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ERGODIC PROPERTIES OF PIECEWISE MONOTONIC TRANSFORMATIONS

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Let $(L_1, \|\ \|)$ be the space of all Lebesgue integrable functions defined on the interval $[0, 1]$. Lebesgue measure on $[0, 1]$ will be denoted by m.

For a real number $M \geq 1 + \frac{1}{M}$ and an integer $q \geq 2$ we define a set $F(M,q)$ of transformations $\tau : [0, 1] \to [0, 1]$ for which there exists a partition $0 = a_o < a_1 < \ldots < a_{q_\tau} = 1$ of the unit interval such that the restriction τ_i of τ to the open interval (a_{i-1}, a_i) can be extended to a C^1 + Lipschitz function $\bar{\tau}_i$ on $[a_{i-1}, a_i]$, $1 + \frac{1}{M} \leq |\dot{\tau}_i| \leq M$, $L_{\dot{\tau}_i} \leq M$, $i = 1,\ldots,q_\tau$, $q_\tau \leq q$.

Here $L_{\dot{\tau}_i}$ denotes the Lipschitz constant for $\dot{\tau}_i$ and the number q_τ is chosen minimal one.

Denote by Q_τ the operator acting in L_1 such that for $f \in L_1$

$$Q_\tau f = \lim_{n\to\infty} \frac{1}{n} \sum_{k=o}^{n-1} P_\tau^k f$$

Here P_τ denotes the Frobenius - Perron operator for $\tau \in F(M,q)$. The function $f^* = Q_\tau f$ has the following properties :

i) $f^* \geq 0$ for $f \geq 0$

ii) $\int_0^1 f^* \, dm = \int_0^1 f \, dm$

iii) $P_\tau f^* = f^*$ and consequently the measure $d\mu^* = f^* \, dm$ is τ-invariant

iv) the function f^* is of bounded variation

v) $f^*(x^-) = f^*(x)$ for $x \in [0 , 1]$ and $f^*(0) = f^*(o^+)$

Let $F = \bigcup_{M \geq 1+\frac{1}{M}} \bigcup_{q \geq 2} F(M , q)$

The proof of existence of the operator Q_τ with properties i - iv for τ piecewise C^2 is due to Lasota and Yorke [4] . By a simple modification we generalize the above result to the class F.

Properties of $Q_\tau f$ for $f \in L_1$, especially iv, allow us to study ergodic properties of τ and τ-invariant measures.

<u>Theorem 1</u>. [1,2]

Let τ *be an element of* F . *Then the interval* $[0 , 1]$ *can be decomposed into a finite number of open sets which are* τ*-invariant and* τ*-ergodic. The number* p_τ *of these is not greater then*

$$\left[\frac{q_\tau - 1}{\inf|\dot{\tau}|-1}\right] \quad \textit{if} \quad \inf|\dot{\tau}| \quad \textit{is an integer}$$

and $$p_\tau \leq \left[\frac{q_\tau - 1}{[\inf|\dot{\tau}|]}\right] \quad \textit{otherwise} .$$

Here $[x]$ denotes the integer part of x .

The related result was announced by J.A. Yorke and T Li on Dynamical Systems Conference - Providence 1974.

We put

$\beta(M,q) = \{\tau : \tau \in F (M,q) \ , \ q_\tau = q \ , \ \beta_\tau \cap \tau^{-k}(\bar{\beta}_\tau) = \phi$ for every k , $o < k \leq k(\tau) - 1\}$

where $\beta_\tau = \{a_o , a_1 , \dots , a_{q_\tau}\}$, $\overline{\tau}^{-1}(x) = \bigcup_{i=1}^{q_\tau} \overline{\tau}_i^{-1}(x)$,

$\overline{\beta}_\tau = \beta_\tau - \{0 , 1\}$ and $k(\tau)$ is the smallest integer such that $\inf|\dot{\tau}|^{k(\tau)} > 2$.

The metric $\rho(\tau_1 , \tau_2) = \int_o^1 |\tau_1 - \tau_2| \, dm$ for $\tau_1 , \tau_2 \in F(M,q)$ gives the compact topology on $F(M,q)$. By uniqueness of invariant measure for τ-ergodic and by continuity of the operator P_τ as a function of parameter τ we get ;

<u>Theorem 2</u>. [1]

If $\tau_n \to \tau$, $\tau_n \in F(M,q)$, $\tau \in B(M,q)$ *and* τ *is ergodic with respect to the measure* m *then for any* $f \in L_1$

$Q_{\tau_n} f \to Q_\tau f$ *in* L_1*-norm* .

<u>Theorem 3</u>. [1]

Let $\tau_n \to \tau$ *and* $\tau_n \in F(M,q)$, $\tau \in B(M,q)$. *Then* $Q_{\tau_n} f \to Q_\tau f$ *for every* $f \in L_1$ *iff* $\lim_{n\to\infty} p_{\tau_n} = p_\tau$.

Let $H(M,q)$ be the subset of $F(M,q)$ such that for every sequence $\tau_n \in F(M,q)$ and $\tau \in H(M,q)$ if $\tau_n \to \tau$ if ρ then $Q_{\tau_n} f \to Q_\tau f$ in L_1 for every $f \in L_1$. By theorem 2, if τ is ergodic and $\tau \in B(M,q)$ then $\tau \in H(M,q)$.

There existsan ergodic transformation τ such that $\tau \in H(M,q) - B(M,q)$.

For example $\tau(x) = \frac{\sqrt{5} + 1}{2} x \bmod (1)$ belongs to $H(M,2) - B(M,2)$.

The above is a consequence of the following theorem ;

Theorem 4. [3]

Transformation $\beta(x) = \beta x \bmod (1)$ *for* $\beta > 1$ *belongs to* $H(M, q_\beta)$.

Let $D_\tau = \{x : Q_\tau 1\,(x) > 0\}$.

Theorem 5. [3]

For every $\tau \in F(M,q)$ *there exist disjoint sets* C_i , $i = 1, \ldots, p_\tau$, *such that* C_i *is finite sum of intervals,* $\tau(C_i) \simeq C_i$, τ *is ergodic on* C_i *and*

$$\bigcup_{i=1}^{p_\tau} C_i = D_\tau \quad .$$

Theorem 5 gives a description of the support for τ-invariant measure. The related result in piecewise C^2-case has been obtained, but by a mistaken way, by A.A. Kosjakin and E.A. Sandler (Izv. Wyss. Uceb. Zav. N.3 118 1972). By construction of C_i in proof of theorem 5 we get ;

Theorem 6. [1,3]

If $\tau \in H(M,q)$ *then* $q_\tau = q$ *in nonergodic case and if* $M > 2$ *then* $q_\tau \geq q - 2$ *for* τ*-ergodic* .

This gives examples of ergodic elements of the set $F(M,q) - H(M,q)$ for $q > 3$.

Let λ be a real such that $1 < \lambda < \sqrt[3]{\frac{7}{4}}$ and let δ be any small positive number.

We choose a transformation $\tau \in F(M,2)$ such that $\dot{\tau}(x) = -\lambda$ for $x \in (0\ ,\ \delta) \cup (\frac{3}{4} - \delta,\ \frac{3}{4})$ $\dot{\tau}(x) = \lambda$ for $x \in (1 - \delta,\ 1) \cup (\frac{3}{4}\ ,\ \frac{3}{4} + \delta)$, $\tau(o) = 1$, $\tau(\frac{3}{4}) = 0$ and $\tau(1) = \frac{3}{4}$.

In [3] it has been shown that $\tau \in F(M,2) - H(M,2)$. At last we get the following fixpoint Theorem ;

Theorem 7. [3]

For $\tau \in F$ *there exists a positive integer* k *such that* τ^k *has a fixpoint.*

References

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[3] Z.S. Kowalski, Piecewise monotonic transformations and their invariant measures (to appear).

[4] A. Lasota and J.A. Yorke, On the existence of invariant measures for piecewise monotonic transformations, Trans. Amer. Math. Soc. 186 (1973), pp. 481-488.

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