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Entropy of piecewise monotone mappings

by

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The aim of the paper is to study topological entropy of piecewise monotone mappings of intervals and their invariant closed subsets. Here we present only results without proofs. The full proofs will be published in Studia Mathematica.

In Section 1 there are some formulas connecting the topological entropy of a map f with: 1° the asymptotic behaviour of the numbers c_n of maximal intervals on which f^n is monotone; 2° the asymptotic behaviour of variation of f^n. If f is a piecewise strictly monotone mapping of an interval, then c_n is the number of points at which f^n has extrema. The obtained formulas are as follows:

$$h(f) = \lim_{n\to\infty} \frac{1}{n} \log c_n = \lim_{n\to\infty} \frac{1}{n} \log \mathrm{Var}\, f^n \quad .$$

In Section 2 the action of piecewise monotone mappings with positive entropy is studied. It turns out that there is a subset on which the phenomenon similar to the horseshoe effect is observed. In the case of a map of an interval it gives a possibility of estimating the asymptotic behaviour of the number of periodic points. Namely, the following inequality holds:

$$h(f) \leqslant \limsup_{n\to\infty} \frac{1}{n} \log \mathrm{Card} \{x : f^n(x) = x\} \quad .$$

In Section 3 the topological entropy is regarded as a function of f , where f belongs to the set of all (C^0 or C^2) mappings of an interval.

There are given some sufficient conditions for f to be a point of semi-continuity and continuity of the function $h(\cdot)$.

In Section 4 there are presented some examples showing that the assumption that mappings are piecewise monotone is essential for some theorems in Sections 1 and 3.

The results mentioned above are also true in the case where f is a piecewise monotone mapping of the circle.

One of the examples of Section 4 in the case of the circle indicates some difficulties which may occur if one attempts to prove the entropy conjecture.

Some results of the paper are related to certain Bowen's [4] and Block's [2] results.

1. We shall use the following notations. By X we denote the studied space. The first capital letters of the alphabet: A,B,...,G denote covers or partitions of X ; their elements are denoted by a,b,...,e . Some fixed subsets of X are denoted by the next capital letters: J,K,...,Y . Mappings are denoted by f, g, φ, Φ ; the letter h is reserved for entropy only. Numbers are denoted by Greek letters $\alpha, \beta, \ldots, \varepsilon$ and by Latin letters i,...,u (also c_n).

We assume that the reader is familiar with the common definitions of the topological entropy [1] , [3] .

Let X be a compact Hausdorff space, $f : X \longrightarrow X$ - a continuous mapping, $Y \subset X$ - an arbitrary subset of X, $\mathfrak{A}(X)$ - the set of all finite open covers of X, A and B - two finite covers of X (not necessarily open). We set

$$A^n = \bigvee_{i=0}^{n-1} f^{-i}(A) \quad .$$

Now we shall recall some notions and results from [5], which will be used in the further investigations.

$$N(Y,A) = \min \left\{ \text{Card } C : C \subset A \ , \ Y \subset \bigcup_{c \in C} c \right\}$$

$$N(A|B) = \max_{b \in B} N(b,A)$$

$$h(f,A|B) = \lim_{n \to \infty} \frac{1}{n} \log N(A^n|B^n)$$

$$h(f|B) = \sup_{A \in \mathcal{O}(X)} h(f,A|B)$$

$$h^*(f) = \inf_{B \in \mathcal{O}(X)} h(f|B) \quad .$$

The number $h^*(f)$ is called the topological conditional entropy of f . If $B = \{X\}$, then $h(f,A|B) = h(f,A)$. We shall use the following results from [5] :

(1.1) $h(f,A) \leq h(f,B) + h(f,A|B)$

(1.2) $h(f) \leq h(f,A) + h(f|A)$.

From now on we assume X to be a closed subset of the interval $I = \langle 0,1 \rangle$ and f - a continuous mapping: $f : X \to X$. Denote by $\mathcal{I}$ the set of all possible subintervals of I (open, closed, half-open, degenerated). For a family of sets C and a set Y we denote by $C|_Y$ the family of sets $\{c \cap Y : c \in C\}$. In particular, $\mathcal{I}|_Y$ denotes the family of all subintervals of $\langle 0,1 \rangle$ restricted to Y .

<u>Definition 1.</u> A cover A is called f-mono if A is finite, $A \subset \mathcal{I}|_X$ and for any $a \in A$ the map $f|_a$ is monotone.

<u>Definition 2.</u> A map f is called piecewise monotone (in abbreviation p.m.) if there exists an f-mono cover of X .

It is easy to prove that the composition of two p.m. mappings is a p.m. mapping.

Let $f : X \to X$ be a piecewise monotone continuous (in abbreviation p.m.c.) mapping. Let

$$c_n = \min\{\text{Card } A : A \text{ is an } f^n\text{-mono cover}\} \quad .$$

The following theorem holds:

Theorem 1. If $f : X \to X$ is a p.m.c. mapping, then

$$\lim_{n\to\infty} \frac{1}{n} \log c_n = h(f)$$

and $\frac{1}{n} \log c_n \geqslant h(f)$ for any n .

The proof bases on the fact that $(\log c_n)_{n=1}^{\infty}$ is a subadditive sequence and on the inequality (1.1). From this proof we can also deduce

Proposition 1. If $f : X \to X$ is a p.m.c. mapping and A is an f-mono cover, then $h(f,A) = h(f)$.

Using Theorem 1 , (1.1) and (1.2) we prove the next theorem:

Theorem 2. If $f : X \to X$ is a p.m.c. mapping, then $h^*(f) = 0$.

Applying some results from [5] and Theorem 2 we get the following corollary:

Corollary 1. If $f : X \to X$ is a p.m.c. mapping, then the measure entropy of f , regarded as a function of measure, is upper semi-comtinuous. In particular, there exists a measure with maximal entropy for f .

Now we shall study the growth of the variation of the iterations of f under the assumption that f has Darboux property, i.e. that for any $J \in \mathcal{I}|_X$ the image $f(J)$ also belongs to $\mathcal{I}|_X$. Of course, this condition is fulfilled in the case when X is an interval. Using the definition of the entropy by (n,ε)-separated sets we prove straightforward the following lemma:

Lemma 1. If $f : X \to X$ is a continuous surjection and f has Darboux property, then

$$\liminf_{n\to\infty} \frac{1}{n} \log \operatorname{Var} f^n \geq h(f) \quad .$$

From that lemma and Theorem 1 it follows easily the next theorem:

Theorem 3. Let $f : X \longrightarrow X$ be a p.m.c. mapping having Darboux property. Then

$$\lim_{n\to\infty} \frac{1}{n} \log \operatorname{Var} f^n = h(f) \quad .$$

2. Now we shall investigate the nature of p.m.c. mappings having Darboux property. In their action one can distinguish a phenomenon which is very similar to Smale's horseshoe effect.

Let $f : X \longrightarrow X$ be a p.m.c. map. Then tnere exists an f-mono cover which is also a partition. In this case the reader may consider the dynamical system (X,f) in terms of symbolic dynamics. The family of sets A is the alphabet, the elements of A^n are words and f is the shift.

For any $J \in \mathcal{I}|_X$ there are at most two elements of A not contained in J and having with it non-empty intersection, i.e.

$$(2.1) \qquad \operatorname{Card}\{a \in A : a \cap J \neq \emptyset \text{ and } a \setminus J \neq \emptyset\} \leq 2 \quad .$$

Let E be a subfamily of A defined as follows:

$$E = \{a \in A : \limsup_{n\to\infty} \frac{1}{n} \log N(a,A^n) = h(f)\} \quad .$$

The family E is non-empty. Let E^n denote, as for covers, the family of sets

$$\left\{\bigcap_{i=0}^{n-1} f^{-i}(e_i) : e_i \in E \text{ for } i=0,\dots,n-1\right\} \quad .$$

Lemma 2. For any $a \in E$ the following equality holds:

$$\limsup_{n\to\infty} \frac{1}{n} \log \operatorname{Card}(E^n|_a) = h(f) \quad .$$

The proof of Lemma 2 is of combinatorial character.

Now let $a,b \in E$ be two arbitrary elements. We define

$$\gamma(a,b,n) = \mathrm{Card}\,\{e \in E^n|_a : f^n(e) \supset b\} \quad .$$

The next lemma is the crucial point of this section.

Lemma 3. Let $f : X \longrightarrow X$ be a p.m.c. mapping having Darboux property and assume $h(f) > \log 3$. Then there exists an $a_0 \in E$ such that

$$(2.2) \qquad \limsup_{n\to\infty} \frac{1}{n} \log \gamma(a_0,a_0,n) = h(f) \quad .$$

The idea of the proof is as follows. Let u be an arbitrary number such that $\log 3 < u < h(f)$ and fix a set $a \in E$. Using Lemma 2 we conclude that for infinitely many n the following two inequalities hold:

$$(2.3) \qquad \begin{aligned} &\frac{1}{n} \log \mathrm{Card}\,(E^n|_a) > u \quad , \\ &\mathrm{Card}\,(E^{n+1}|_a) \geqslant 3\,\mathrm{Card}\,(E^n|_a) \quad . \end{aligned}$$

Fix a set $e \in E^n|_a$. The set $f^n(e)$ belongs to $\mathcal{I}|_X$. If it has non-empty intersections with r elements of E, then it contains at least $r-2$ of them. On the other hand, $r = \mathrm{Card}\,(E^{n+1}|_e)$. Therefore

$$\mathrm{Card}\,\{b \in E : f^n(e) \supset b\} \geqslant \mathrm{Card}\,(E^{n+1}|_e) - 2 \quad .$$

Summing over $e \in E^n|_a$ we obtain

$$\sum_{b\in E} \gamma(a,b,n) \geqslant \mathrm{Card}\,(E^{n+1}|_a) - 2\,\mathrm{Card}\,(E^n|_a) \quad .$$

In view of (2.3) we conclude that

$$\limsup_{n\to\infty} \frac{1}{n} \log \Big(\sum_{b\in E} \gamma(a,b,n)\Big) \geqslant h(f) \quad ,$$

since u is an arbitrary number less than $h(f)$. Hence we easily get an a_0 stysfying (2.2).

From Lemma 3 we deduce the following theorem:

Theorem 4. Let $f : X \longrightarrow X$ be a p.m.c. mapping having Darboux property. Then there exist:

1^o a set $J \in \mathcal{I}|_X$,

2^o a sequence $(D_n)_{n=1}^{\infty}$ of partitions of J by elements of $\mathcal{I}|_X$,

3^o a sequence $(k_n)_{n=1}^{\infty}$ of positive integers

such that:

(2.4) $\lim\limits_{n\to\infty} \frac{1}{k_n} \log \text{Card } D_n = h(f)$

and $f^{k_n}(d) \supset J$ for any $d \in D_n$.

<u>Corollary 2.</u> If $f : I \to I$ is continuous and $X \subset I$ is a closed invariant set such that $h(f|_X) = h(f)$ and $f|_X$ satisfies the hypotheses of Theorem 4, then

$$\limsup_{n\to\infty} \frac{1}{n} \log \text{Card}\left\{x \in I : f^n(x) = x\right\} \geqslant h(f) .$$

<u>3.</u> Now we shall study the topological entropy as a function of a mapping. We shall consider two cases: 1^o $f \in C^o(I,I)$ -the space of all continuous mappings of the interval I into itself with C^o-topology; 2^o $f \in C^2(I,I)$ - the space of all mappings of class C^2 of the interval I into itself with C^2-topology. The obtained results, namely Theorems 1 and 4, give us a possibility to prove some continuity properties of the entropy. The first continuity property is a slight generalization of a result of Bowen [4] .

<u>Proposition 2.</u> Let $f : I \to I$ be a mapping of class C^2 such that for any $x \in I$ at least one of the numbers $f'(x)$, $f''(x)$ is non-zero. Then the topological entropy regarded as a function $h : C^2(I,I) \to \mathbb{R}$ is upper semi-continuous at f .

The proof of Proposition 2 bases on Theorem 1.

<u>Theorem 5.</u> Let $f : I \to I$ be a continuous mapping and let X be a closed invariant subset of I such that

$h(f|_X) = h(f)$ and $f|_X : X \longrightarrow X$ is p.m. and has Darboux property. Then the topological entropy regarded as a function $h : C^0(I,I) \longrightarrow \mathbb{R}$ is lower semi-continuous at f.

The proof bases strongly on Theorem 4.

As a conclusion of Proposition 2 and Theorem 5 we have

<u>Theorem 6.</u> If $f : I \longrightarrow I$ satisfies the hypotheses of Proposition 2, then f is a point of continuity of topological entropy $h : C^2(I,I) \longrightarrow \mathbb{R}$.

<u>4.</u> Now we shall present two examples showing that some assumptions of Theorem 3 and Proposition 2 (and therefore of Theorem 6) cannot be omitted.

We define two auxiliary functions.

1^o $\varphi : \mathbb{R} \longrightarrow \mathbb{R}$ is given by the formula:

$$\varphi(x) = \frac{1}{4}\left(x - \frac{1}{4}\right)^{2r+1} \sin\frac{1}{x-\frac{1}{4}} + \frac{3}{4} \quad \text{for } x \neq \frac{1}{4}$$

and $\varphi(\frac{1}{4}) = \frac{3}{4}$. The function φ is of class C^r and $\varphi^{(i)}(\frac{1}{4}) = 0$ for $i = 1,\dots,r$.

2^o $\Phi : \mathbb{R} \longrightarrow \langle 0,1 \rangle$ is a function of class C^∞ such that $\Phi(x) = 0$ for $x \in (-\infty,1\rangle \cup \langle 5,+\infty)$ and $\Phi(x) = 1$ for $x \in \langle 2,4 \rangle$.

Let $f : I \longrightarrow I$ be a C^r-mapping such that (see Fig. 1):

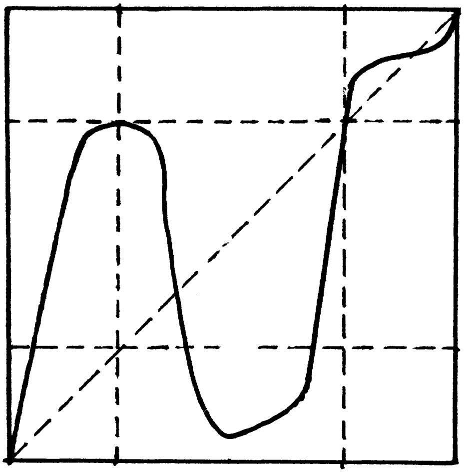

Fig. 1

(i) $f(0) = 0$, $f(1) = 1$, $f(\frac{1}{4}) = \frac{3}{4}$, $f'(\frac{1}{4}) = f'(\frac{1}{2}) = 0$

and $f'(x) \neq 0$ for $x \neq \frac{1}{4}, \frac{1}{2}$; moreover $f^{(n)}(\frac{1}{4}) = 0$ for $n = 1, \dots, r$

(ii) $f(x) = \frac{3}{4} + \alpha(x - \frac{3}{4})$ for $x \in \langle \frac{3}{4} - \frac{1}{2\alpha} , \frac{3}{4} \rangle$, where α is a fixed number such that $\alpha > 3^{2r+1}$.

We consider p.m.c. mappings $\tilde{f}_t$ arbitrary C^r-close to the following ones:

$$f_t(x) = f(x) + (\varphi(x) - f(x)) \cdot \Phi(t(x - \tfrac{1}{4})) .$$

Proposition 3. If $t \longrightarrow +\infty$, then $\tilde{f}_t - f$ tends to 0 in C^r-topology; moreover,

$$\limsup_{t \to +\infty} h(\tilde{f}_t) > h(f) .$$

This example gives us the following theorem:

<u>Theorem 7.</u> For every non-negative integer r the topological entropy regarded as a function of a C^r p.m.c. mapping of the interval I into itself is not upper semi-continuous in C^r-topology.

The second example shows that Theorem 3 is not true without the assumption that f is piecewise monotone.

There exists a C^r-mapping $g : I \longrightarrow I$ such that (see Fig. 2)

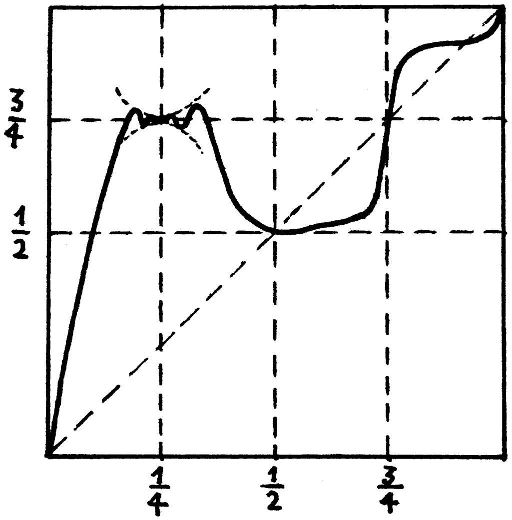

Fig. 2

(i) $g(0) = 0$, $g(1) = 1$, $g(\frac{1}{2}) = \frac{1}{2}$,

(ii) for certain $c > 0$ we have $g(x) = \varphi(x)$ for $x \in \langle \frac{1}{4} - c , \frac{1}{4} + c \rangle$,

(iii) $g(x) > x$ for $x \in (0, \frac{1}{2})$,

(iv) g is increasing on the interval $\langle \frac{1}{2}, 1 \rangle$,

(v) $g(x) = \frac{3}{4} + \alpha(x - \frac{3}{4})$ for $x \in \langle \frac{3}{4} - c , \frac{3}{4} + c \rangle$.

Theorem 8. For the map g defined above the following formulas hold:

(a) $\liminf_{n\to\infty} \frac{1}{n} \log \mathrm{Var}\, g^n \geqslant \frac{\log}{2r+1} > 0$,

(b) $h(g) = 0$.

It is easy to see that the map g is not p.m. The formulas (a) and (b) show that the assertion of Theorem 3 does not hold.

Slightly modifying g in the neighbourhood of the point $\frac{1}{4}$ one can easily obtain a C^r-mapping $\tilde{g} : I \to I$ for which $\lim_{n\to\infty} \frac{1}{n} \log \mathrm{Var}\, \tilde{g}^n$ does not exist.

As we mentioned at the beginning all the theorems are true in the case where X is the circle. Theorem 8 in the case of the circle shows that one cannot hope to prove the entropy conjecture using the way proposed by Sacksteder and Shub in [6] (we have $h_1(g) > 0 = h(g)$, where $h_1(\cdot)$ is a certain number defined in [6]).

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