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SPECTRUM OF MEASURABLE FLOWS

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Let  $\{T_t\}$  be a measurable flow (one parameter group of measure preserving transformations satisfying standard measurability condition) on the (Lebesgue) probability space  $(X, \mathcal{E}, \mu)$ . Denote by  $H$  the generator of the group  $U_t = e^{-itH}$  of induced unitaries and let  $\text{sp}(H)$  be its spectrum.

Let  $P(x)$  be the period of  $x \in X$  (if  $x$  is aperiodic,  $P(x) = \infty$ ). We have the

Proposition.

If  $\|P\|_\infty = \infty$ ,  $\text{sp}(H) = \mathbb{R}$

Corollary. If  $\{T_t\}$  has an aperiodic component and in particular, if  $T^t$  is ergodic and  $T^t = \mathbb{1}$  only if  $t = 0$ , then  $\text{sp}(H) = \mathbb{R}$ .

Proof: We first give an argument which only establishes the latter half of the corollary. The argument, which is a generalization of the standard proof that the discrete spectrum of an ergodic transformation forms a group, may be applied to prove the appropriate results for automorphisms of  $C^*$ -algebras.

Let  $\text{sp } \psi$  denote the support of  $\psi \in L^2(X, \mu)$  in the spectral representation determined by  $H$ . Then

Lemma 1. If  $\psi, \phi \in L^\infty(X, \mu)$  then  $\text{sp}(\psi\phi) \subset \text{sp}\psi + \text{sp}\phi$ .

Proof: For  $\psi \in L^2(X, \mu)$  let  

$$\hat{\psi}^\tau(E) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{iEt} e^{-\frac{\tau}{2}|t|^2} (U_t \psi) dt \quad (\text{Bochner Integral})$$

By the spectral theorem :

$$\hat{\psi}^\tau(E) = \delta_E^\tau(H)\psi$$

where

$$\delta_E^\tau(x) = \frac{1}{\sqrt{2\pi\tau}} e^{-(x-E)^2/2\tau}.$$

We claim that if  $\lim_{\tau \rightarrow 0} \|\hat{\psi}^\tau(E)\|_2 = 0$  for all  $E$  in an open set  $\Theta$ ,

then  $\text{sp}(\psi)$  and  $\Theta$  are disjoint. [By the spectral theorem :

$$\begin{aligned} \int_{\Theta'} dE \|\hat{\psi}^\tau(E)\|_2^2 &= \frac{1}{2\pi\tau} \int_{\Theta'} dE \int d\mu_\psi(E') e^{-(E'-E)^2/2\tau} \\ &= \frac{1}{2\pi\tau} \int d\mu_\psi(E') \int_{\Theta'} dE e^{-\frac{(E'-E)^2}{\tau}} \sim \frac{1}{\sqrt{4\pi\tau}} \mu_\psi(\Theta'). \end{aligned}$$

Therefore if  $\|\hat{\psi}^\tau(E)\|_2 \rightarrow 0$  uniformly on  $\Theta'$ ,  $\mu_\psi(\Theta') = 0$ .

Hence, by Egoroff's theorem,  $\mu_\psi(\Theta) = 0$  (here  $\mu_\psi$  is the spectral measure belonging to  $\psi$ ) ] .

$$\text{Now } (\widehat{\psi\phi})^\tau(E_0) = \hat{\psi}^{\tau/2} * \hat{\phi}^{\tau/2}(E_0) = \int \hat{\psi}^{\tau/2}(E_0-E) \hat{\phi}^{\tau/2}(E) dE,$$

and

$$\|(\widehat{\psi\phi})^\tau(E_0)\|_2 \leq \int \|\hat{\psi}^{\tau/2}(E_0-E) \hat{\phi}^{\tau/2}(E)\|_2 dE \leq \frac{\text{const}}{\tau^2} e^{\delta^2/2\tau} \int e^{-\text{const}'E^2} dE \xrightarrow{\tau \rightarrow 0} 0$$

if  $E_0 \notin \text{sp}\psi + \text{sp}\phi$ , where

$$\delta = \inf_E \text{dist}(E, \text{sp}\phi) \vee \text{dist}(E_0 - E, \text{sp}\psi) = \frac{1}{2} \text{dist}(E_0, \text{sp}\psi + \text{sp}\phi).$$

In the above we have used these estimates :

$$\|\hat{\psi}^\tau(E)\|_\infty \leq \int \left(\frac{1}{2\pi} e^{-\tau|t|^2/2} dt\right) \|\psi\|_\infty = \frac{1}{\sqrt{2\pi\tau}} \|\psi\|_\infty$$

$$\|\hat{\psi}^\tau(E)\|_2 \leq \frac{1}{\sqrt{2\pi\tau}} \|\psi\|_2 e^{-[\text{dist}(E, \text{sp}\psi)]^2/2\tau}$$

(spectral theorem) and

$$\|\psi\phi\|_2 \leq \|\psi\|_\infty \|\phi\|_\infty \|\psi\|_2 \|\phi\|_2$$

Theorem.

Let  $T_t$  be ergodic. Then  $sp(H)$  forms a group.

Proof. Let  $C\psi(x) = \bar{\psi}(x)$ . Then  $C U_t = U_t C$ , i.e.

$CH = -HC$ . Therefore if  $\lambda \in sp(H)$ ,  $-\lambda \in sp(H)$ . Let

$$E_1 \in \sigma(H), \quad sp \psi \left( E_1 - \frac{1}{n}, E_1 + \frac{1}{n} \right), \|\psi\|_2 = 1$$

$$E_2 \in \sigma(H), \quad sp \phi \left( E_2 - \frac{1}{n}, E_2 + \frac{1}{n} \right), \|\phi\|_2 = 1.$$

By ergodicity,

$$\frac{1}{T} \int_0^T dt \int d\mu |\psi| |\phi(T_t x)| \xrightarrow{T \rightarrow \infty} \left( \int d\mu |\psi| \right) \left( \int d\mu |\phi| \right) > 0,$$

so we may assume that  $\psi\phi \neq 0$ . By lemma 1,  $\psi\phi$  is therefore an approximate eigenvector of  $H$  of eigenvalue  $E_1 + E_2$ , so that  $E_1 + E_2 \in sp(H)$ .

Now if  $sp(H)$  is discrete, it must be of the form

$$\{n E_0 \mid n = \dots, -2, -1, 0, 1, 2, \dots\}$$

and  $T_\tau = \mathbf{1}$  for  $\tau = 2\pi/E_0$ . If  $sp(H)$  is not discrete, it must be  $\mathbb{R}$ , since it is closed.

The proposition itself should follow from the decomposition of  $T_t$  into its ergodic components. We instead prove it directly, using an entirely different approach.

Assume first that  $T_t$  has a nontrivial aperiodic component. This may be represented as a special flow  $[ ]$ , i.e. as a flow built under a function  $f \geq \delta > 0$  and aperiodic automorphism  $(X_0, \mu_0, T_0)$ .  $(X, \mu)$  is identified with the "region under the function  $f$  on  $(X_0, \mu_0)$ " with measure given by  $\mu \times$  Lebesgue measure.

For  $t > 0$ ,  $T_t(x,s) = (x,s+t)$  for  $s+t \leq f(x)$  and

$$T_t(x,s) = (T_0 x, s+t-f(x)) \text{ otherwise.}$$

Since  $T_0$  is aperiodic, for any  $n > 0$  there exists a set  $A$  with  $\mu_0(A) > 0$  and  $A, T_0(A), \dots, T_0^{n-1}(A)$  disjoint [ ]. Therefore there exist in  $X$  "rectangular tubes" of arbitrarily large length in which  $T_t$  induces a uniform translation. Consequently we may construct approximate eigenvectors (approximate plane waves  $e^{its}$ ) corresponding to any  $\lambda \in \mathbb{R}$ . Finally, if  $T_t$  has no aperiodic component, we may construct approximate eigenfunctions as above by employing the representation of  $T_t$ , as the flow built under the function  $\pi(x)$  and the identity automorphism. Here  $\pi(x)$  is the period of  $x \in X_0$ . Since  $\|P(x)\|_\infty = \infty$ , we obviously have "tubes" of arbitrarily large length on which the motion is a uniform translation.

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