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MUTUAL SINGULARITY OF SPECTRA OF DYNAMICAL SYSTEMS

GIVEN BY "SUMS OF DIGITS" TO DIFFERENT BASES

Teturo Kamae

0. Summary

In [3] , it was proved that if $(p,q) = 1$ and a and b are irrational numbers, then the following two arithmetic functions α and β have mutually singular spectral measures :

$$\begin{aligned}\alpha(n) &= \exp (2\pi i a s_p(n)) \\ \beta(n) &= \exp (2\pi i b s_q(n))\end{aligned}\quad (n \in \mathbb{N}) \quad ,$$

where $s_p(n)$ ($s_q(n)$) is the sum of digits in the p -adic (q -adic) representation of n . Here we prove a slightly stronger result that the two shift dynamical systems corresponding to the strictly ergodic sequences α and β have mutually singular spectral measures.

That is to say that for any $f \in L_2(\mu_\alpha)$ and $g \in L_2(\mu_\beta)$ such that $\int f d\mu_\alpha = \int g d\mu_\beta = 0$, where μ_α and μ_β are the measures on $\mathbb{T}^{\mathbb{N}}$ ($\mathbb{T}$ being the unit circle in the complex plane) for which α and β are generic with respect to the shift, respectively, the spectral measures $\Lambda_{\alpha,f}$ and $\Lambda_{\beta,g}$ are mutually singular, where $\Lambda_{\alpha,f}(\Lambda_{\beta,g})$ is the measure Λ on $\mathbb{R}/\mathbb{Z}$ determined by the relation

$$(T^n f, f)_{\mu_\alpha} = \int e^{2\pi i \lambda n} d\Lambda(\lambda) \quad ((T^n g, g)_{\mu_\beta} = \int e^{2\pi i \lambda n} d\Lambda(\lambda)) \quad \text{for all } n \in \mathbb{N}$$

(T denoting the shift as well as the isometry on L_2 induced by the shift).

1. Mutual singularity of spectra and disjointness

Given two dynamical systems $X = (X, \mu, S)$ and $Y = (Y, \nu, T)$. We consider, in the obvious way, $L_2(\mu)$ and $L_2(\nu)$ as subspaces of $L_2(\mu \times \nu)$. For $f \in L_2(\mu \times \nu)$, $H(f)$ denotes the closed subspace of $L_2(\mu \times \nu)$ spanned by f , $(S \times T)f$, $(S \times T)^2 f, \dots$. The following theorem is essentially due to A.N. Kolmogorov.

Theorem A.

X and Y have mutually singular spectral measures if and only if

- (1) X and Y are disjoint in the sense of H. Furstenberg, and
- (2) for any $f \in L_2(\mu)$ and $g \in L_2(\nu)$ such that $\int f d\mu = \int g d\nu = 0$, $f \in H(f+g)$.

Proof :

We prove only that the mutual singularity of spectra implies the disjointness, since the other parts follows easily from [4]. Assume that X and Y are not disjoint. Then there exists a probability measure $\xi \neq \mu \times \nu$ on $X \times Y$ which is $S \times T$ -invariant and satisfies that $\xi|_X = \mu$ and $\xi|_Y = \nu$. Take $f \in L_2(\mu)$ and $g \in L_2(\nu)$ such that $\int f d\mu = \int g d\nu = 0$ and $(f, g)_\xi \neq 0$. Since

$$\frac{1}{N} \left| \sum_{n=1}^N e^{-2\pi i \lambda n} S^n f \right|_\mu^2 d\lambda \rightarrow \Lambda_{X, f}$$

$$\frac{1}{N} \left| \sum_{n=1}^N e^{-2\pi i \lambda n} T^n g \right|_\nu^2 d\lambda \rightarrow \Lambda_{Y, g} \quad (\text{weakly})$$

and the property of the affinity $\rho[2]$, we have

$$\begin{aligned}
 & \rho(\Lambda_{x,f}, \Lambda_{y,g}) \\
 & \geq \overline{\lim}_N \int \frac{1}{N} \left| \left| \sum_{n=1}^N e^{-2\pi i \lambda n} S^n f \right| \right|_\mu \left| \left| \sum_{n=1}^N e^{-2\pi i \lambda n} T^n g \right| \right|_\nu d\lambda \\
 & \geq \overline{\lim}_N \int \frac{1}{N} \left| \left(\sum_{n=1}^N e^{-2\pi i \lambda n} S^n f, \sum_{n=1}^N e^{-2\pi i \lambda n} T^n g \right)_\xi \right| d\lambda \\
 & \geq \overline{\lim}_N \frac{1}{N} \left| \int \left(\sum_{n=1}^N e^{-2\pi i \lambda n} S^n f, \sum_{n=1}^N e^{-2\pi i \lambda n} T^n g \right)_\xi d\lambda \right| \\
 & = |(f, g)_\xi| > 0
 \end{aligned}$$

Thus $\Lambda_{x,f}$ and $\Lambda_{y,g}$ are not mutually singular.

2. Disjointness of α and β

To prove the disjointness of the two dynamical systems given by α and β in §0, it is sufficient to prove that any γ and δ in the orbit closures of α and β , respectively, with respect to the shift are independent of each other. The proof by J. Besineau [1] for the independency of α and β works well for these γ and δ . Thus, we have the disjointness of α and β .

3. Mutual singularity of dynamical systems given by α and β

Let (X, μ, S) be a dynamical system. Let f and g be in $L_2(\mu)$. Then we have

Lemma

- (1) $\Lambda_{cf} = |c|^2 \Lambda_f$, where c is a constant.
- (2) $\Lambda_{f+g} \leq 2\Lambda_f + 2\Lambda_g$.
- (3) $||\Lambda_f - \Lambda_g|| < ||f-g||^2 + 2||f|| ||f-g||$, where $||\Lambda_f - \Lambda_g||$ is the total variance of the measure $\Lambda_f - \Lambda_g$.

Proof :

(1) is clear. To prove (2), we have

$$\begin{aligned}\Lambda_{f+g} &= \text{w-lim}_N \frac{1}{N} \left| \left| \sum_{n=1}^N e^{-2\pi i n \lambda} S^n(f+g) \right| \right|^2 d\lambda \leq \\ &\leq \text{w-lim}_N \frac{2}{N} \left(\left| \left| \sum_{n=1}^N e^{-2\pi i n \lambda} S^n f \right| \right|^2 + \left| \left| \sum_{n=1}^N e^{-2\pi i n \lambda} S^n g \right| \right|^2 \right) d\lambda = \\ &= 2\Lambda_f + 2\Lambda_g\end{aligned}$$

(3) follows from the fact that

$$\begin{aligned}\left| \Lambda_f - \Lambda_g \right| &\leq \frac{1}{N} \int \left| \left| \sum_{n=1}^N e^{-2\pi i n \lambda} S^n f \right| \right|^2 - \left| \left| \sum_{n=1}^N e^{-2\pi i n \lambda} S^n g \right| \right|^2 \right| d\lambda \leq \\ &\leq \frac{1}{N} \int \frac{1}{N} \left(\left| \left| \sum_{n=1}^N e^{-2\pi i n \lambda} S^n(f-g) \right| \right|^2 + \right. \\ &+ 2 \left| \left| \sum_{n=1}^N e^{-2\pi i n \lambda} S^n(f-g) \right| \right| \left| \left| \sum_{n=1}^N e^{-2\pi i n \lambda} S^n f \right| \right| \right) d\lambda \leq \\ &< \left| f-g \right|^2 + 2 \left| f-g \right| \left| f \right|\end{aligned}$$

Because of this lemma, to prove the mutual singularity of dynamical systems given by α and β , it is sufficient to show that $\Lambda_{\alpha, f}$ and $\Lambda_{\beta, g}$ are mutually singular for f and g of the form

$$\begin{aligned}f(\gamma) &= \gamma^{M_0} (T\gamma)^{M_1} \dots (T^k \gamma)^{M_k} - C \\ g(\gamma) &= \gamma^{N_0} (T\gamma)^{N_1} \dots (T^k \gamma)^{N_k} - D\end{aligned}$$

($k=1, 2, \dots, M_i, N_i \in \mathbb{Z}$; C, D are constants such that

$$\int f d\mu_\alpha = \int g d\mu_\beta = 0)$$

Let ϕ and ψ are sequences such that

$$\begin{aligned}\phi(n) &= \exp 2\pi i (M_0 s_p(n) + M_1 s_p(n+1) + \dots + M_r s_p(n+k)) - C \\ \psi(n) &= \exp 2\pi i (N_0 s_q(n) + N_1 s_q(n+1) + \dots + N_r s_q(n+k)) - D\end{aligned}$$

Then $\Lambda_{\alpha, f}$ and $\Lambda_{\beta, g}$ are the spectral measures Λ_ϕ and Λ_ψ of the sequences ϕ and ψ , respectively, in the sense of [2]. Let

$$\phi_L(n) = e^{2\pi i E a \sum_{p=0}^L s_p\left(\left[\frac{n}{p}\right]\right)} A(n-p^L \left[\frac{n}{p}\right]) - C$$

$$\psi_L(n) = e^{2\pi i F b \sum_{q=0}^L s_q\left(\left[\frac{n}{q}\right]\right)} B(n-q^L \left[\frac{n}{q}\right]) - D$$

$$\text{where } E = \sum_{i=0}^k M_i, \quad F = \sum_{i=0}^k N_i \quad \text{and}$$

$$A(\ell) = \exp 2\pi i (M_0 s_p(\ell) + \dots + M_k s_p(\ell))$$

$$B(\ell) = \exp 2\pi i (N_0 s_q(\ell) + \dots + N_k s_q(\ell)) .$$

Then, it is easy to see that ϕ_L and ψ_L converge to ϕ and ψ , respectively, as $L \rightarrow \infty$ in the sense of Besicovich norm. Therefore $\Lambda_{\phi_L}(\Lambda_{\psi_L})$ converges to $\Lambda_{\phi}(\Lambda_{\psi})$ in the sense of total variance (cf. Lemma). Therefore our conclusion follows from the statement that Λ_{ϕ_L} and Λ_{ψ_L} are mutually singular. The last statement can be proved in the following way.

Case 1 : $E = F = 0$. Then ϕ_L and ψ_L are cyclic sequences whose cycles are coprime. Thus Λ_{ϕ_L} and Λ_{ψ_L} are mutually singular

Case 2 : $E \neq 0, F = 0$. Since

$$(*) \quad d\Lambda_{\phi_L+C}(\lambda) = \left| \frac{1}{p^L} \sum_{\ell=0}^{p^L-1} A(\ell) e^{-2\pi i \lambda \ell} \right|^2 d\Lambda_{\eta}(p^L \lambda)$$

where $\eta(n) = e^{2\pi i E a \sum_{p=0}^L s_p(n)}$ is known [2] to have a continuous spectral measure, Λ_{ϕ_L+C} is continuous. This implies that $C = 0$ and Λ_{ϕ_L} is continuous. Since Λ_{ψ_L} is discrete, Λ_{ϕ_L} and Λ_{ψ_L} are mutually singular.

Case 3 : $E = 0, F \neq 0$. Parallely as in case 2.

Case 4 : $E \neq 0$, $F \neq 0$. Then as was shown in case 2, $C = D = 0$.

Let η be as in case 2 and $\zeta(n) = e^{2\pi i Fb s_q(n)}$. It is known [3] that Λ_η and Λ_ζ are mutually singular. Since (*) and

$$d\Lambda_\eta(p^L_\lambda) = \left| \frac{1}{p^L} \sum_{\ell=0}^{p^L-1} e^{2\pi i (Ea s_p(\ell) - \ell\lambda)} \right|^{-2} d\Lambda_\eta(\lambda) ,$$

Λ_{ϕ_L} is absolutely continuous with respect to Λ_η .

Parallely, Λ_{ψ_L} is absolutely continuous with respect to Λ_ζ . Thus Λ_{ϕ_L} and Λ_{ψ_L} are mutually singular. Thus we proved

Theorem B.

The two dynamical systems given by α and β in §0 have mutually singular spectral measures.

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