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NONWANDERING POINTS OF ANOSOV DIFFEOMORPHISMS.

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There are two well-known conjectures in the Anosov diffeomorphism (C-diffeomorphism) theory. Let f be a C-diffeomorphism of a smooth compact Riemannian manifold M^n. Then (see [1] and [2]):

1. The set NW(f) of nonwandering points of f is M^n.

2. The covering manifold $\overline{M}$ is homeomorphic to R^n.

Many of the results in the C-diffeomorphism theory were got assuming that all the points are nonwandering. The condition $NW(f) = M^n$ implies for instance that a C-diffeomorphism f is topologically transitive (or simply "transitive"), the set Per(f) of all the periodic points of f is dense in M^n, every stable or unstable layer is dense in M^n.

All the existing examples of C-diffeomorphisms act on nilmanifolds (in particular on tori) and on their generalizations - infranilmanifolds. That's why proving the second statement (or finding the conditions under which it is valid) is the natural fist step in the classification of C-diffeomorphisms.

The sufficient conditions for the set NW(f) to coincide with M^n and for the covering manifold $\overline{M}$ to be homeomorphic to R^n are stated in this paper.

Every diffeomorphism f of a compact Riemannian manifold M induces a linear operator f_* in the space of continuous vector fields: $(f_* v)(x) = df_{f^{-1}x} v(f^{-1}x)$. C-diffeomorphisms are characterized (see [3]) by the fact that the spectrum S of the complexification of the operator f_* doesn't intersect the unit circle. So S is contained in the interiors of two rings with the radii $0<r_1<r_2<1$ and $1<R_2<R_1<\infty$. There exist such a constant $0<C_1<\infty$ that for every positive integer n

$$C_1^{-1}\, r_1^n\, \|v\| < \|df^n v\| < C_1\, r_2^n\, \|v\| \qquad \text{if } v\in E^s$$
$$C_1^{-1}\, R_2^n\, \|v\| < \|df^n v\| < C_1\, R_1^n\, \|v\| \qquad \text{if } v\in E^u$$

here E^s and E^u are respectively the stable and unstable subbundles of the tangent bundle TM.

Let's say that the correspondence mapping for the stable W^s and unstable W^u foliations can be infinitely extended if for every three points $x\in M$, $y\in W^s(x)$, $z\in W^u(x)$ there exists such a continuous mapping g of the unit square I^2 into M that: 1) $g(0,0) = x$, $g(0,1) = y$, $g(1,0) = z$; 2) $g(t,.)$ is a continuous curve on a stable layer for every fixed $t\in I$; 3) $g(.,t)$ is a continuous curve on an unstable layer for every fixed $t\in I$. The existence of such a mapping is obvious if the distancies between x,y,z measured along the corresponding layers are small enough. According to the given definition this mapping can be extended beyond the boundaries of a small neibourhood.

<u>PROPOSITION 1.</u> If $1 + \frac{\ln R_2}{\ln R_1} > \frac{\ln r_1}{\ln r_2}$ (*) or $1 + \frac{\ln r_2}{\ln r_1} > \frac{\ln R_1}{\ln R_2}$ (**) then the correspondence mapping for the foliations W^s and W^u can be infinitely extended.

Let d_s and d_u denote the distancies induced by the internal metrics of stable and unstable layers, and $l(c)$ be the length of a piecewise smooth curve c.

<u>LEMMA 2.</u> Let c be a smooth curve on an unstable layer $W^u(x_1)$ (respectively $W^s(x_1)$) connecting the points x_1 and y_1, and let $x_2\in W^s(x_1)$ $(W^u(x_1))$. Suppose that there exists such a continuous mapping $g(c,x_2) = g$ of the unit square I^2 into M that: 1) $g(t,0) = c(t)$,

$g(0,1) = x_2$; 2) the points $g(t,\bar{t}_1)$ and $g(t,\bar{t}_2)$ belong to the stable (unstable) layer; 3) $g(t,1) \in W^u(x_2)$ $(W^s(x_2))$; 4) the restriction of g to the set $I \times \{0\}$ is bijective.

Then there are such constants $b_1, b_2 > 0$ that if

$$m_s(g) = \max_{0 \leq t \leq 1,\ 0 \leq \bar{t}_1 \leq \bar{t}_2 \leq 1} d_s(g(t,\bar{t}_1), g(t,\bar{t}_2)) \leq b_1 \cdot \min(l(c), b_2)$$

(respectively for d_u) then there is such a curve c' connecting x_2 and $y_2 = g(1,1)$ on the layer $W^u(x_2)$ $(W^s(x_2))$ that $l(c') < 2 \cdot l(c)$.

Proof. If $l(c)$ is sufficiently small then the statement of the lemma follows from the transversality and continuity of the stable and unstable foliations. I.e. there are such $b_1 > 0$ and $q > 0$ that the statement of lemma is true if $l(c) < q$ and $m_s(g) < b_1 l(c)$. Let $b_2 = \frac{1}{2}q$. The curve c can be divided into segments c_i with the ends $g(z_i, 0)$ and $g(z_{i+1}, 0)$, $\frac{1}{2}q \leq l(c_i) < q$. Let's consider for each segment c_i in the capacity of c_i' the shortest geodesic on the layer $W^u(x_2)$ $(W^s(x_2))$ connecting $g(z_i, 1)$ and $g(z_{i+1}, 1)$, Since $l(c_i') < 2 \cdot l(c_i)$ we have $l(c') < 2 \cdot l(c)$. Q.e.d.

Let's say that two unstable layers $W^u(x_1)$ and $W^u(x_2)$ are (ε, r)-close at points x_1 and $x_2 \in W^s(x_1)$ if there exists such a continuous mapping $h: B^k \times I \to M$ of the direct product of the unit k-ball and unit segment into M that: 1) $h(0,0) = x_1$, $h(0,1) = x_2$; 2) $h(0,t)$ is the geodesic segment connecting x_1 and x_2 on the layer $W^s(x_1)$; 3) $h(B^k, 0)$ is the r-ball on the layer $W^u(x_1)$ with the centre x_1; 4) $h(B^k, t) \subset W^u(h(0,t))$; 5) $h(y, I) \subset W^s(h(y,0))$ and $d_s(h(y,t_1), h(y,t_2)) \leq \varepsilon$ for every $y \in B^k$, $t_1, t_2 \in I$; 6) the restriction of h on either $(B^k \times \{0\})$ or $(\{0\} \times I)$ is bijective.

Let $d(r,\varepsilon)$ be the supremum of those $\bar{d}$ that for every x_1 and $x_2 \in W^s(x_1)$ with $d_s(x_1,x_2) \leq \bar{d}$ the layers $W^u(x_1)$ and $W^u(x_2)$ are (ε,r)-close at the points x_1 and x_2. It follows from the continuity of the stable and unstable foliations that $d(r,\varepsilon) > 0$. Let's denote by ε_0 the diameter of the neighbourhood with product structure for f (see [4]).

LEMMA 3. Let r be greater than ε_0 and $\mathcal{E} = b_1 \min(\varepsilon_0, b_2)$. Then there is such a constant $C > 0$ independent of r that

$$d(r,\mathcal{E}) \geq C \exp\left(-\ln\left(\frac{r_1}{r_2}\right) \frac{\ln r}{\ln R_2}\right).$$

Proof. It follows from the compactness of M and continuity of the foliations that there are such points x_1, $x_2 \in W^s(x_1)$, $y_i \in W^u(x_i)$, $i=1,2$ that 1) $d_s(x_1,x_2) = d(r,\varepsilon) = d$; 2) the layers $W^u(x_1)$ and $W^u(x_2)$ are $(\mathcal{E},r)$-close at the points x_1 and x_2; 3) $d_u(x_1,y_1) \leq r$, $d_s(y_1,y_2) = \mathcal{E}$

Let $c_s = c_s(x_1,x_2)$ be the geodesic segment connecting x_1 and x_2 on the layer $W^s(x_1)$ and let $c_u = c_u(x_1,y_1)$ be the geodesic segment connecting x_1 and y_1 on the layer $W^u(x_1)$. The points x_1, y_1, $h(0,t)$, the curve $c_u(x_1,y_1)$ and the restriction of h on the set $c_s \times c_u$ satisfy the condition of Lemma 2 for every $t \in I$. Let $h(z,0) = y_1$. It follows then from Lemma 2 that $d_u(h(0,t),h(z,t)) < 2r$ for every $t \in I$. Suppose n is the minimal integer for which

$$d_u(f^{-n}h(0,t), f^{-n}h(z,t)) \leq b_1 \cdot \mathcal{E} = \mathcal{E}_1, \quad t \in I.$$

There is such a constant $C' = C'(\mathcal{E}_1)$ that $n \leq (\ln r)\cdot(\ln R_2)^{-1} + C'$. The points $f^{-n}x_1$, $f^{-n}x_2$, $f^{-n}y_1$ and the curve $c = f^{-n}(c_s)$ satisfy the condition of Lemma 2. That's why there's a piecewise smooth curve c', $l(c') < 2 \cdot l(c)$ connecting the points $f^{-n}y_1$ and $f^{-n}y_2$ and $c \subset W^s(f^{-n}y_1)$. Now $l(c') > C_1^{-1} r_2^{-n} d_s(y_1,y_2)$ and $l(c) \leq C_1 r_1^{-n} d_s(x_1,x_2)$.

These inequalities imply the statement of Lemma 3.

Proof of Proposition 1. Let $y \in W^s(x)$, $z \in W^u(x)$, $d_s(x,y) = a$, $d_u(x,z) = b$. The distancies between $f^n x$, $f^n y$, and $f^n z$ satisfy the following inequalities: $d_s(f^n x, f^n y) \leq C_1 r_2^n a$, $d_u(f^n x, f^n z) \leq C_1 R_1^n b$. Let's verify that for sufficiently large n one has

$$d(d_u(f^n x, f^n z), \varepsilon) \geq d_s(f^n x, f^n y).$$

Indeed in accordance with Lemma 3

$$\frac{d(d_u(f^n x, f^n z), \varepsilon)}{d_s(f^n x, f^n y)} \geq \frac{C \exp(-\ln\frac{r_2}{r_1} \ln(C_1 R_1^n b)(\ln R_2)^{-1})}{C_1 r_2^n a} \geq$$

$$\geq \frac{C\ C_2}{C_1 a} \exp\left(-n\left(\frac{\ln R_1}{\ln R_2} \ln\frac{r_2}{r_1} + \ln r_2\right)\right).$$

It is clear now that the statement of Proposition 1 follows from (*). The inequality (**) is treated by analogy. Q.e.d.

<u>REMARK 4.</u> If the correspondence mapping can be infinitely extended then every stable layer $W^s(x)$ intersects every unstable layer $W^u(y)$. Indeed let's connect x and y by a smooth curve c and divide this curve into arcs c_i (with the ends x_i and x_{i+1}), $l(c_i) \leq \frac{1}{2}\ \varepsilon_c$ (ε_c is the diameter of the product structure neighbourhood). If the intersection $W^u(x_i) \cap W^s(x)$ isn't empty then neither is the intersection $W^u(x_{i+1}) \cap W^s(x)$. So the induction argument shows that $W^u(y) \cap W^s(x) \neq \emptyset$.

<u>THEOREM 5.</u> Let $f: M^n \to M^n$ be a C-diffeomorphism. Suppose that the correspondence mapping for the stable and unstable foliations can be infinitely extended. Then the set NW(f) of nonwandering points coincides with M^n and the universal covering manifold $\overline{M}$ is homeomorphic to R^n.

Proof. In accordance with the Smale spectral theorem (see [2]) the set $NW(f)$ can be represented as a finite union of disjoint closed sets called basic sets. Let B_1 be a repeller and B_2 - an attractor. It's well known that B_1 consists of stable layers and B_2 consists of unstable layers. If $x \in B_1$ and $y \in B_2$ then $W^s(x) \subset B_1$ and $W^u(y) \subset B_2$ But $W^s(x) \cap W^u(y) \neq \emptyset$ (see Remark 4), hence $B_1 \cap B_2 \neq \emptyset$. It follows that there exists only one basic set $B = M^n$.

Now let $x \in M^n$. There are (see [1]) two diffeomorphisms $F^s\colon R^k \longrightarrow W^s(x)$ and $F^u\colon R^{n-k} \longrightarrow W^u(x)$. If $y \in W^u(x)$ and $z \in W^s(x)$ then according to the infinite extendability of the correspondence mapping there's a continuous function $g\colon I^2 \longrightarrow M^n$. Let's denote $g(1,1) = q(y,z)$. It's easy to verify that although ~~xx~~ g isn't uniquely defined the point $q(y,z)$ is independent of the choise of g. This statement is obvious if the distancies $d_u(x,y)$ and $d_s(x,z)$ are sufficiently small. Indeed $q(y,z)$ is the unique point of intersection of the local layers in this case. Almost the same argument shows that $q(y,z)$ is independent of g if either $d_u(x,y)$ or $d_s(x,z)$ is small enough, and to achieve such a situation it is sufficiently to apply the iterations of f.

Let's define the mapping $p\colon R^n = R^k \times R^{n-k} \longrightarrow M^n$ by the formula $p(v,w) = q(F^u(w), F^s(v))$. It follows from the continuity and transversality of the stable and unstable foliations that p is a local homeomorphism. I.e. for every point $\bar{x} \in R^n$ there's such a neighbourhood $U(\bar{x})$ that the restriction $p|U(\bar{x})$ maps it homeomorphically onto a neighbourhood of $p(\bar{x})$. Let's show that for every curve $c(t)$, $t \in [0,1]$ on M^n and for every point $\bar{x} \in p^{-1}(c(0))$ there is the unique curve $\bar{c}(t)$ on R^n with $p(\bar{c}(t)) = c(t)$ and $\bar{c}(0) = \bar{x}$. Since p is a local homeomorphism it is sufficient to prove this statement for piecewise smooth curves with the length less than ε_0. Let c be such a curve, $c(0) = x$, $p(\bar{x}) = x$, $\bar{x} = (\bar{x}_s, \bar{x}_u)$, $\bar{x}_s \in R^k$, $\bar{x}_u \in R^{n-k}$. Since c is contained in the product structure neighbourhood with the centre x there are such curves $c_s \subset W^s(x)$ and $c_u \subset W^u(x)$ that the intersection point of the local layers $W^u_{loc}(c_s(t))$ and $W^s_{loc}(c_u(t))$ coincides with $c(t)$ for every $t \in [0,1]$. To construct the curve $\bar{c}$ it is sufficient toapply the property of the infinite correspondence mapping extension to the points $c(t)$, $c_s(t)$ and $p(0,\bar{x}_u)$. The uniqueness of this curve follows from the fact that p is a local homeomorphism. So p is the covering mapping. Q.e.d.

COROLLARY 6. Let f be a C-diffeomorphism of M^n and

$$\text{either} \quad 1 + \frac{\ln R_2}{\ln R_1} > \frac{\ln r_1}{\ln r_2} \ (*) \quad \text{or} \quad 1 + \frac{\ln r_2}{\ln r_1} > \frac{\ln R_1}{\ln R_2} \ (**).$$

Then $NW(f) = M^n$ and the covering manifold for M^n is homeomorphic to R^n.

Let A be a hyperbolic automorphism of a nilpotent Lie algebra N inducing a hyperbolic diffeomorphism of a compact nilmanifold. The eigenvalues of A are contained in two rings with the radii $0 < r_1 < r_2 < 1$ and $1 < R_2 < R_1 < \infty$.

PROPOSITION 7. If either

1. a) $1 + \dfrac{\ln R_2}{\ln R_1} > \dfrac{\ln r_1}{\ln r_2}$; b) $r_2 R_1 \geqslant 1$

or

2. a) $1 + \dfrac{\ln r_2}{\ln r_1} > \dfrac{\ln R_1}{\ln R_2}$; b) $r_1 R_2 \leqslant 1$

then N is a commutative algebra.

Proof. Let's denote $V_q = \{x \in N \,/\, \exists k: (A-qE)^k = 0\}$. It is easy to show that $[V_q, V_{\bar{q}}] \subseteq V_{q\bar{q}}$. Let condition 1 be true and N be noncommutative. The intersection of the uniform discrete subgroup and the commutator group is a uniform discrete subgroup of the commutator group (see [5]). That's why among the eigenvalues of the restriction of A on the commutator algebra there are those numbers of modulo less and greater than 1. It is easy to show that there exists such a non-zero vector $x \in V_r$, $|r| < 1$ that $x = [y,z]$, $y \in V_q$, $z \in V_{\bar{q}}$. There is an alternative: either $|q|, |\bar{q}| < 1$ or one of these numbers is greater than 1. In the first case we get $r_1 < r_2^2$ which contradicts condition 1. Let's consider the second case. Let $|q| > 1$. Since $\ln|r| = \ln|q| + \ln|\bar{q}|$ then $\ln r_2 > \ln r_1 + \ln R_2$ which also contradicts condition 1. Condition 2 is treated on the analogy. The proposition is proven.

It seems that the conditions $r_2 R_1 \geqslant 1$ and $r_1 R_2 \leqslant 1$ are inessential. A.Katok noted that conditions 1a and 2a aren't valid for the well-known Smale examples of C-diffeomorphisms of nilmanifolds.

CONJECTURE. If a C-diffeomorphism $f: M \longrightarrow M$ possesses the property (*) or (**) (see proposition 1 and corollary 6) then M is a torus.

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