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On a generalization of topological conditional entropy

by

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§ 0. Introduction

We introduce the notion of topological conditional entropy for a subset of a compact space and a continuous transformation. It is a generalization of the topological entropy defined by Bowen ([1]) and the topological conditional entropy introduced by Misiurewicz ([3]).

§ 1. Definition

The following notations and terminology will be used:

X - a compact space

$f : X \rightarrow X$ - a continuous transformation

$Y \subset X$ - a subset of X

$\mathcal{P}(X)$ - the set of all finite covers of X

$\mathcal{U}(X)$ - the set of all open finite covers of X

N - positive integers

$c \prec A$ - if a subset c of X is contained in some member of $A \in \mathcal{P}(X)$

$A \succcurlyeq B$ - if $a \prec B$ for every $a \in A$ and $A, B \in \mathcal{P}(X)$

$$A^n = \bigvee_{i=0}^{n-1} f^{-i}A, \quad A^\infty = \bigvee_{i=0}^{\infty} f^{-i}A \quad \text{for } A \in \mathcal{P}(X), \quad n \in \mathbb{N}$$

Now, for $c \subset X$ and $A \in \mathcal{P}(X)$ let (see [1])

$$n_{f,A}(c) = \begin{cases} \sup \{k \in \mathbb{N} : c \prec A^k\} & \text{if } c \prec A \\ 0 & \text{if } c \not\prec A \end{cases}$$

$$D_A(C) = e^{-n_{f,A}(C)}$$

$$D_A(C, \lambda) = \sum_{i=1}^{\infty} D_A(c_i)^\lambda \quad \text{where } C = (c_i)_{i=1}^{\infty} \text{ and } \lambda > 0$$

For $b \in B \in \mathcal{P}(X)$ and $\epsilon > 0$ we define

$$F_{A,b,\lambda,\epsilon}(Y) = \inf_C \left\{ D_A(C, \lambda) : \bigcup_{i=1}^{\infty} c_i \supset b \cap Y, D_A(c_i) < \epsilon \right\}$$

and

$$F_{A|B,\lambda,\epsilon}(Y) = \max_{b \in B} F_{A,b,\lambda,\epsilon}(Y)$$

It is easy to see that

$$(1.1) \quad F_{A_1|B_1,\lambda_1,\epsilon_1}(Y) \geq F_{A_2|B_2,\lambda_2,\epsilon_2}(Y) \quad \text{for } A_1 \geq A_2, B_1 \leq B_2, \\ \lambda_1 \leq \lambda_2, \epsilon_1 \leq \epsilon_2$$

Now we can define

$$m_{A|B,\lambda}(Y) = \lim_{\epsilon \rightarrow 0} \lim_{n \rightarrow \infty} F_{A|B^n,\lambda,\epsilon}(Y).$$

Notice that

$$m_{A_1|B_1,\lambda_1}(Y) \geq m_{A_2|B_2,\lambda_2}(Y) \quad \text{for } A_1 \geq A_2, B_1 \leq B_2, \lambda_1 \leq \lambda_2$$

and $m_{A|B,\lambda}(Y) \notin \{0, +\infty\}$ for at most one λ

Define

$$h_{A|B}(f, Y) = \inf \left\{ \lambda; m_{A|B,\lambda}(Y) = 0 \right\} \quad (\inf \emptyset = +\infty)$$

From the definition it follows that

$$(1.2) \quad h_{A_1|B_1}(f, Y) \geq h_{A_2|B_2}(f, Y) \quad \text{for } A_1 \geq A_2, B_1 \leq B_2$$

Now we can take limits (finite or infinite) (see [3])

$$\lim_{A \in \mathcal{U}(X)} h_{A|B}(f, Y) = \sup_{A \in \mathcal{U}(X)} h_{A|B}(f, Y) = h(f|B, Y)$$

$$\lim_{B \in \mathcal{U}(X)} h_{A|B}(f, Y) = \inf_{B \in \mathcal{U}(X)} h_{A|B}(f, Y) = h^{\mathbb{K}}(f, Y)$$

$h^{\mathbb{K}}(f, Y)$ will be called the topological conditional entropy of f for a subset Y of X .

§ 2. Basic properties

The proofs of the following propositions are simple and therefore we omit them

Proposition 1

- a) If $f_1 : X_1 \rightarrow X_1$ and $f_2 : X_2 \rightarrow X_2$ are topologically conjugate (i.e. there exists a homeomorphism $\Pi : X_1 \rightarrow X_2$ such that $\Pi \circ f_1 = f_2 \circ \Pi$) then $h^{\mathbb{K}}(f_1, Y_1) = h^{\mathbb{K}}(f_2, \Pi(Y_1))$ for $Y_1 \subset X_1$
- b) If $f : X \rightarrow X$ is a homeomorphism, then $h^{\mathbb{K}}(f, f^{-1}(Y)) = h^{\mathbb{K}}(f, Y) = h^{\mathbb{K}}(f, f(Y))$.

Proposition 2

If $n \in \mathbb{N}$ and $A, B \in \mathcal{Q}(X)$, then

- a) $h_{A^n|B}(f^n, Y) = nh_{A|B}(f, Y)$
- b) $h(f^n|B, Y) = nh(f|B, Y)$
- c) $h^{\mathbb{K}}(f^n, Y) = nh^{\mathbb{K}}(f, Y)$

Proposition 3

If $n \in \mathbb{N}$ and $A, B \in \mathcal{Q}(X)$, then

- a) $h_{A|B}(f, \bigcup_{i=1}^{\infty} Y_i) = \sup_i h_{A|B}(f, Y_i)$
- b) $h^{\mathbb{K}}(f, \bigcup_{i=1}^{\infty} Y_i) = \max_{1 \leq i \leq n} h^{\mathbb{K}}(f, Y_i)$

From Proposition 3 it follows that if Y is countable, then $h^{\mathbb{K}}(f, Y) = 0$.

§ 3. Connection with topological conditional entropy

First let us recall the definition of the topological conditional entropy $h^{\mathbb{K}}(f)$, ([3]).

$$h^{\mathbb{K}}(f) = \lim_{B \in \mathcal{U}(X)} h(f|B) = \inf_{B \in \mathcal{U}(X)} h(f|B)$$

$$\text{where } h(f|B) = \lim_{A \in \mathcal{U}(X)} h(f, A|B) = \sup_{A \in \mathcal{U}(X)} h(f, A|B)$$

$$\text{and } h(f, A|B) = \lim_{n \rightarrow \infty} \frac{1}{n} \log N(A^n|B^n).$$

$$\text{Moreover for } A, B \in \mathcal{Q}(X) \quad N(A|B) = \max_{b \in B} N(b, A),$$

$$\text{where for } b \subset X, \quad b \neq \emptyset \quad N(b, A) = \min \{ \text{Card } C; C \subset A, \bigcup C_i \supset b \\ \text{and } N(\emptyset, A) = 1.$$

Theorem 1

$$h^{\mathbb{K}}(f, X) = h^{\mathbb{K}}(f)$$

Proof

It is sufficient to prove that

$$\text{a) if } A, B \in \mathcal{Q}(X), \text{ then } h_{A|B}(f, X) \leq h(f, A|B)$$

$$\text{b) } h^{\mathbb{K}}(f, X) \geq h^{\mathbb{K}}(f)$$

For the proof of a) let $\lambda > 0$, $\epsilon > 0$ and $e^{-n} < \epsilon$,

where $n \in \mathbb{N}$. Moreover let $b^n \in B^n$ be such, that

$$F_{A|B^n, \lambda, \epsilon}(X) = F_{A, b^n, \lambda, \epsilon}(X) \quad \text{and let } C \subset A^n \text{ be a cover of}$$

b^n with $N(b^n, A^n)$ members. Then

$$D_A(C, \lambda) \leq \text{Card } C \cdot e^{-n\lambda} \leq N(A^n|B^n)e^{-n\lambda}$$

and

$$\begin{aligned} F_{A|B^n, \lambda, \epsilon}^{(X)} &\leq F_{A|B^n, \lambda, e^{-n\lambda}}^{(X)} \leq D_A(C, \lambda) \leq \\ &\leq (e^{\frac{1}{n} \log N(A^n|B^n)} - \lambda)^n \end{aligned}$$

If $\lambda > h(f, A|B)$, then $\lim_{n \rightarrow \infty} F_{A|B^n, \lambda, \epsilon}^{(X)} = 0$. Since ϵ was arbitrary, it follows that

$$m_{A|B, \lambda}^{(X)} = 0, \text{ hence } h(f, A|B) \geq h_{A|B}(f, X)$$

For the proof of b) let $m_{A|B, \lambda}^{(X)} = 0$ where $A, B \in \mathcal{O}(X)$ and $\lambda > 0$ then for fixed $0 < \eta < 1$ and $0 < \epsilon < 1$ there exists $n_0 \in \mathbb{N}$ such that

$$F_{A|B^{n_0}, \lambda, \epsilon}^{(X)} < \eta.$$

Hence for every $b^{n_0} \in B^{n_0}$ there exists a cover $C(b^{n_0}) = (c_i)_{i=1}^\infty$ of b^{n_0} with $D_A(c_i) < \epsilon$ (so $n_{f, A}(c_i) > 0$)

and $D_A(C(b^{n_0}), \lambda) < \eta$. We may assume that c_i is open.

Let $D \in \mathcal{O}(X)$ and $\bar{D} \supset B^{n_0}$, where $\bar{D} = \{\bar{d} : d \in D\}$.

Since for $d \in D$ $\bar{d}$ is compact, therefore there exists a finite, open cover $C^*(d)$ of $\bar{d}$ such that $C^*(d) \subset C(b^{n_0})$ for some $b^{n_0} \in B^{n_0}$.

If $C = \{c \in C^*(d) : d \in D\}$ and $K = \text{Card } C$, then $0 < K < \infty$.

Let $n \in \mathbb{N}$ and for $c_1, \dots, c_s \in C$ such that

$$n_{f,A}(c_1) + \dots + n_{f,A}(c_{s-1}) < n \quad \text{and}$$

$$n \leq n_{f,A}(c_1) + \dots + n_{f,A}(c_s)$$

$$Z(c_1, \dots, c_s) = \left\{ x \in c_1: f^{n_{f,A}(c_1) + \dots + n_{f,A}(c_r)}(x) \in c_{r+1} \right. \\ \left. \text{for } r = 1, \dots, s-1 \right\}$$

then

$$Z(c_1, \dots, c_s) \subset A^n$$

If $d^n \in D^n$ and $d^n = \bigcap_{j=0}^{n-1} f^{-j} d_j$, where $d_j \in D$ for $j=0, \dots, n-1$ then

$$Z = \left\{ Z(c_1, \dots, c_s): s \geq 1, \quad c_1 \in C^{\mathbb{K}}(d_0), \right. \\ \left. c_{r+1} \in C^{\mathbb{K}}(d_{n_{f,A}(c_1) + \dots + n_{f,A}(c_r)}) \right. \\ \left. \text{for } r = 1, \dots, s-1 \right\} \quad \text{covers } d^n.$$

Hence

$$N(d^n, A^n) e^{-\lambda n} \leq \text{Card } Z e^{-\lambda n} \leq \\ \leq \sum_{c \in C} \left(1 + \sum_{s \geq 2} \sum_{Z(c_1, \dots, c_{s-1}, c) \in Z} e^{-\lambda(n_{f,A}(c_1) + \dots + n_{f,A}(c_s))} \right) \\ \leq \sum_{c \in C} \left(1 + \sum_{s \geq 2} \sum_{c_1} e^{-\lambda n_{f,A}(c_1)} (\dots (\sum_{c_{s-1}} e^{-\lambda n_{f,A}(c_{s-1})}) \dots) \right) \\ \leq K \sum_{s=0}^{\infty} \eta^s = M < \infty$$

and

$$(e^{-\lambda + \frac{1}{n} \log N(A^n | D^n)})^n \leq M$$

Therefore if $m_{A|B, \lambda}(X) = 0$, then $h(f, A|D) \leq \lambda$
(for some D which depends on B).

This implies that

$$h(f, A|D) \leq h_{A|B}(f, X) \text{ and therefore } h^*(f) \leq h^*(f, X).$$

Remark.

If Y is closed and f - invariant then $h^*(f, Y) = h^*(f|Y)$.
The proof is like that of Theorem 1 and is close to the proof of Proposition 1 [1].

§ 4. Connection with topological entropy introduced by
Bowen ([1]).

R. Bowen defined the topological entropy $h(f, Y)$ in the following way:

$$h(f, Y) = \sup_{A \in \mathcal{O}(X)} h_A(f, Y)$$

where for $A \in \mathcal{O}(X)$ $h_A(f, Y) = \inf \{ \lambda ; m_{A, \lambda}(Y) = 0 \}$
and for $\lambda > 0$

$$m_{A, \lambda}(Y) = \lim_{\epsilon \rightarrow 0} \inf \{ D_A(C, \lambda) : \bigcup_{i=1}^{\infty} c_i \supset Y, D_A(c_i) < \epsilon \}$$

It is easy to see that $h(f, Y) \geq h^*(f, Y)$

Theorem 2 gives another relation between these notions

Theorem 2

Let $A \in \mathcal{O}(X)$ and $B \in \mathcal{P}(X)$. Then

$$h_{A|B}(f, Y) \geq \sup_{b \in B^{\infty}} h_A(f, Y \cap b^{\infty}).$$

If in addition Y is closed and B is a closed cover then the reverse inequality also holds.

Proof.

If $B \in \mathcal{P}(X)$ and $n \in \mathbb{N}$ then for any $b^\infty \in B^\infty$ exists $b^n \in B^n$ such that $b^\infty \subset b^n$.

Therefore for $\epsilon > 0$

$$\inf \{ D_A(C, \lambda) : \bigcup_{i=1}^{\infty} c_i \supset Y \cap b^\infty, D_A(c_i) < \epsilon \} \leq F_{A, b^n, \lambda, \epsilon}(Y) \leq F_{A|B^n, \lambda, \epsilon}(Y)$$

$$\text{and } h_A(f, Y \cap b^\infty) \leq h_{A|B}(f, Y)$$

This gives the required inequality.

For the proof of the second part let $\epsilon > 0$ and $\lambda > 0$.

Then for any $b^\infty \in B^\infty$ there exists a cover $C(b^\infty) = \{c_i(b^\infty)\}_{i=1}^{\infty}$ of $b^\infty \cap Y$ such that $D_A(c_i(b^\infty)) < \epsilon$ for $i = 1, 2, \dots$ and $D_A(C(b^\infty), \lambda) \leq F_{A, b^\infty, \lambda, \epsilon}(Y) + \epsilon$

We may assume that $c_i(b^\infty)$ are open.

Let $U(b^\infty) = \bigcup_{i=1}^{\infty} c_i(b^\infty)$ and $U = \{U(b^\infty)\}_{b^\infty \in B^\infty}$, then

U covers Y .

For $W = \{d^n \in Y \cap B^n : n \in \mathbb{N}\}$ and $V = \{d^n \in W : d^n \not\subset U\}$

we define $g : V \rightarrow W$ in the following way:

if $d^{n+1} \in V$, where $d^{n+1} = d^n \cap f^{-n}b_n$ for $d^n \in W$ and $b_n \in B$, then

$$g(d^{n+1}) = d^n.$$

A sequence $(d^n)_{n=1}^m$ with $g(d^{n+1}) = d^n$ for $n = 1, \dots, m-1$ is called a branch of length m with initial vertex d^1 .

We notice that for every $d^n \in W$ $g^{-1}(d^n)$ is finite and d^1 is from a finite family $Y \cap B$.

Suppose that for every $m \in \mathbb{N}$ there is a branch of length m . By means of König's Lemma ([2], p.104) there exists an infinite branch, that is a sequence $(d^n)_{n \in \mathbb{N}}$ with $d^{n+1} \subset d^n$ and $d^n \not\subset U$ for $n > 1$. There exists $b^\infty \in B^\infty$ such that $\bigcap_{n \in \mathbb{N}} d^n = Y \cap b^\infty \subset U(b^\infty)$.

Since each d^n is closed, so $d^n \subset U(b^\infty)$ for sufficiently large n . This is impossible, because $d^n \in V$ for $n > 1$.

We have proved that there exists $n_0 \in \mathbb{N}$ such that all branches have length less than n_0 . Hence for $b^{n_0} \in B^{n_0}$ such that $F_{A|B^{n_0}, \lambda, \epsilon}(Y) = F_{A, b^{n_0}, \lambda, \epsilon}(Y)$ we have $Y \cap b^{n_0} \subset U$, so $Y \cap b^{n_0} \subset U(b^\infty)$ for some $b^\infty \in B^\infty$.

From this and (1.1) it follows that for $n \geq n_0$

$$\begin{aligned} F_{A|B^n, \lambda, \epsilon}(Y) &\leq F_{A|B^{n_0}, \lambda, \epsilon}(Y) \leq D_A(C(b^\infty), \lambda) \leq \\ &\leq F_{A, b^\infty, \lambda, \epsilon}(Y) + \epsilon \end{aligned}$$

This implies that $m_{A|B, \lambda}(Y) \leq m_{A, \lambda}(Y \cap b^\infty)$

and this gives the required inequality.

Corollary 1.

$$h^*(f, Y) \geq \inf_{B \in \mathcal{U}(X)} \sup_{b^\infty \in B^\infty} h(f, Y \cap b^\infty) \text{ for any } Y \subset X$$

Corollary 2.

$$\text{If } h^*(f, \bar{Y} - Y) = 0, \text{ then } h^*(f, Y) = \inf_{B \in \mathcal{U}(X)} \sup_{b^\infty \in B^\infty} h(f, Y \cap b^\infty)$$

Proof

In view of Proposition 3 b) § 2, Proposition 2 c) [1] and Corollary 1 it is sufficient to prove that if $\bar{Y} = Y$ then

$$h^*(f, Y) = \inf_{B \in \mathcal{O}(X)} \sup_{b^\infty \in B^\infty} h(f, Y \cap b^\infty) .$$

Let $\bar{Y} = Y$. For any $B \in \mathcal{O}(X)$ there exists $C_B \in \mathcal{O}(X)$ such that $\bar{C}_B \supseteq B$. Then by Theorem 2b) and (1.2) we obtain

$$\begin{aligned} h^*(f, Y) &\leq \inf_{B \in \mathcal{O}(X)} \sup_{A \in \mathcal{O}(X)} h_A(f, Y) \leq \inf_{B \in \mathcal{O}(X)} \sup_{A \in \mathcal{O}(X)} \sup_{C \in \mathcal{O}_B^\infty} h_A(f, Y \cap C) \\ &\leq \inf_{B \in \mathcal{O}(X)} \sup_{A \in \mathcal{O}(X)} \sup_{b^\infty \in B^\infty} h_A(f, Y \cap b^\infty) = \\ &= \inf_{B \in \mathcal{O}(X)} \sup_{b^\infty \in B^\infty} h(f, Y \cap b^\infty) \end{aligned}$$

In view of Corollary 1 this completes the proof.

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