

Astérisque

MAGDA KOMORNIKOVA

JOZEF KOMORNIK

On sequential entropy of measure preserving transformation

Astérisque, tome 49 (1977), p. 141-144

[<http://www.numdam.org/item?id=AST_1977__49__141_0>](http://www.numdam.org/item?id=AST_1977__49__141_0)

© Société mathématique de France, 1977, tous droits réservés.

L'accès aux archives de la collection « Astérisque » (<http://smf4.emath.fr/Publications/Asterisque/>) implique l'accord avec les conditions générales d'utilisation (<http://www.numdam.org/conditions>). Toute utilisation commerciale ou impression systématique est constitutive d'une infraction pénale. Toute copie ou impression de ce fichier doit contenir la présente mention de copyright.

NUMDAM

Article numérisé dans le cadre du programme
Numérisation de documents anciens mathématiques
<http://www.numdam.org/>

ON SEQUENTIAL ENTROPY OF
 MEASURE PRESERVING TRANSFORMATION

Madga Komornikova

Jozef Komornik

The aim of this article is to present a simple proof of a slightly more general result than those ones obtained in [2] and [4].

Definition

Let $A = \{t_1, \dots, t_n, \dots\}$ be a subset of the set Z_+ of the set Z_+ of nonnegative integers. For a given positive integer n, k we denote

$$A_{n,k} = \bigcup_{i=1}^n \{t_i, t_{i+1}, \dots, t_i + k-1\}$$

$$S_{n,k}(A) = \text{card } (A_{n,k})$$

$$\overline{S}_k(A) = \overline{\lim}_{n \rightarrow \infty} S_{n,k}(A) \quad \text{and} \quad K(A) = \lim_{k \rightarrow \infty} \overline{S}_k(A)$$

Newton in [2] defined $K(A)$ in another way, but the definitions are equivalent. Let $h_A(T)$ denote the sequential entropy introduced by Kushnirenko [1].

Theorem 1.

The equation

$$h_A(T) = K(A) \cdot h(T)$$

holds for any measure transformation T except of the case $h(T) = 0$ and $K(A) = \infty$. In case $K(A) = 0$, $h_A(T) = 0$.

For the proof we recall some properties of conditional entropy.

Let ξ be a finite measurable partition. We denote

$$\xi^K = \bigvee_{i=0}^{k-1} T^{-i}(\xi) \quad \text{for any positive integer } K.$$

Lemma 1.

$$\begin{aligned} H(\xi^k) &= H(\xi) + \sum_{i=1}^{k-1} H(T^{-i}(\xi) \mid \bigvee_{j=0}^{i-1} T^{-j}(\xi)) = \\ &= H(\xi) + \sum_{i=1}^{k-1} H(\xi \mid \bigvee_{j=1}^i T^{-j}(\xi)) \end{aligned}$$

Lemma 2.

For $k \leq q$ we have

$$\frac{1}{q} \cdot H(\xi^q) \leq \frac{1}{k} \cdot H(\xi^k)$$

Proof.

$$\begin{aligned} H(\xi^q) &= H(\xi) + \sum_{i=1}^{q-1} H(\xi \mid \bigvee_{j=1}^i T^{-j}(\xi)) = H(\xi) + \sum_{i=1}^{k-1} H(\xi \mid \bigvee_{j=1}^i T^{-j}(\xi)) + \\ &+ \sum_{i=k}^{q-1} H(\xi \mid \bigvee_{j=1}^i T^{-j}(\xi)) \leq H(\xi^k) + (q-k) \cdot H(\xi \mid \bigvee_{j=1}^{k-1} T^{-j}(\xi)) \leq \\ &\leq H(\xi^k) + \frac{q-k}{k} \cdot [H(\xi) + \sum_{i=1}^{k-1} H(\xi \mid \bigvee_{j=1}^i T^{-j}(\xi))] = \frac{q}{k} \cdot H(\xi^k) \end{aligned}$$

Proof of the theorem.

The fact that in case $K(A) = 0$ $h_A(T) = 0$ was proved in [3].

Suppose that the product $K(A) \cdot h(T)$ is well defined. For any measurable partition ξ and any positive number k we have

$$H_A(T, \xi) \leq H_A(T, \xi^k)$$

For a given k we can write the set $A_{n,k}$ as a union of its maximal connected subsets.

$$A_{n,k} = \bigvee_{j=1}^m C_j$$

We denote

$$q_j = \text{card } (C_j) \quad , \quad n_j = \min (C_j)$$

From the construction of $A_{n,k}$ we have

$$q_j \geq k \quad , \quad \sum_{j=1}^m q_j = S_{n,k}(A)$$

According the definition of sequential entropy

$$\begin{aligned} H_A(T, \xi) &\leq H_A(T, \xi^k) = \overline{\lim} \frac{1}{n} \cdot H\left(\bigvee_{i \in A_{n,k}} T^{-i}(\xi)\right) \leq \\ &\leq \overline{\lim} \frac{1}{n} \cdot \sum_{j=1}^m H\left(\bigvee_{i \in C_j} T^{-i}(\xi)\right) = \overline{\lim} \frac{1}{n} \cdot \sum_{j=1}^m H(T^{-n_j}(\xi^{q_j})) \leq \\ &\leq \overline{\lim} \frac{1}{n} \cdot \sum_{j=1}^m \frac{q_j}{k} \cdot H(\xi^k) = \frac{1}{k} \cdot H(\xi^k) \cdot \overline{\lim}_{n \rightarrow \infty} \frac{S_{n,k}(A)}{n} = \\ &= \overline{S}_k(A) \cdot \frac{1}{k} \cdot H(\xi^k) \quad . \end{aligned}$$

Taking limit with respect to k we get

$$H_A(T, \xi) \leq K(A) \cdot H(\xi)$$

Therefore the inequality

$$h_A(T) \leq K(A) \cdot h(T)$$

holds.

The converse inequality was proved in [3].

References

- [1] Kushnirenko, A.G. : O matricěskich invariantach tipa entropii, UMN 22(5), 1967, 57-62

- [2] Newton D. : On sequential entropy I - MST 4, 119-125 (1970)
- [3] Newton D. : On sequential entropy II - MST 4, 126-128 (1970)
- [4] Krug E. , Newton D. : On sequence entropy of Automorphisms of Lebesgue space . Zeits. f. Wahrsch. 24, 211-214 (1972)

M. Komornikova
Katedra Kybernetiky
EFSVŠT Bratislava
Gottwaldovo Naim 5
Czechoslovakia

J. Komornik
Matematický pavilón PFUK
Mlynská Dolina
81631 Bratislava
Czechoslovakia