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ON INVARIANT MEASURES ON THE SPACE OF BILATERAL SEQUENCES

Jan Kwiatkowski

Let $S = \{1,2,\dots,s\}$, $X = \prod_{-\infty}^{+\infty} S$. Denote by $\mathcal{B}$ the σ-algebra generated by the cylinder sets and by T the shift transformation on X. For $n \geq 1$, we put $X_n = \prod_1^n S$. An element $B = (i_1, i_2, \dots, i_n) \varepsilon X_n$ is called a block. We shall identify B with the cylinder $\{x \varepsilon X ; x_o = i_1, x_2 = i_2, \dots, x_{n-1} = i_n\}$. Let us denote $M(X)$ the set of all T-invariant measures on $\mathcal{B}$. For given $\mu \varepsilon M(X)$ we define μ_n on X_n as $\mu_n(B) = \mu(B)$. The measure μ_n may be considered as a point of the space R^{s^n} in this sense that the coordinates of μ_n are indexed by the blocks B , and B-th coordinate of μ_n is equal $\mu_n(B)$.

Denote by K_n the set of all vectors of the form $\langle \mu_n(B) \rangle_{B \varepsilon X_n}$, when μ runs over all invariant measures on X . The set K_n may be described by the following conditions :

$$\begin{cases} \sum_{i=1}^{s} \mu_n(iC) = \sum_{i=1}^{s} \mu_n(C_i) \quad , \quad C \varepsilon X_{n-1} \quad (n > 1) \\ \sum_{B \varepsilon X_n} \mu_n(B) = 1 \quad , \quad \mu_n(B) \geq 0 \quad . \end{cases}$$

In this way K_n is polygon in R^{s^n} . It is easy to see that $\dim K_n = S^{n-1}(S-1)$.

Now, we define a mapping $f_n : K_{n+1} \longrightarrow K_n$ as follows :

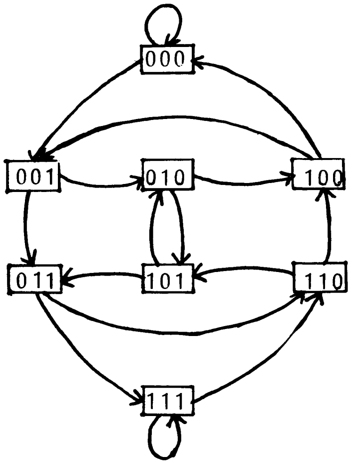
$$(f_n \mu_{n+1})(B) = \sum_{i=1}^{s} \mu_{n+1}(iB) ,$$

for $B \varepsilon X_n$ and $\mu_{n+1} \varepsilon K_{n+1}$. We obtain the sequence of the polygons K_n and the sequence of the mapping f_n

$$K_1 \xleftarrow{f_1} K_2 \xleftarrow{f_2} K_3 \xleftarrow{f_3} \quad \cdots\cdots$$

The set $M(X)$ may be identified with $\xleftarrow{\lim} K_n$.

My first aim is to describe the extremal points of the polygons K_n . In order to do this we use a graph Y_n defined as follows : the vertices of Y_n are the blocks C of the length $n-1$, and two blocks $C_1 = (i_1,i_2,\ldots,i_{n-1})$, $C_2 = (j_1,j_2,\ldots,j_{n-1})$ are joined by oriented edge iff $(i_2,\ldots i_{n-1}) = (j_1\ldots j_{n-2})$. If $S = \{0,1\}$ and $n = 4$ then we have



Y_4

We remark that the edges of Y_n one can identify with the blocks of the length n . Let $\gamma = \{B_1,B_2,\ldots,B_l\}$ $(1 \le l \le S^{n-1})$ be a closed path in Y_n such that γ does not contain any loop. Define $\bar{p}\gamma \varepsilon K_n$:

$$p_\gamma(B) = \begin{cases} \frac{1}{l} & , \quad B \in \gamma \\ 0 & , \quad B \notin \gamma \end{cases}$$

Theorem 1.

The vector $\mu_n \in K_n$ is an extremal point of K_n iff $\mu_n = \bar{p}\gamma$ for some closed path γ not having any loop.

Now, I would like to describe extremal points of K_n that may be used in a decomposition of Bernoulli measures. We say that two blocks $B, \bar{B} \in X_n$ are ϕ-equivalent iff $\bar{B} = (i_t\ i_{t+1} \cdots i_n i_1 \ldots i_{t-1})$ for some $1 \leq t \leq n$, where $B = (i_1\ i_2 \ldots i_n)$. It is easy to verify that ϕ is an equivalence relation and therefore ϕ induce a partition on disjoint subsets of X_n. Let $\mathcal{L}_n$ be the set of all classes of X_n. If $\gamma \in \mathcal{L}_n$, γ $\{B_1, B_2, \ldots, B_{\ell}\}$, then $1/n$ and γ is a closed path not having any loop.

Take a Bernoulli measure μ given by a probability vector $q = (q_1, q_2, \ldots, q_s)$. Then we have

$$\mu_n = \sum_{n_1 + \ldots + n_s = n} q_1^{n_1}\ q_2^{n_2} \ldots q_s^{n_s} \sum_{\substack{\gamma \subset X(n_1, \ldots, n_s) \\ \gamma \in \mathcal{L}_n}} l_\gamma \cdot \bar{p}_\gamma ,$$

Where $n_1, n_2, \ldots, n_s$ is non-negative integers, $X(n_1, \ldots, n_s)$ is the set of all blocks B of the length n with the frequency of symbols $(n_1, n_2, \ldots, n_s)$, and l_γ is the length of γ.

Note that if $\gamma \in \mathcal{L}_n$ then $\gamma \subset X(n_1, n_2, \ldots, n_s)$ since the relation ϕ does not change of the numbers of symbols.

Let L_n be the convex hull spanned by the extremal points $\bar{p}\gamma$, $\gamma \in \mathcal{L}_n$. Now, we shall describe all invariant measures μ satisfying conditions $\mu_n \in L_n$ for $n = 1, 2, \ldots$

Let $T_s = \{(x_1,x_2,\dots,x_s) ; \sum_{i=1}^{s} x_i=1 , x_i \geq o\}$. Let us consider a probability Borel measure $\overline{\mu}$ on T_s . The measure $\overline{\mu}$ allows to define a measure $\mu \in M(X)$ as follows

$$(*) \quad \mu(B) = \int_{T_s} x_1^{n_1} \dots x_s^{n_s} \overline{\mu}(dx) , \ x = (x_1,x_2,\dots,x_s) \in T_s ,$$

where B is a block with frequency of symbols $(n_1,n_2,\dots,n_s)$.

Theorem 2.

Let $\mu \in M(X)$. Then $\mu_n \in L_n$, $n = 1,2,\dots,$ iff there exists a Borel probability measure $\overline{\mu}$ on T_s such that μ has the form (*)

We shall write $\mu = \phi(\overline{\mu})$.

Consider the dynamical system $Z(\overline{\mu}) = (X,\psi(\overline{\mu}),T)$.

Put $H(x_1,x_2,\dots,x_s) = -\sum_{i=1}^{s} x_i \log x_i$, $(x_1,\dots,x_s) \in T_s$. Then we have $\mathfrak{I}mH = \langle o,\log s) = I_s$. Let $\xi = H^{-1}(\varepsilon)$, where ε is the partition of I_s on points. Therefore the measure $\overline{\mu}$ determines the measure ν on I_s and a conditional measures $\overline{\mu}_a$, $a \in Is$. Let $m_n(a)$ be the type of $\overline{\mu}a$ that is $\{m_n(a)\}$ is a sequence of measurable functions on I_s such that $\sum_{n=1}^{\infty} m_n(a) \leq 1$, $m_{n+1}(a) \leq m_n(a)$, $m_n(a) \geq 0$ for $n = 1,2,\dots$ and for a.e. $a \in Is$.

We obtain a pair $\Theta(\nu,\{m_n(a)\}_{a\in Is}) = \Theta(\overline{\mu})$.

Theorem 3.

Given two Borel probability meausres $\overline{\mu}_1$, $\overline{\mu}_2$ on T_s . The dynamical systems $Z(\overline{\mu}_1)$, $Z(\overline{\mu}_2)$ are isomorphic iff $\Theta(\overline{\mu}_1) = \Theta(\overline{\mu}_2)$.

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