

Astérisque

S. G. HANCOCK

Orbits of paths under hyperbolic toral automorphisms

Astérisque, tome 49 (1977), p. 93-96

http://www.numdam.org/item?id=AST_1977__49_93_0

© Société mathématique de France, 1977, tous droits réservés.

L'accès aux archives de la collection « Astérisque » (<http://smf4.emath.fr/Publications/Asterisque/>) implique l'accord avec les conditions générales d'utilisation (<http://www.numdam.org/conditions>). Toute utilisation commerciale ou impression systématique est constitutive d'une infraction pénale. Toute copie ou impression de ce fichier doit contenir la présente mention de copyright.

NUMDAM

Article numérisé dans le cadre du programme
Numérisation de documents anciens mathématiques
<http://www.numdam.org/>

ORBITS OF PATHS UNDER HYPERBOLIC TORAL AUTOMORPHISMS

S. G. Hancock

In [1], M. Hirsch considers the existence of compact sets invariant under hyperbolic toral automorphisms (h.t.a.'s), and mentions the question :

Can a h.t.a. $f: T^n \rightarrow T^n$ have a compact invariant set of dimension 1 ?

J. Franks [2] went some way towards providing a negative answer when he proved that a compact f -invariant set which contains a C^2 arc must contain a coset of an invariant toral subgroup of dimension at least 2. If we impose the condition that the characteristic polynomial of f be irreducible over $\mathbb{Z}$, then there are no proper invariant toral subgroups, so every C^2 arc must have a dense orbit. The following simple result shows that this is usually the case for C^0 arcs, even without the irreducibility assumption :

Proposition :

Let $f: T^n \rightarrow T^n$ be a h.t.a. . Then $\{\sigma: I \rightarrow T^n : 0(\sigma) \text{ is dense}\} = D$ is a Baire set in $C(I, T^n)$.

Proof.

Let $\{U_m\}$ be a countable open base for T^n , and $D_m = \{\sigma: \sigma(I) \cap 0(U_m) \neq \emptyset\}$. Then D_m is open since $0(U_m)$ is open, and

$O(U_m)$ is dense since f is ergodic, so D_m is also dense, and $D = \bigcap D_m$.

Against this we have :

Theorem

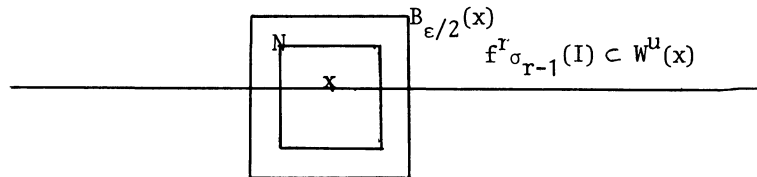
Let $f: T^n \rightarrow T^n$ be a h.t.a. with $u > 1$, $s > 1$, where u and s denote the respective dimensions of the unstable and stable manifolds of f . Then $\{\sigma: I \rightarrow T^n : O(\sigma) \text{ is not dense}\}$ is dense in $C(I, T^n)$.

Sketch of proof.

Given $\sigma \in C(I, T^n)$ and $\epsilon > 0$, we take $x \in T^n$ and a closed neighbourhood N of x of diameter less than ϵ , and first construct a sequence of paths $\sigma_{-1} = \sigma, \sigma_0, \sigma_1, \sigma_2, \dots$ such that

- 1) $f^r_{\sigma_r}(I) \cap N = \emptyset \quad (r \geq 0)$
- 2) $f^r_{\sigma_r}(t) \in W^u_\epsilon(f^r_{\sigma_{r-1}}(t)) \quad (t \in I, r \geq 0)$

Thus given σ_{r-1} , we obtain $f^r_{\sigma_r}$ by moving each point $f^r_{\sigma_{r-1}}(t)$ by a small amount in its own unstable manifold to a point $f^r_{\sigma_r}(t) \notin N$. The hypothesis $u > 1$ is necessary at this stage, for suppose $W^u(x)$ were 1-dimensional and $f^r_{\sigma_{r-1}}$ passed through N along $W^u(x)$:



It is clearly impossible to move $f^r_{\sigma_{r-1}}$ by at most ϵ along $W^u(x)$ and obtain a continuous path $f^r_{\sigma_r}$ avoiding N . With the con-

dition $u > 1$, it is always possible to make this construction.

From 2), $d(\sigma_r, \sigma_{r-1}) \leq \alpha^r d(f^r \sigma_r, f^r \sigma_{r-1}) \leq \alpha^r \varepsilon$, where α gives the contraction of the unstable manifolds under f^{-1} , and if α is small enough (as can be ensured by taking a power of f) it follows that the sequence (σ_r) converges uniformly to a path τ , with $d(\sigma, \tau) \leq 2\varepsilon$, whose forward orbit misses a neighbourhood $N' \subset N$ of x . Now using the same method with f replaced by f^{-1} we move the path τ by an even smaller amount, say at most δ , $2\delta < \text{diam } N'$, in the direction of the stable manifolds of f , to get a path ρ with $0^-(\rho) \cap N'' = \emptyset$ for some neighbourhood N'' of x , $N'' \subset N'$.

Since $\rho(t) \in W_\delta^S(\tau(t))$, the forward orbit of ρ will be within δ of that of τ and so $0^+(\rho) \cap N'' = \emptyset$. Thus ρ is a path within 3ε of σ and $0(\rho)$ is not dense.

Remarks:

1) If $u > 1$ and $s = 1$, the first half of the proof goes through to give a path τ with a non-dense forward orbit. If the original path σ lies in $W_Y^u(p)$ for a fixed point p , then $\tau(I)$ will lie in $W_{Y+2\varepsilon}^u(p)$ and the backward iterates of $\tau(I)$ will remain in this set. We can thus obtain paths with non-dense orbits in this case. In particular, suppose $u = 2$, $s = 1$ and τ is such a path. Then $K = \overline{0(\tau)}$ is a compact invariant set of dimension at least one. By a result of Hirsch and R. Williams, $\dim K \leq 1$, so the original question is answered.

For $n > 3$, it would be interesting to know whether the resulting invariant sets can be 1-dimensional.

2) It seems to be possible, using a similar method, to prove the theorem for maps $\sigma: I^m \rightarrow T^n$ if $m < \min\{u, s\}$.

3) The results probably go through largely unchanged if f is any Anosov diffeomorphism with $\Omega(f) = M$.

4) Using Markov partitions, we can answer Hirsch's question more directly. For, in the notation of [3], if $\mathcal{C}$ is a Markov partition for an Anosov diffeomorphism $f:M \rightarrow M$ with $\Omega(f) = M$ and $\dim M = n$, then $\overline{O(\partial^S \mathcal{C} \cap \partial^u \mathcal{C})}$ is an invariant set of dimension $n-2$. This follows from Hirsch and Williams's result and the product structure of rectangles in $\mathcal{C}$.

References

- [1] M. Hirsch "On Invariant Subsets of Hyperbolic Sets" in Essays on Topology and Related Topics, Springer (1970)
- [2] J. Franks "Invariant Sets of Hyperbolic Toral Automorphisms" preprint, Northwestern University (1976)
- [3] R. Bowen "Markov Partitions for Axiom A Diffeomorphisms" American Journal of Maths, 92, (1970)

Prof. S.G. Hancock
Mathematics Institute
University of Warwick
Coventry CV4 7AL
England