $o(p)$, we let $H(p, f)$ be the set of transverse homoclinic points of p . Thus $H(p, f)$ is the set of transverse intersections of $W^u(o(p), f)$ and $W^s(o(p), f)$ where $W^u(o(p), f)$ and $W^s(o(p), f)$ are the unstable and stable manifolds of the orbit $o(p)$. Then the closure $\overline{H(p, f)}$ of $H(p, f)$ is a closed f -invariant set on which f has a dense orbit [4].

If E is a closed subset of M and $\alpha > 0$, $\epsilon > 0$ are positive real numbers, let

$$H_\epsilon^\alpha(E) = \inf \left\{ \sum_i (\text{diam } U_i)^\alpha : \{U_i\} \text{ is a countable open covering of } E \text{ each of whose elements has diameter less than } \epsilon \right\}.$$

The Hausdorff α -outer measure of E is the number $H^\alpha(E) = \lim_{\epsilon \rightarrow 0} H_\epsilon^\alpha(E)$. The Hausdorff dimension of E , denoted $HD(E)$, is the number

$$\inf\{\alpha : H^\alpha(E) = 0\} = \sup\{\beta : H^\beta(E) = \infty\}.$$

If $\dim E$ is the topological dimension of E , then $HD(E) \geq \dim E$. Also, $m(E) > 0$ implies $HD(E) = 2$, but not conversely, where $m(E)$ is the Lebesgue measure of E .

A closed f -invariant set Λ is hyperbolic if there are a continuous splitting $T M|_\Lambda = E^s \oplus E^u$, a Riemann norm $|\cdot|$, and a constant $0 < \lambda < 1$ such that $Tf(E^s) = E^s$, $Tf(E^u) = E^u$, $|Tf|E^s| < \lambda$, and $|Tf^{-1}|E^u| < \lambda$. The hyperbolic set Λ is a hyperbolic basic set if $f|_\Lambda$ has a dense orbit and there is a compact neighborhood U of Λ such that $\bigcap_{-\infty < n < \infty} f^n U = \Lambda$. For $g \in C^1$ near f , there is a

hyperbolic basic set $\Lambda(g) = \bigcap_{-\infty < n < \infty} g^n U$ for g such that $f|_{\Lambda(f)}$ and $g|_{\Lambda(g)}$ are topologically equivalent [3].

If M is a hyperbolic set for f , then f is called Anosov.

Theorem. There is a residual set $\mathcal{B} \subset \text{Diff}^1 M$ such that if f is in \mathcal{B} , then each set $\overline{H(p, f)}$ has Hausdorff dimension two. In addition, if f is in \mathcal{B} and f is not Anosov, then

$$(*) \quad \underline{h(f)} > \underline{s(f)}$$

Recall that a residual set is one which contains a countable intersection of dense open sets. Properties true for residual sets are called generic, and a generic diffeomorphism is defined to be an element of some residual set.

Remarks 1. For an Anosov diffeomorphism f , each $\overline{H(p, f)} = M$, so the first statement of our theorem is trivially true. On the other hand, it is easily seen that there are open sets of Anosov diffeomorphisms for which $(*)$ fails. For instance, if f is linear, then $h(f) = \log |\lambda(p)|$ where $f(p) = p$. However, with a small perturbation, we can increase the expansion at non-fixed periodic points to make $(*)$ fail. With a bit more work one can show that $(*)$ fails for an open dense set of Anosov diffeomorphisms. To see this, consider the function ϕ^u of Bowen and Ruelle [1]. We may suppose that f is C^2 , so Lebesgue measure is the unique equilibrium state for ϕ^u . Let μ be the unique invariant measure of maximal entropy for f . Then, $-\int \phi^u d\mu \leq s(f)$. As μ and m are ergodic f -invariant probability measures, they are either equivalent or mutually singular. Using Proposition 4.5 of [1] and simple perturbation techniques, one can show that C^2 generically, μ is singular with respect to m . Then,

$$\begin{aligned} 0 = P_m(\phi^u) &= h_m(f) + \int \phi^u dm \\ &> h_\mu(f) + \int \phi^u d\mu \\ &= h(f) + \int \phi^u d\mu, \end{aligned}$$

so $h(f) < s(f)$. Since $h(f) < s(f)$ is a C^1 open condition for Anosov diffeomorphisms, (*) fails for a C^1 open dense set.

2. It would be nice to know if C^1 generically each set $\overline{H(p,f)}$ has positive measure or if $f|_{\overline{H(p,f)}}$ has positive measure theoretic entropy. Also, what analogs of our results hold for the C^r topology, $r \geq 2$?

We proceed to the proof of the theorem.

In view of remark 1 our theorem only has content for non-Anosov diffeomorphisms. Let A be the set of Anosov diffeomorphisms on M and let $\mathcal{D} = \text{Diff}_\omega^1 M - A$. Of course, A is open in $\text{Diff}_\omega^1 M$ and is empty unless M is the two-dimensional torus.

For positive integers n and m , let $B_{n,m}$ be the set of diffeomorphisms f in \mathcal{D} such that there are a p in $\text{Hyp}_n f$ and a hyperbolic basic set $\Lambda \subset \overline{H(p,f)}$ satisfying $h(f|_\Lambda) > s_n(f) - \frac{1}{m}$. Analogously, we let $B'_{n,m}$ be the set of diffeomorphisms f in \mathcal{D} such that $\text{Hyp}_n f \neq \emptyset$, and, for each p in $\text{Hyp}_n f$, there is a hyperbolic basic set $\Lambda \subset \overline{H(p,f)}$ so that $\text{HD}(\Lambda) > 2 - \frac{1}{m}$.

We assert that (1) $B_{n,m}$ and $B'_{n,m}$ are dense open sets in \mathcal{D} .

The theorem follows from (1) by taking $B = A \cup \bigcap_{n,m} B_{n,m} \cap B'_{n,m}$.

The main step in the proof of (1) is the next result.

Proposition. Suppose p is a hyperbolic periodic point of the diffeomorphism f and $W^u(o(p))$ is tangent to $W^s(o(p))$ at some point. Given $\epsilon > 0$ and any neighborhood N of f in \mathcal{D} , there is a g in N such that p is a hyperbolic periodic point for g , and

(a) g has a hyperbolic basic set Λ in $\overline{H(p,g)}$ on which

$$h(g|_\Lambda) > \frac{1}{\tau(p)} \log |\lambda(p)| - \epsilon$$

(b) each g_1 near g has a hyperbolic basic set $\Lambda(g_1)$ in $\overline{H(p(g_1),g_1)}$ such that $\text{HD}(\Lambda(g_1)) > 2 - \epsilon$.

Before proving the proposition, let us show how we can use it to prove assertion (1).

Let $f \in \mathcal{D}$, and let n and m be positive integers. We may perturb f to f_1 so that the hyperbolic and elliptic periodic points of f_1 are dense in M by theorems (1.3) and Corollary (3.2) in [5]. Using Takens [10], we may also assume $W^u(p, f_1) \cup W^s(p, f_1) \subset \overline{H(p, f_1)}$ for each hyperbolic periodic point p of f_1 . Choose $p \in \text{Hyp}_n(f_1)$ so that $\frac{1}{\tau(p)} \log|\lambda(p)| > s_n(f_1) - \frac{1}{2m}$.

Since f_1 has elliptic periodic points, it is in \mathcal{D} . If $\overline{H(p, f_1)}$ were hyperbolic, it would have interior (since $W^u(p) \cup W^s(p) \subset \overline{H(p, f_1)}$). But then local product structure [9, Theorem (7.4)] and topological transitivity would imply that $\overline{H(p, f_1)}$ is open and closed in M . So $\overline{H(p, f_1)}$ would equal M , making f_1 Anosov and giving a contradiction. Thus, $\overline{H(p, f_1)}$ is not hyperbolic. Using [5], we can find $f_2 \in C^1$ near f_1 so that $p \in \text{Hyp}_n f_2$, and $W^u(o(p))$ has a tangency with $W^s(o(p))$. Applying statement (a) in the proposition enables us to find $f_3 \in C^1$ near f_2 so that f_3 has a hyperbolic basic set Λ with entropy larger than $\frac{1}{\tau(p)} \log|\lambda(p)| - \frac{1}{4m}$. Also, $s_n(\cdot)$ is continuous, so if f_3 is near f_1 and f' is near f_3 , we have $s_n(f') < s_n(f_1) + \frac{1}{4m}$. But Λ continues to topologically equivalent hyperbolic sets for perturbations f' of f_3 . Hence, for f' near f_3 ,

$$\begin{aligned} h(f') &> \frac{1}{\tau(p)} \log|\lambda(p)| - \frac{1}{4m} \\ &> s_n(f_1) - \frac{3}{4m} \\ &> s_n(f') - \frac{1}{m} . \end{aligned}$$

This proves that $B_{n,m}$ is dense and open in \mathcal{D} . Similarly, we can use statement (b) of the proposition to prove that $B'_{n,m}$ is dense and open in \mathcal{D} .

It remains to prove the proposition. All of our estimates will be with respect to the C^r norm induced from a fixed finite covering by symplectic coordinate charts, $r = 1$ and 2 . The C^r norm of a function f will be the maximum of the r^{th} order partial derivatives computed in that covering, and we denote it by $|f|_{C^r}$.

All of our approximations are local and will be done in local coordinates using generating functions. Let us recall the main properties of these functions.

Suppose (x,y) are coordinates in \mathbb{R}^2 and $f(x,y) = (\xi(x,y), \eta(x,y))$ is an area preserving C^1 diffeomorphism with $f(o,o) = (o,o)$ and $\frac{\partial \eta}{\partial y}$ nowhere zero. Then we may solve for y as a C^1 function of x and η in the equation $\eta = \eta(x,y)$, and the mapping $(x,\eta) \rightarrow (x,y(x,\eta))$ allows us to use x and η as coordinates on \mathbb{R}^2 . Since f preserves area, the 1-form $\alpha = \xi d\eta + y dx$ is closed, and we may find a unique C^2 function $S(x,\eta)$ so that $S(o,o) = 0$, $S_x = y$, $S_\eta = \xi$, and $S_{x\eta}$ never vanishes. The function S is called the generating function of f . Conversely, given a C^2 function $S(x,\eta)$ so that $S(o,o) = 0$ and $S_{x\eta}(x,\eta)$ is never zero, we may solve for η as a function of x and y in the equation $S_x(x,\eta) = y$, and obtain an area preserving diffeomorphism by

$$f(x,y) = (S_\eta(x,\eta(x,y)), \eta(x,y)) .$$

If g is an area preserving diffeomorphism C^1 near f , then its generating function \bar{S} is C^2 near S , and conversely, The generating function for the identity transformation is $S(x,\eta) = x\eta$.

We now begin the proof of the proposition. Let us assume, at first, for notational simplicity, that p is a fixed point of f , so $\tau(p) = 1$.

Suppose $W^u(p,f)$ is tangent to $W^s(p,f)$ at z_0 . With a preliminary C^1 approximation we may make $W^u(p,f)$ and $W^s(p,f)$ coincide on a small curve, say I , around z_0 in $W^u(p,f)$. The picture is as follows:

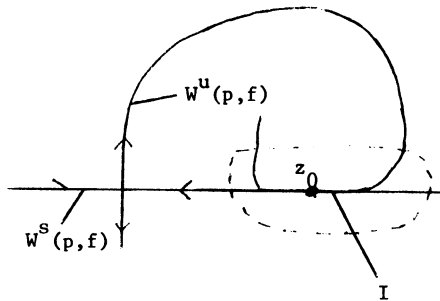


Figure 1

Let U be a small neighborhood around z_0 in M with $f^{-1}U \cap U = \emptyset$ and assume I small enough to be in U . Introduce local symplectic coordinates $z = (x,y)$ about $z_0 = (0,0)$ in U so that I is contained in $(y = 0)$. Thus there is a diffeomorphism $\phi : U \rightarrow \mathbb{R}^2$ so that $\phi(z_0) = (0,0)$, $\phi(I) \subset \{(x,y) : y = 0\}$, and $\phi^*(dx \wedge dy) = \omega$. Let $a > 0$ be such that $\phi^{-1}([-2a,2a]) \subset U$.

We identify \mathbb{R}^2 with U via ϕ in the sequel.

Let $\epsilon > 0$. We will produce an area preserving C^1 perturbation g of f with $g(z) = f(z)$ for $z \notin f^{-1}U$ such that g has a hyperbolic basic set $\Lambda \subset \overline{H(p,g)}$ such that $h(g|\Lambda) > \log|\lambda(p)| - \epsilon$.

Intuitively, we obtain Λ in the following way. Introduce a large number of bumps in $W^u(p,g)$ over the interval $[-a,a]$ in I without disturbing the fact that $I \subset W^s(p,g)$. Letting I' denote the piece of $W^u(p,g)$ over I , we arrange for I' to be the graph of the function $x \rightarrow A \cos(\frac{\pi Nx}{2a})$ with $-a \leq x \leq a$, N a large positive integer, and A a small positive number. The maximum height of I' is A , the minimum is $-A$, and I' has N intersections with I . This gives the next figure

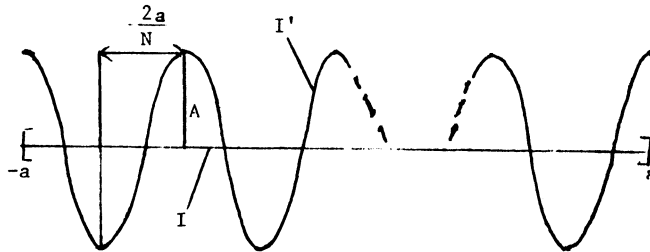


Figure 2

To do this with $g \delta - C^1$ close to f we will need $2\Lambda \left(\frac{2a}{N}\right)^{-1} < K_1 \delta$ for some constant K_1 independent of N . Suppose we take $A = \frac{K_1 \delta a}{2N}$. Since $I \subset W^s(p, g)$ and $I' \subset W^u(p, g)$, we will be able to find a rectangle D_A with distance around $\frac{A}{4}$ units from I whose image under g^n for some large n is around $\frac{A}{4}$ units from I' as in the next figure

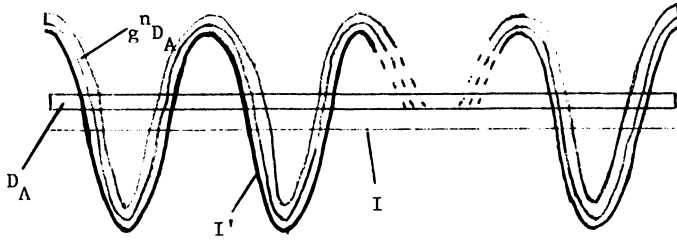


Figure 3

The "around" in the preceding statement means we are ignoring constants independent of N . Then, if Λ_1 is the largest invariant set for $g^n|_{D_A}$, Λ_1 will be hyperbolic for g^n and $h(g^n|_{\Lambda_1}) = \log N$. This gives us $\Lambda = \bigcup_{0 < j < n} g^{-j} \Lambda_1$ hyperbolic for g and $h(g|_{\Lambda}) = \frac{1}{n} \log N$. From the construction, g has a periodic point in Λ which is homoclinically related to p , so $\Lambda \subset \overline{H(p, g)}$. Except, for constants independent of N , we will have $|\lambda(p)|^{-n} = \frac{A}{4} = \frac{K_1 \delta a}{2N}$. Thus, $-n \log |\lambda(p)| = \log \frac{K_1 \delta a}{8} - \log N$ or $\log |\lambda(p)| = -\frac{1}{n} \log \frac{K_1 \delta a}{8} + \frac{\log N}{n}$. Choosing N very large forces n to be large, so we can get

$$h(g|_{\Lambda}) = \frac{1}{n} \log N > \log |\lambda(p)| - \epsilon.$$

Let us now specify more precisely how we obtain g .

Let $\alpha(x, y)$ be a C^∞ function from U to \mathbb{R} so that $\alpha(x, y) = 1$ on a neighborhood U_1 of I and $\alpha(x, y) = 0$ off a slightly larger neighborhood contained

in U . Given the neighborhood N of f , let $\delta > 0$ be small enough so that any g which is $\delta - C^1$ -close to f must be in N . Let A be a small constant, and consider the area preserving transformation $\xi(x,y) = (x, A \cos \frac{\pi x N}{2a} + y)$. It carries the line segment $-a \leq x \leq a, y = 0$ onto a curve I' as described earlier.

The generating function for ξ is $S(x,\eta) = x\eta - \int_0^x A \cos(\frac{\pi s N}{2a}) ds$ where $\xi(x,y) = x$ and $\eta(x,y) = A \cos(\frac{\pi x N}{2a}) + y$. Note that $S_{x\eta} = 1$ throughout the region, so (x,η) is a good coordinate system throughout.

Let $\beta(x,\eta) = \alpha(x,y(x,\eta)) = \alpha(x,\eta - A \cos \frac{\pi x N}{2a})$, and let $S_1(x,\eta) = \beta(x,\eta)(S(x,\eta) - x\eta) + x\eta$. The reader may check that as AN approaches 0, the function $S(x,\eta) \sim x\eta$ approaches 0 in the C^2 topology. Thus, for AN small, $S_{1x\eta}(x,\eta) \neq 0$ for all x,η . We may find a C^1 function $\eta_1(x,y)$ so that $S_{1x}(x,\eta_1(x,y)) = y$, and $\eta_1(x,y)$ approaches $\eta(x,y)$ in the C^1 topology as $AN \rightarrow 0$. Let $\psi(x,y) = (S_{1\eta}(x,\eta_1(x,y)), \eta_1(x,y))$ be the area preserving transformation induced by S_1 , and let $g = \psi \circ f$. For some small constant $K_1 > 0$, if we put $A = \frac{K_1 \delta a}{2N}$, then $|g - f|_{C^1} < \delta$ and $g = f$ off $f^{-1}U_1$ as required.

We now construct the rectangle D_A . Let $W_{loc}^S(p,g)$ be a closed interval in $W^S(p,g)$ containing p and I in its interior, and let V be a tubular neighborhood of $W_{loc}^S(p,g)$. We assume that U is contained in V . For a set E and a point z in E , let $C(z,E)$ be the connected component of E which contains z . Let γ_1 be the curve in U given by $x = -a, 0 \leq y \leq 2A$, and let γ_2 be the curve given by $x = a, 0 \leq y \leq 2A$. Set $\{z_1\} = \gamma_1 \cap I'$ and $\{z_2\} = \gamma_2 \cap I'$. Since $I' \subset W^U(p,g)$, parts of backward iterates of γ_1 and γ_2 will accumulate on $W_{loc}^S(p,g)$ by the λ -lemma [8]. Also, there are constants $K_2, K_3 > 0$ so that if $g^j(z) \in V$ for $0 \leq j \leq m$, then

$$K_2 |\lambda(p)|^{-m} \leq \text{dist}(z, W_{loc}^S(p,g)) \leq K_3 |\lambda(p)|^{-m},$$

and if $g^{-j}(z) \in V$ for $0 \leq j \leq m$, then

$$K_2 |\lambda(p)|^{-m} \leq \text{dist}(z, C(p, W^U(p,g) \cap V)) \leq K_3 |\lambda(p)|^{-m}.$$

For this step it is convenient to assume via a preliminary approximation that f is C^2 . Then g is C^2 as well and hence C^1 linearizable on $W^S(p,g)$ and $W^U(p,g)$ near p .

For n large the curves $\gamma_1, \gamma_2, C(g^{-n}z_1, g^{-n}\gamma_1 \cap V)$, and $C(g^{-n}z_2, g^{-n}\gamma_2 \cap V)$ will enclose a rectangle R_n in U near I . Let γ'_1 and γ'_2 be the pieces of γ_1 and γ_2 in that rectangle as indicated in figure 4.

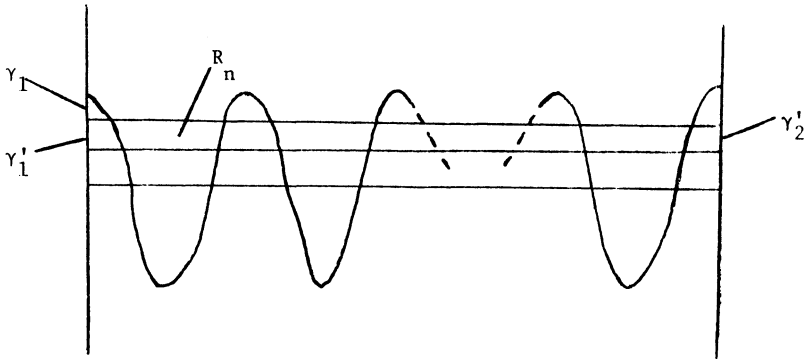


Figure 4

Let n be the smallest positive integer such that $C(g^{-n}z_1, g^{-n}\gamma_1 \cap V)$ and $C(g^{-n}z_2, g^{-n}\gamma_2 \cap V)$ are C^1 closer to $W_{loc}^s(p, g)$ than $\frac{A}{4}$ and $g^n\gamma'_1$ and $g^n\gamma'_2$ are C^1 closer to I' than $\frac{A}{4}$. There are constants $K_4, K_5 > 0$ so that $K_4|\lambda(p)|^{-n} \leq A \leq K_5|\lambda(p)|^{-n}$. Set $D_A = R_n, \Lambda_1 = \bigcap_{-\infty < j < \infty} g^{jn}D_A$, and $\Lambda = \bigcup_{0 \leq j \leq n} g^j\Lambda_1$. For N large, the reader may verify, with estimates similar to those in [7] and [6], that Λ is hyperbolic basic set for g . Clearly, $\Lambda \subset \overline{H(p, g)}$ and, as we have indicated, $h(g|\Lambda) = \frac{1}{n} \log N > \log|\lambda(p)| - \epsilon$. This proves statement (a) of the proposition when $\tau(p) = 1$.

When $\tau(p) > 1$, the proof is analogous except that z_0 will be in $W^s(p, f) \cap W^u(f^k p, f)$, $[0 \leq k < \tau(p)]$. The n above may then be chosen of the form $n = \tau(p)n_1 + k$, and we have the estimate $K_4|\lambda(p)|^{-n_1} \leq A \leq K_5|\lambda(p)|^{-n_1}$. We obtain Λ and g near f so that $h(g|\Lambda) = \frac{1}{n} \log N = \frac{1}{\tau(p)n_1 + k} \log N$, and $\frac{1}{\tau(p)n_1 + k} \log N \rightarrow \frac{1}{\tau(p)} \log|\lambda(p)|$ as $N \rightarrow \infty$.

We now move on to statement (b) of the proposition. We assume $\tau(p) = 1$ leaving the remaining generalization to the reader.

Consider the rectangle D_A and the mapping g^n . It is clear from figure 3 that $g^n D_A \cap D_A$ has N components. These are slanted "rectangles" joining the top and bottom of D_A as in the next figure.

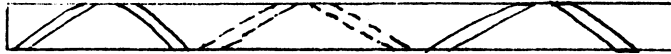


Figure 5

Also, $g^{-n}(D_A) \cap D_A$ consists of N rectangular strips stretching across D_A . In the standard way, this implies that for $k > 0$, $\bigcup_{-k < j < 0} g^j D_A$ consists of N^k thin rectangular strips joining the sides of D_A , and $\bigcup_{0 < j < k} g^j D_A$ consists of N^k thin slanted rectangular strips joining the top and bottom of D_A . Each component of $\bigcup_{-k < j < k} g^j D_A$ is a small disk whose diameter is larger than $(K_6 |\lambda(p)|^{-n})^k$ with $K_6 > 0$ independent of N . There are N^{2k} such components and their diameters approach zero as $k \rightarrow \infty$.

From this it follows that the Hausdorff dimension α of $\bigcup_{-\infty < j < \infty} g^j D_A$ satisfies

$$\alpha \geq \alpha_1 = \inf_{k \geq 0} \{ \beta : \inf_{k \geq 0} N^{2k} (K_6 |\lambda(p)|^{-n})^{k\beta} = 0 \}.$$

Now α_1 is given by $N^2 (K_6 |\lambda(p)|^{-n})^{\alpha_1} = 1$ or $\alpha_1 = \frac{2 \log N}{n \log |\lambda(p)| - \log K_6}$.

But for some constant $K_7 > 0$ independent of N , $n \log |\lambda(p)| < K_7 + \log N$,

so $\alpha_1 > \frac{2 \log N}{K_7 + \log N - \log K_6}$. Thus $\alpha_1 \rightarrow 2$ as $N \rightarrow \infty$, so $\alpha \rightarrow 2$. Given $\epsilon > 0$, we choose N_1 large enough so that $\frac{2 \log N_1}{K_7 + \log N_1 - \log K_6} > 2 - \epsilon$. Then,

$HD(\Lambda) > 2 - \epsilon$ with $\Lambda = \bigcup_{0 \leq j \leq n} g^j \left(\bigcap_{-\infty < k < \infty} g^{nk} D_A \right)$. For g_1 near g , each component of $\bigcap_{-k \leq j \leq k} g_1^{jn} D_A$ has diameter larger than $(K_6 |\lambda(p)|^{-n} - \epsilon_1)^k$ with ϵ_1 small, so we can insure that $HD(\Lambda(g_1)) > 2 - \epsilon$. This completes the proof of the proposition.

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