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The algebraic approach to the study of Statistical Mechanics suggests an expansion of classical ergodic theory to a noncommutative ergodic theory. We indicate here how one can proceed in this spirit to generalize the classical concept of a Kolmogorov-Sinai flow.

Firstly we recall that a classical K-flow is constituted by: a probability space (Ω,Σ,μ) , a measurable group $\{T(t)\,|\,t\in R\}$ of measure preserving transformations of (Ω,Σ,μ) , and a partition $\xi\subset\Sigma$ such that:

(i) $\xi \subseteq T(t)[\xi]$ $\forall t \geq 0$; (ii) $\bigwedge_{t\in R} T(t)[\xi] = \hat{0}$; (iii) $\bigvee_{t\in R} T(t)[\xi] = \hat{1}$.

Secondly we construct from these elements: a separable Hilbert space $\mathfrak{H} = L^2(\Omega,\mu)$, a maximal abelian von Neumann algebra $\mathcal{R} = L^\infty(\Omega,\mu)$ acting on $\mathfrak{H}$, a cyclic and separating vector $\Phi(\omega) = 1$ $\forall\omega\in\Omega$ for $\mathcal{R}$ in $\mathfrak{H}$, a faithful normal state on $\mathcal{R}$ $\phi: f\in\mathcal{R} \mapsto \langle\phi;f\rangle = (f\Phi,\Phi) = \int f(\omega)\,d\mu(\omega)$, a continuous group of automorphisms of $\mathcal{R}$ $\alpha(t): f\in\mathcal{R} \to \alpha(t)[f] = f\circ T(t)$, and a von Neumann subalgebra $\mathcal{A}$ of $\mathcal{R}$ $\mathcal{A} = \{\chi_\Delta\,|\,\Delta\in\xi\}''$ with the properties:

(i) $\mathcal{A} \subseteq \alpha(t)[\mathcal{A}]$ $\forall t \geq 0$; (ii) $\bigcap_{t\in R}\alpha(t)[\mathcal{A}] = \mathbb{C}I$; (iii) $\bigvee_{t\in R}\alpha(t)[\mathcal{A}] = \mathcal{R}$.

Thirdly we notice that a standard representation theorem allows to get back to the original definition from $(\mathcal{R},\phi,\alpha,\mathcal{A})$ where: $\mathcal{R}$ is an <u>abelian</u> von Neumann algebra acting on a separable Hilbert space $\mathfrak{H}$, ϕ is a faithful normal state on $\mathcal{R}$, $\alpha: R \to \mathrm{Aut}(\mathcal{R},\phi)$, and $\mathcal{A}$ is a completely selfrefining , generating von Neumann subalgebra of $\mathcal{R}$.

Fourthly the generalisation to the noncommutative domain now consists exactly in taking the above as an alternative definition of a classical K-flow, and in dropping from this definition the condition that $\mathfrak{N}$ be abelian. For reasons which would be too long to make explicit here we also impose that $\mathcal{A}$ be stable under the modular automorphism group $\{\sigma_\phi(t) \mid t \in R\}$ canonically associated to ϕ, and that every maximal abelian subalgebra $\mathfrak{X}$ of the centralizer $\mathfrak{N}_\phi$ of $\mathfrak{N}$ be already maximal abelian in $\mathfrak{N}$. (Notice that both of the last two conditions are redundant when $\mathfrak{N}$ is abelian since $\sigma_\phi(t) = id \ \forall \ t \in R$ in this case.)

We now can emphasize [1] that algebraic proofs can be given to several theorems which are well-known in the classical case, and which thus do generalize to the new situation just defined. For instance the system $(\mathfrak{N}, \phi, \alpha, \mathcal{A})$ is ergodic (i.e. $N \in \mathfrak{N}$ and $\alpha(t)[N] = N \ \forall \ t \in R \Rightarrow N = \lambda I$ with $\lambda \in C$); it is mixing (i.e. $<\phi; N\alpha(t)[M]> \rightarrow <\phi;N><\phi;M>$ as $t \rightarrow \pm\infty$ for all $N, M \in \mathfrak{N}$); it has Lebesgue spectrum (i.e. $\alpha(t)$ is spatial and the generator H of the corresponding unitary group on $\mathfrak{H}$ has the property $Sp(H) = Sp_d(H) \cup Sp_{ac}(H)$ with $Sp_d(H) = \{0\}$ simple and $Sp_{ac}(H) = R$ has countable muliplicity). Furthermore a noncommutative entropy can be defined [2] which is strictly positive for all such systems.

We next remark that the generalization is genuine in the sense that, in addition to the classical case where $\mathfrak{N}$ is abelian, there exist [1,3,4] K-flows where $\mathfrak{N}$ is of type II_1, $III_\lambda (0 < \lambda < 1)$, or III_1.

Finally a link has been established [3] between certain quantum transport phenomena, governed by an evolution equation of the diffusion type, and the generalized K-flows presented here.

REFERENCES

[1] Nonabelian Special K-flows
Journ. Funct. Analysis 19 (1975) 1-12. Zblatt Math.1975 # 501130

[2] Positivity of the K-entropy on Non-Abelian K-Flows
Z. Wahrscheinlichkeitstheorie verw. Gebiete 29 (1974) 241 - 252.

[3] The Minimal K-Flow associated to a Quantum Diffusion Process
in Physical Reality and Mathematical Description, Enz Mehra, Eds., D. Reidel Publishing Company, Dordrecht-Holland, 1974, 477 - 493.

[4] Generalized K-Flows, Commun. math.Phys. 49 (1976) 191-215.

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