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A NOTE ABOUT HEDLUND'S THEOREM

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In this note, we propose a new proof of Hedlund's theorem for an arbitrary group without using skew-products.

Let  $X$  be a compact space and  $T$  a minimal homeomorphism of  $X$ . Hedlund proved the following theorem (1)

Theorem

Let  $f$  be a continuous function on  $X$ . If the sums  $\sum_{i=0}^{n-1} f \circ T^{-i}$  are uniformly bounded there exists a continuous function  $F$  such that

$$f = F \circ T^{-1} - F$$

One can remark that the sums of iterates appearing in this theorem are the values of a cocycle for the powers of transformation  $T$ . More generally, let  $X$  be a compact space and  $G$  a group of homeomorphisms of  $X$ . An application  $V$  from  $G$  to the space  $C(X)$  of continuous functions on  $X$  ( $g \mapsto V_g$ ) is said to be a cocycle if

$$\forall h, g \in G \quad V_{gh} = V_g + V_h \circ g^{-1}$$

Cocycles appear in the study of quasi-invariant probabilities (2) and in the construction of skew-products.

Theorem

Let  $X$  be a compact space and  $G$  a group of homeomorphisms of  $X$  whose action is minimal i.e. every orbit is dense.

Let  $V$  be a cocycle on  $X$  for the action of  $G$ .

If the functions  $V_g$  are uniformly bounded there exists a continuous function  $F$  such that

$$\forall g \in G \quad V_g = F \circ g^{-1} - F$$

Proof

1. Let  $a$  be a point of  $X$  and  $O(a)$  its orbit under  $G$  (which is dense).

Let us define on  $O(a)$  a function  $f$  by  $f(g^{-1}a) = V_g(a)$

We can do it because  $V_g(a)$  does not depend on  $g$  but only on the point  $g^{-1}a$  of  $O(a)$  :

If  $g^{-1}a = k^{-1}a$ ,  $k = sg$  where  $s$  belongs to the group  $St(a)$  of all  $s$  such that  $s(a) = a$ .

Then

$$V_{sg}(a) = V_s(a) + V_g(a)$$

But the application from  $St(a)$  to  $\mathbb{R}$ ,  $s \mapsto V_s(a)$ , is a group homomorphism and from the hypothesis of boundedness is the zero homomorphism.

The function  $f$  is bounded.

2. Let  $F$  the lower semi-continuous function defined by

$$F(x) = \inf_{y \rightarrow x} \lim_{y \in O(a)} f(y)$$

$F$  is bounded.

Let us show that the increment of  $F$  is  $V$  i.e.

$$\forall h \in G \quad F \circ h^{-1} - F = V_h$$

This relation is true for  $f$  for all points in  $O(a)$  :

$$f(h^{-1}b) - f(b) = V_{gh}(a) - V_g(a) = V_h(g^{-1}a) = V_h(b) \quad \text{where } b = g^{-1}a$$

$$\text{Thus} \quad f(h^{-1}b) = f(b) + V_h(b) \quad \forall b \in O(a) \quad (1)$$

$$F(h^{-1}x) = \inf_{y \rightarrow h^{-1}x} \lim_{y \in O(a)} f(y) = \inf_{b \rightarrow x, b \in O(a)} \lim_{b \in O(a)} f(h^{-1}b)$$

with  $b = hy$ , because  $h$  is continuous.

Taking the inf limit in (1)

$$F(h^{-1}x) \geq F(x) + V_h(x)$$

and

$$\forall x \in X \quad \forall h \in G \quad F(h^{-1}x) - F(x) \geq V_h(x)$$

$$\text{Changing } h \text{ in } h^{-1} \text{ and } x \text{ in } h^{-1}x : \quad F(hh^{-1}x) - F(h^{-1}x) \geq V_h(h^{-1}x)$$

$$\text{So} \quad F(h^{-1}x) - F(x) \leq V_h(x) \quad \text{Whence the equality.}$$

3. Let  $OF$  be the oscillation function of  $F$

$$OF(c) = \limsup_{x, y \rightarrow c} F(x) - F(y)$$

$OF$  is a bounded upper semi-continuous function.

$$OF(h^{-1}c) = \sup_{x,y \rightarrow c} \lim F(h^{-1}x) - F(h^{-1}y) \quad \text{for } h^{-1} \text{ is continuous.}$$

$$\text{But as } F(x) - F(y) = F(h^{-1}x) - F(h^{-1}y) - (V_h(x) - V_h(y))$$

Taking the sup limit  $OF(c) \leq OF(h^{-1}c)$  because the oscillation of the continuous function  $V_h$  is zero.

That shows that the function  $OF$  is invariant under  $G$ .

By minimality, an upper semi-continuous function invariant is a constant.

4. To complete the proof of the theorem it remains to show that the constant function  $OF$  is zero. This result comes from the lemma :  
 Let  $g$  a lower semi-continuous function on a topological space  $E$ ,  $Og$  its oscillation. If  $Og(a)$  is finite then  $\inf \lim_{x \rightarrow a} Og(x) = 0$

Proof  $g$  being a l.s.c. function

$$Og(x) = \sup_{y \rightarrow x} \lim g(y) - \inf_{y \rightarrow x} \lim g(y) = \sup_{y \rightarrow x} \lim g(y) - g(x)$$

If  $Og(a)$  is finite there exists an open neighbourhood  $V_o$  of  $a$  in which  $g$  is bounded. So that for every open neighbourhood  $V \subset V_o$ , for every  $\varepsilon$ , there exists an  $x_V$  in  $V$  with

$$g(x_V) \geq \sup_{y \in V} g(y) - \varepsilon$$

Then

$$Og(x_V) \leq \sup_{y \in V} g(y) - g(x_V) \leq \varepsilon \quad \text{q.e.d.}$$

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