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A REMARK ON VECTOR FIELDS ON OPEN MANIFOLDS

by

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Summary . A partial answer is given to the question in what circumstances for a vector field X on an open manifold M there exists a neighborhood U in C^0 - Whitney topology , such that for every $Y \in U$ and compact $K \subset M$ the closure of positive semitrajectory $\bigcup_{t \geq 0} Y_t(K)$ is compact .

Let M be an open differentiable manifold / i.e. non compact without boundary , with a countable basis / and let $X^1(M)$ be the class of all C^1 vector fields on M endowed with C^0 -Whitney topology which is given by the neighborhoods of zero

$$\{X \in X^1(M) : \|X(p)\| \leq \varepsilon(p)\}$$

where ε is a real positive continuous function on M and

$\| \cdot \|$ is a Riemannian complete metric on M .

Throughout the paper X will denote a complete C^1 vector field on M and $\{X_t\}_{t \in \mathbb{R}}$ will be the corresponding flow generated by X . The positive semitrajectory of a point $p \in M$ will be denoted by $O_X(p) = \{X_t(p) : t \geq 0\}$, its ω -limit set by $\omega^X(p)$ and α -limit set by $\alpha^X(p)$.

Most of the notations and definitions used here are as in [1], [3]

DEFINITION 1. We say that $X \in X^1(M)$ is C-stable iff there exists a neighborhood U of X such that for every $Y \in U$ and every compact $K \subset M$ the closure of semitrajectory of K $\overline{\bigcup_{t \geq 0} Y_t(K)}$ is compact.

EXAMPLES. Let $M = \mathbb{R}^2$ and let flow $\{X_t\}_{t \in \mathbb{R}}$ be given by

$$\frac{d}{dt}x^1 = x^2 \quad \frac{d}{dt}x^2 = -x^1$$

The vector field $(x^2, -x^1)$ is not C-stable. It is easy to see that the vector field on $\mathbb{R}^2$ given by

$$\frac{d}{dt}x^1 = -x^1 \quad \frac{d}{dt}x^2 = -x^2$$

is C-stable.

We ask : Which conditions imply C-stability of X ?

In this paper we give a partial answer to this question. Proofs and ideas we use in this paper are similar to those used in studying the phenomenon of ∞ -explosion / see [2] / .

DEFINITION 2 . A compact invariant subset A of M is called an attractor iff there exists a neighborhood V of A such that for every $p \in V$ its ω -limit set $\omega^X(p)$ is contained in A . The domain of attraction of A is a maximal subset D of M such that for every $p \in D$, $\omega^X(p) \subset A$.

DEFINITION 3 . Let A be an attractor. We say that A is uniformly asymptotically stable if for every neighborhood U of A there exists neighborhood V of A such that $V \subset U$ and $X_t(V) \subset V$ for every $t \geq 0$.

Wilson [4] proved that if A is uniformly asymptotically stable and D is a domain of attraction of A then there exists a smooth function $L : D \rightarrow \mathbb{R}^+$ such that :

I. $L|_A = 0$

II. For any $c \in \mathbb{R}^+$ $\text{dist}(L^{-1}(c), A) < +\infty$ and

$\lim_{c \rightarrow 0} \text{dist}(L^{-1}(c), A) = 0$ here dist denotes the Hausdorff distance .

III. $\lim_{p_n \rightarrow \infty} L(p_n) = +\infty$ and $\lim_{p_n \rightarrow \partial D} L(p_n) = +\infty$
and $p_n \rightarrow \infty$ means that $\text{dist}(p_n, A)$ tends to $+\infty$

IV. If $p \in D \setminus A$ then $\frac{d}{dt} L(X_t(p)) < 0$

The function L is called Lyapunov function .

DEFINITION 4 . A filtration for X is a collection

$$\{M_i : i = 1, 2, \dots\}$$

of submanifolds of M / with boundaries / such that for every i

1. M_i is compact and $M_i \subset \text{Int } M_{i+1}$
2. $X_t(M_i) \subset M_i$ for every $t \geq 0$
3. X is transverse to the boundary ∂M_i of M_i
4. $\bigcup_{i \in \mathbb{N}} M_i = M$

It is clear that if a filtration for X exists then for every compact $K \subset M$ $\bigcup_{t \geq 0} X_t(K)$ is compact.

The following property which follows immediately from Definition 4 shows that the existence of filtration is an open property .

LEMMA 1 . If $\{M_i : i = 1, 2, \dots\}$ is a filtration for X then there is a C^0 -neighborhood U of X such that $\{M_i : i = 1, 2, \dots\}$ is also a filtration for every $Y \in U$.

In virtue of this Lemma the existence of a filtration implies C-stability of X .

LEMMA 2. Let $X \in X^1(M)$, $\bigcup_{t \geq 0} X_t(K)$ be compact for every compact $K \subset M$ and let A be an attractor with domain of attraction M . Then there is an uniformly asymptotically stable set B such that $A \subset B$.

PROOF. We present in detail an argument from [2].

Let $B = \{p \in M : \alpha^X(p) \cap A \neq \emptyset\}$. By definition, B is invariant and closed. Let W be the compact neighborhood of A . If $p \in M$ and $\alpha^X(p) \cap A \neq \emptyset$ then there exists $t < 0$ such that $X_t(p) \in W$ hence $p \in \bigcup_{t \geq 0} X_t(W)$ so the set B is compact / being a closed subset of compact $\bigcup_{t \geq 0} X_t(W)$ /. It is clear that B is an attractor.

We show that B is uniformly asymptotically stable. Let U be the compact neighborhood of A . By definition of the set B , all points of $U \setminus B$ have their α -limit sets empty. For every positive real number r denote $A_r = \bigcap_{0 \leq t \leq r} X_t(U)$. We note that the compact sets A_r are nested. We show that for sufficiently large r , $X_t(A_r) \subset \text{Int } A_1$ for $0 \leq t \leq 1$. Consider the sets

$$V_r = \bigcup_{0 \leq t \leq 1} X_t(A_r) \setminus \text{Int } A_1$$

which are a nested family of compact sets with empty intersection

hence there exists r such that $V_r = \emptyset$. For such r and $0 \leq T \leq 1$

$X_T(A_r) \subset A_r$. This implies that $X_T(A_r) \subset A_r$ for every $T \geq 0$. We put $\text{Int } A_r = V$. Then $B \subset V \subset U$ and $X_t(V) \subset V$ for every $t \geq 0$.

THEOREM 1 . Let $X \in X^1(M)$ and let for every compact $K \subset M$ the closure of positive semitrajectory $\overline{\bigcup_{t \geq 0} X_t(K)}$ and the set $F = \bigcup_{p \in M} \omega^X(p)$ be compact. Then the vector field X is C-stable .

PROOF. F is an attractor with domain of attraction M . By Lemma 2

there is an uniformly asymptotically stable set B with domain

of attraction M . Therefore [4] there exists a smooth Lyapunov

function L for B . We define $M_1 = L^{-1}([0, 1])$. M_1 is a compact

/ being a closed , bounded subset of Riemannian manifold with

complete metric / submanifold and $X_t(M_1) \subset M_1$ for every $t \geq 0$.

Now define a sequence of submanifolds $\{M_i : i = 1, 2, \dots\}$

by putting $M_i = X_{-1}(M_1)$. It is then clear that $\{M_i : i = 1, 2, \dots\}$

is a filtration for X , and by Lemma 1 , X is C stable .

Suppose that for a vector field X the set $F = \bigcup_{p \in M} \omega^X(p)$ is a union of compact , invariant , isolated subset ω_i i.e.

$$(*) \quad F = \omega_1 \cup \omega_2 \cup \omega_3 \cup \dots$$

Let $W^s \omega_i = \{p \in M : \omega^X(p) \subset \omega_i\}$ and $W^u \omega_i = \{p \in M : \alpha^X(p) \subset \omega_i\}$

and define in $\{\omega_i : i = 1, 2, \dots\}$ the relation

$$\omega_i \leq \omega_j \text{ iff } \overline{W^s \omega_i} \cap \overline{W^u \omega_j} \neq \emptyset$$

LEMMA 3. Let $\omega_1, \omega_2, \omega_3$ be any sets appearing in $(*)$ and

suppose that there exists a point $p_0 \in \omega_2$ such that $\overline{\alpha_X(p_0)} = \omega_2$

If $\omega_1 \geq \omega_2 \geq \omega_3$ then in every neighborhoods U and V of X and

ω_1 respectively there are $Y \in U$ and $p \in V$ such that $\omega^Y(p) \subset \omega_3$

This Lemma is consequence lemma 9 from [2] .

THEOREM 2 . Let X be a vector field , $F = \bigcup_{p \in M} \omega^X(p) = \omega_1 \cup \omega_2 \cup \dots$

be a union of infinitely many compact , invariant , isolated sets and let for every ω_i there exists $p_i \in \omega_i$ such that $\overline{O_X(p)} = \omega_i$. Moreover let $\bigcup_{t \geq 0} X_t(K)$ be compact for every compact K . Then X is stable iff no infinite sequence

$$\omega_{i_1} \supseteq \omega_{i_2} \supseteq \omega_{i_3} \supseteq \dots \quad \text{exists .}$$

PROOF. It suffices to show the existence of a filtration for X

We define a set

$$A_1 = \left\{ \omega_k : \text{there is a sequence } \omega_{i_1}, \dots, \omega_{i_j} \right.$$

$$\left. \begin{array}{l} \text{such that } \omega_{i_1} = \omega_1, \quad \omega_{i_j} = \omega_k \text{ and} \\ \omega_{i_1} \supseteq \omega_{i_2} \supseteq \dots \supseteq \omega_{i_j} \end{array} \right\}$$

Due to our assumption there is no infinite sequence

$$\omega_{i_1} \supseteq \omega_{i_2} \supseteq \omega_{i_3} \supseteq \dots$$

and since $\bigcup_{t \geq 0} X_t(K)$ is compact for every compact K , A_1 is finite and $C_1 = \bigcup_{i \in A_1} \omega_i$ is compact, therefore there is a neighborhood V of C_1 such that for every $p \in V$ $\omega^X(p)$ is contained in C_1 . Hence C_1 is an attractor with a domain of attraction $D_1 = \bigcup_{i \in A_1} W^s \omega_i$. Let $B_1 = \{p \in D_1 : \omega^X(p) \cap C_1 \neq \emptyset\}$

The set B_1 is compact invariant / see Lemma 2 / and $B_1 \subset D_1$ by definition of the relation $\preceq$. In the same way as in Lemma 2 we may show that B_1 is uniformly asymptotically stable . Let L_1 be a Lyapunov function for B_1 and $M_1 = L_1^{-1}([0, 1])$.

Starting with M_1 we construct a filtration $M_1, M_2, M_3, \dots$ by induction . Suppose that the submanifolds $M_1, M_2, M_3, \dots, M_k$ are already done . To define M_{k+1} , put $N_k = X_{-1}(M_k)$.

We choose a set ω_{i_0} not contained in N_k and define

$$A_{k+1} = \left\{ \omega_k : \text{there is a sequence } \omega_{i_1}, \dots, \omega_{i_j} \text{ such that } \omega_{i_1} = \omega_{i_0} \text{ and } \omega_k = \omega_{i_j} \text{ and } \omega_{i_1} \geq \omega_{i_2} \geq \dots \geq \omega_{i_j} \right\}$$

Again

$$C_{k+1} = \bigcup_{i \in A_{k+1}} \omega_i \cup B_k \text{ is an attractor.}$$

Let B_{k+1} be the uniformly asymptotically stable set such that

$C_{k+1} \subset B_{k+1}$ and let L_{k+1} be a Lyapunov function for B_{k+1} .

Put $M_{k+1} = L_{k+1}^{-1}([0, c_k])$ where c_k is chosen such that $N_k \subset M_{k+1}$

The sequence $M_1, M_2, M_3, \dots$ of compact submanifolds is a

filtration for X , for it is clear that $X_t(M_1) \subset M_1$ for $t \geq 0$

and X is transverse to ∂M_1 . To verify that $\bigcup_{i=1}^{\infty} M_i = M$,

take $p \in M$, by assumption $\omega^X(p) \neq \emptyset$, hence there exists k such that p is contained in domain of attraction of B_k . By

our construction $X_{-1}(M_k) \subset M_{k+1}$ so there exists l such that $p \in M_l$, hence $\bigcup_{i=1}^{\infty} M_i = M$.

Suppose now that there exists an infinite sequence

$$\omega_{i_1} \geq \omega_{i_2} \geq \omega_{i_3} \geq \dots. \text{ We will show then, that in}$$

every neighborhood of the vector X there is a vector field Y

and a point $p \in M$ such that $\omega^Y(p) = \emptyset$. Since the closure of

the semitrajectory of every compact set is compact, we may

choose from the sequence $\omega_{i_1}, \omega_{i_2}, \dots$ a sequence

$$\tilde{\omega}_{i_1} \geq \tilde{\omega}_{i_2} \geq \tilde{\omega}_{i_3} \geq \dots \text{ such that if only } k > l+1 \text{ then}$$

$$\overline{W^s \tilde{\omega}_{i_k}} \cap \overline{W^u \tilde{\omega}_{i_l}} = \emptyset$$

Then choose for every $\tilde{\omega}_{i_k}$ a neighborhood V_{i_k} of $\tilde{\omega}_{i_k}$ such that $V_{i_k} \cap V_{i_l} = \emptyset$ if $k \neq l$. By Lemma 3 there are a vector field Y_1 and $p_{i_1} \in V_{i_1}$ such that $\omega^{Y_1}(p_{i_1}) \subset \tilde{\omega}_{i_3}$. $Y_1 = X$ off V_{i_1} and $\sup_{p \in M} \|Y_1(p) - X(p)\|$ is arbitrarily small. Similarly we change the vector field Y_1 on V_{i_3} so that for this new vector field Y_2 $\omega^{Y_2}(p_{i_1}) \subset \tilde{\omega}_{i_4}$ and then repeat this procedure for $V_{i_4}, V_{i_5}, \dots$. In this way we get a vector field Y which is arbitrarily near to X and such that $\omega^Y(p_{i_1}) = \emptyset$. This proves the necessity part of our theorem.

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