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ALGEBRAIC K-FLOWS

Gérard G. Emch

The algebraic approach to the study of Statistical Mechanics suggests an expansion of classical ergodic theory to a noncommutative ergodic theory. We indicate here how one can proceed in this spirit to generalize the classical concept of a Kolmogorov-Sinai flow.

Firstly we recall that a classical K-flow is constituted by: a probability space (Ω, Σ, μ) , a measurable group $\{T(t) | t \in \mathbb{R}\}$ of measure preserving transformations of (Ω, Σ, μ) , and a partition $\xi \subset \Sigma$ such that:

$$(i) \xi \subseteq T(t)[\xi] \quad \forall t \geq 0; \quad (ii) \bigcap_{t \in \mathbb{R}} T(t)[\xi] = \widehat{0}; \quad (iii) \bigvee_{t \in \mathbb{R}} T(t)[\xi] = \widehat{1}.$$

Secondly we construct from these elements: a separable Hilbert space $\mathfrak{H} = L^2(\Omega, \mu)$, a maximal abelian von Neumann algebra $\mathfrak{K} = L^\infty(\Omega, \mu)$ acting on \mathfrak{H} , a cyclic and separating vector $\Phi(\omega) = 1 \quad \forall \omega \in \Omega$ for \mathfrak{K} in \mathfrak{H} , a faithful normal state on \mathfrak{K} $\phi: f \in \mathfrak{K} \mapsto \langle \phi, f \rangle = \int f(\omega) d\mu(\omega)$, a continuous group of automorphisms of \mathfrak{K}

$\alpha(t): f \in \mathfrak{K} \mapsto \alpha(t)[f] = f \circ T(t)$, and a von Neumann subalgebra \mathcal{A} of \mathfrak{K} $\mathcal{A} = \{\chi_\Delta | \Delta \in \xi\}$ with the properties:

$$(i) \mathcal{A} \subseteq \alpha(t)[\mathcal{A}] \quad \forall t \geq 0; \quad (ii) \bigcap_{t \in \mathbb{R}} \alpha(t)[\mathcal{A}] = \text{CI}; \quad (iii) \bigvee_{t \in \mathbb{R}} \alpha(t)[\mathcal{A}] = \mathfrak{K}.$$

Thirdly we notice that a standard representation theorem allows to get back to the original definition from $(\mathfrak{K}, \phi, \alpha, \mathcal{A})$ where: \mathfrak{K} is an abelian von Neumann algebra acting on a separable Hilbert space \mathfrak{H} , ϕ is a faithful normal state on \mathfrak{K} , $\alpha: \mathbb{R} \rightarrow \text{Aut}(\mathfrak{K}, \phi)$, and \mathcal{A} is a completely selfrefining, generating von Neumann subalgebra of \mathfrak{K} .

Fourthly the generalisation to the noncommutative domain now consists exactly in taking the above as an alternative definition of a classical K-flow, and in dropping from this definition the condition that \mathcal{N} be abelian. For reasons which would be too long to make explicit here we also impose that \mathcal{A} be stable under the modular automorphism group $\{\sigma_\phi(t) | t \in \mathbb{R}\}$ canonically associated to ϕ , and that every maximal abelian subalgebra \mathcal{K} of the centralizer \mathcal{N}_ϕ of \mathcal{N} be already maximal abelian in \mathcal{N} . (Notice that both of the last two conditions are redundant when \mathcal{N} is abelian since $\sigma_\phi(t) = \text{id} \forall t \in \mathbb{R}$ in this case.)

We now can emphasize [1] that algebraic proofs can be given to several theorems which are well-known in the classical case, and which thus do generalize to the new situation just defined. For instance the system $(\mathcal{N}, \phi, \alpha, \mathcal{A})$ is ergodic (i.e. $N \in \mathcal{N}$ and $\alpha(t)[N] = N \forall t \in \mathbb{R} \Rightarrow N = \lambda I$ with $\lambda \in \mathbb{C}$); it is mixing (i.e. $\langle \phi; N \alpha(t)[M] \rangle \rightarrow \langle \phi; N \rangle \langle \phi; M \rangle$ as $t \rightarrow \pm\infty$ for all $N, M \in \mathcal{N}$); it has Lebesgue spectrum (i.e. $\alpha(t)$ is spatial and the generator H of the corresponding unitary group on \mathfrak{H} has the property $\text{Sp}(H) = \text{Sp}_d(H) \cup \text{Sp}_{ac}(H)$ with $\text{Sp}_d(H) = \{0\}$ simple and $\text{Sp}_{ac}(H) = \mathbb{R}$ has countable multiplicity). Furthermore a noncommutative entropy can be defined [2] which is strictly positive for all such systems.

We next remark that the generalization is genuine in the sense that, in addition to the classical case where \mathcal{N} is abelian, there exist [1,3,4] K-flows where \mathcal{N} is of type II_1 , III_λ ($0 < \lambda < 1$), or III_1 .

Finally a link has been established [3] between certain quantum transport phenomena, governed by an evolution equation of the diffusion type, and the generalized K-flows presented here.

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