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NORMAL EXTENSIONS DEFINED BY A BINOMIAL EQUATION

by

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Let F be a field and α a root of $x^n - a \in F[x]$. When is $F(\alpha)$ normal over F ? When is $F(\alpha)$ the splitting field of $x^n - a$ over F ? Consider the following

Examples (over the rational field $\mathbb{Q}$).

- (1) $x^3 - 2$ ($\mathbb{Q}(\alpha)$ is not normal for any root α).
- (2) $x^{12} - 1$ ($\mathbb{Q}(\alpha)$ is normal for any root α ; there exists a root β such that $\mathbb{Q}(\beta)$ is the splitting field).
- (3) $x^6 + 3$ ($\mathbb{Q}(\alpha)$ is the splitting field for every root α).
- (4) $x^6 + 27$ ($\mathbb{Q}(\alpha)$ is the splitting field for every root α).
- (5) $x^{42} - 21^7$ (If $\sqrt[6]{21}$ is a real 6-th root of 21 and ζ a primitive 42-th root of 1, then $\sqrt[6]{21}$ and $\zeta\sqrt[6]{21}$ are roots of the binomial; $\mathbb{Q}(\sqrt[6]{21})$ is not normal; $\mathbb{Q}(\zeta\sqrt[6]{21})$ is the splitting field).
- (6) $x^4 - 9$ ($\mathbb{Q}(\alpha)$ is normal for every root; for no root is $\mathbb{Q}(\alpha)$ the splitting field).

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Background. Darbi [1] found all irreducible, normal binomials over $\mathbb{Q}$. Example (3) is from his list. Mann and Véléz [3] extended this list to include all binomials sharing the property of examples (3) and (4) : $\mathbb{Q}(\alpha)$ is the splitting field for every root α . They call such binomials uniformly normal. In case the exponent is a prime power, Schinzel [4 ; Proposition 1] has determined for an arbitrary field those binomials which are products of normal polynomials (thus including irreducible and uniformly normal binomials).

In this exposé we consider a binomial $x^n - a \in \mathbb{Q}[x]$ satisfying the property (shared by examples (2) - (5)) :

There exists a root α of $x^n - a$ such that $\mathbb{Q}(\alpha)$ is the splitting field of $x^n - a$.

Such a binomial is called partially normal and the special root α a generating root.

Results. In the theorem below we list all partially normal binomials (over $\mathbb{Q}$). Without loss of generality, we consider only those binomials $x^n - a$ with a an integer. For positive integer b , let $s(b)$ be the largest square integer dividing b and $f(b) = b/s(b)$, the square-free part of b . If c is a square-free integer ($s(c) = 1$), we denote by $\ell(c)$ the number of prime factors of the form $4k+3$. Let $\mathbb{N}$ denote the natural numbers.

THEOREM. The partially normal binomials over $\mathbb{Q}$ are

- (A) $x^m \pm b^m$, $m, b \in \mathbb{N}$.
- (B) $x^{2m} \pm b^m$, $m, b \in \mathbb{N}$ such that $f(b) > 1$ and, in case sign is negative and
 $m = 2m'$ with m' odd, then $f(b) \mid m'$ and $\ell(f(b))$ odd.
- (C) $x^{4m} - b^m$, $m, b \in \mathbb{N}$, m odd, $f(b) \mid m$, and $\ell(f(b))$ odd.
- (D) $x^{4m} + b^m$, $m, b \in \mathbb{N}$, $f(b) > 1$, and
 (a) if m is odd, then $f(b) \mid m$ and $\ell(f(b))$ even,

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(b) if m is even, then $f(b) \mid (m/2)$.

(E) $x^{6m} - b^m$, $m, b \in \mathbb{N}$, $(m, 6) = 1$, $3 \mid f(b)$, $f(b) \nmid 3m$, and $\ell(f(b))$ even.

(F) $x^{6m} + b^m$, $m, b \in \mathbb{N}$, $(m, 3) = 1$, $4 \nmid m$, $3 \mid f(b)$ and

(a) if m is odd, then $f(b) \mid 3m$ and $\ell(f(b))$ even,

(b) if m is even, then $2 \mid f(b)$ and $f(b) \nmid 6m$.

Moreover, the Galois group of a partially normal binomial is abelian iff it falls under case (A) or (B) .

Remarks. A more detailed version of theorem (with proof) together with some consequences of the notion of partially normal binomial over real fields can be found in [2] .

An investigation of normal extensions (of a field F) defined by a binomial equation might start with γ algebraic over F , $F(\gamma)$ normal over F , and $\gamma^m \in F$ for some m . From these assumptions, if n is the smallest positive integer such that $\gamma^n \in F$, one can show that a primitive n -th root of 1 is in $F(\gamma)$. It then follows that $x^n - \gamma^n$ is partially normal over F . The converse is not true : if $x^n - a$ is partially normal over F with generating root α , then it may be the case that $\alpha^t \in F$ for some $t < n$. [2, p. 21] .

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BIBLIOGRAPHY

- [1] G. DARBI.- "Sulla riducibilità delle equazioni algebriche". Annali di Mat. pur. e appl., Ser. 4, 4 (1926), 185-208.
- [2] D. GAY, A. Mc. DANIEL, W. VÉLEZ.- "Partially normal radical extensions of the rationals". Battelle Mathematics Report N° 97 (1975).
- [3] H. MANN, W. VÉLEZ.- "On normal radical extensions of the rationals". Lin. and Multilin. Alg., 3 (1975), 73-80.
- [4] A. SCHINZEL.- "Les résidus de puissances et les congruences exponentielles". Rapport, Journées Arithmétiques, Caen (1976).

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