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A COMMON GENERALIZATION OF TOPOLOGICAL AND MEASURE-THEORETIC ENTROPY

Günther Palm

Nowadays ergodic theory is split into two branches: measure-theoretic and topological, according to the methods used.
In both branches there are similar results proved using similar ideas. Therefore it is natural to look for a common generalization.

For theorems connecting spectral and mixing properties of dynamical systems Nagel [2],[3],[4] has found an approriate generalization in terms of Banach lattices: an abstract dynamical system is a triple (E,u,T), where E is a Banach lattice with quasi-interior point $u \in E_+$ and $T:E \longrightarrow E$ is a lattice homeomorphism satisfying Tu=u (this definition is slightly different from that given in [2]).

For theorems concerning entropy and related questions, other mathematical structures are used: If one looks into the entropy sections of Walters' book [8], for example, the measure-theoretical and topological proofs of many analogous theorems look very similar and these proofs are based on lattice methods. Therefore I have defined entropy for a dynamical lattice (see definition 1.1.).

This definition has two advantages:

1) Given an abstract dynamical system (E,u,T), the lattice of all closed ideals in E yields a dynamical lattice (see 1.3.), whose entropy reduces to the usual entropy in both the measure-theoretic and the topological case (see 1.4.).

2) In this definition of entropy it is necessary to define the entropy for not necessarily disjoint covers, even in the measure-theoretic case. But this fact allows an easy proof of Goodwyn's theorem [1] by means of a generalized version of the Kolmogoroff-Sinai theorem (see 3.4.).

In the following I want to give the basic definitions and theorems for the entropy of dynamical lattices and to sketch the proof of Goodwyn's theorem.

1. Dynamical Lattices.

1.1. Definition.

A dynamical lattice is a triple (V,m,f), where

V is a distributive lattice with 0 and 1,

$m:V\longrightarrow\mathbb{R}_+$ satisfies $m(0)=0$ and:

$m(a)=0 \Rightarrow m(a\vee b)=m(b)$ for every $a,b\in V$,

$f:V\longrightarrow V$ satisfies $f(0)=0$, $f(1)=1$ and:

$m(a)=0 \Rightarrow m(f(a))=0$ for every $a\in V$.

1.2. Definition.

Two dynamical lattices (V,m,f) and (V',m',f') are called isomorphic, if there is a lattice isomorphism $\phi:V\longrightarrow V'$ satisfying $\phi\cdot f = f'\cdot\phi$ and $m'\cdot\phi = m$.

1.3. Definition.

Let (E,u,T) be an abstract dynamical system. Let V be the lattice of all closed (lattice-)ideals in E (see [6]),

$$m:\begin{cases} V\longrightarrow\mathbb{R}_+ \\ I\longrightarrow\sup\{\|x\| : x\in I\wedge[0;u]\}\end{cases},\quad f:\begin{cases} V\longrightarrow V \\ I\longmapsto\langle T(I)\rangle\end{cases},$$

where $\langle A \rangle$ denotes the closed ideal generated by A . Then (V,m,f) is called the <u>dynamical lattice of closed ideals</u> associated to (E,u,T) .

By the entropy of (E,u,T) we mean the entropy of the associated dynamical lattice of closed ideals.

<u>1.4.</u>

In the <u>topological case</u> we have a <u>topological dynamical system</u> (X,ϕ), i.e. a compact Hausdorff space X and a continuous mapping $\phi:X\longrightarrow X$. Here we set $E:=C(X)$, $u=1$ and $T(f):=f\circ\phi$. For this abstract dynamical system we get (using 1.3)

$$V = \{\text{open sets in } X\} \ , \ m(a) = m_1(a) := \begin{cases} 0 & \text{if } a=0 \\ 1 & \text{if } a\neq 0 \end{cases} \quad \text{and } f=\phi^{-1} \ .$$

In the <u>measure-theoretic case</u> we have a <u>dynamical system</u> (X,Σ,μ,ϕ) , i.e. a probability space (X,Σ,μ) and a measurable, measure-preserving mapping $\phi:X\longrightarrow X$. Here we set $E:L^1(X,\Sigma,\mu)$,and again $u=1, T(f):=f\circ\phi$. For this abstract dynamical system we get V isomorphic to the measure algebra $\Sigma/\mathcal{N}$ ($\mathcal{N}$ denoting the μ-nullsets), $m=\mu$ and $f=\phi^{-1}$.

<u>2. Entropy</u>

<u>2.1. Definition</u>.

Let (V,m,f) be a dynamical lattice.

1) A finite subset α of V is called a <u>cover</u>, if $\sup \alpha=1$.

2) The set $\tilde{V}$ of all covers is ordered by:

$\alpha\leq\beta$ (β is a <u>refinement</u> of α) if and only if for every $b\in\beta$ there is an $a\in\alpha$ such that $b\leq a$.

3) $\alpha\vee\beta := \{a\wedge b : a\in\alpha, b\in\beta\}$ and $\alpha^n := \bigvee_{i=0}^{n-1} f^i(\alpha)$.

4) Let α be a cover and $k:=\sum_{a\in\alpha} m(a)$, then we set $h^*(\alpha):=-\sum_{a\in\alpha} \frac{m(a)}{k} \log \frac{m(a)}{k}$.

5) $\hat{h}(\alpha) := \sup\{h^*(\beta) : \beta \geqslant \alpha \ , N(\beta) \leq N(\alpha)\}$, $N(\alpha)$ denoting the number of elements $a \in \alpha$ such that $m(a) \neq 0$.

6) $h(\alpha) := \inf\{ \sum_{i=1}^{n} \hat{h}(\beta_i) : \bigvee_{i=1}^{n} \beta_i \geq \alpha \ , n \in \mathbb{N}\}$.

7) $h(f,\alpha) := \underline{\lim}\ h(\alpha^n)/n$, $H(f,\alpha) := \overline{\lim}\ h(\alpha^n)/n$.

8) $h(V,m,f) := \sup\{h(f,\alpha) : \alpha \in \tilde{V}\}$, $H(V,m,f) := \sup\{H(f,\alpha) : \alpha \in \tilde{V}\}$.

$h(V,m,f)$ is called the <u>entropy</u> of (V,m,f) .

2.2. Remarks.

a) It can be proved; that in many cases $h(f,\alpha) = H(f,\alpha)$ holds for every cover α [5] .

b) Step 5 of the definition should be explained:
In the measure-theoretic case we want to get the measure entropy, therefore it should be sufficient to consider <u>disjoint</u> covers Now if V is a Boolean algebra and α any cover, there is a disjoint refinement β of α with $N(\beta) \leq N(\alpha)$, but if α is already disjoint, then α is the only such refinement. Therefore in step 6 we have

$$h(\alpha) = \inf\{ \sum_{i=1}^{n} \hat{h}(\beta_i) : \bigvee_{i=1}^{n} \beta_i \geq \alpha, \ \beta_i \text{ disjoint}, \ n \in \mathbb{N}\} \text{ and}$$

$$\hat{h}(\beta) = h^*(\beta) \quad \text{for disjoint } \beta .$$

c) In this general context the entropy still has many of the well-known properties of the usual entropies:

2.3. Theorem [5] .

a) If (V,m,f) *and* (V',m',f') *are isomorphic, they have the same entropy.*

b) Let (V,m,f) *be a dynamical lattice, where* f *is a lattice isomorphism such that* $m \circ f = m$ *, then*

$h(V,m,f) = H(V,m,f)$ *and* $h(V,m,f^k) = |k|\cdot h(V,m,f)$ *for* $k \in \mathbb{Z}$

c) *In the topological case (see* 1.4.) $h(V,m,f)$ *is equal to the topological entropy.*

d) *In the measure-theoretic case* $h(V,m,f)$ *is equal to the measure entropy.*

3. Generators.

Let me define pseudometrics on V and $\tilde{V}$:

3.1. Definition.

a) Given $a,b \in V$ let $\delta(a,b) := \inf\{m(d) : d \vee a = d \vee b\}$.

b) Given $\alpha,\beta \in \tilde{V}$ with $|\alpha| \le |\beta|$ (say) let $d(\alpha,\beta) = d(\beta,\alpha) :=$

$= \inf\{ \sum_{a \in \alpha} \delta(a,\pi(a)) + \sum_{b \notin \pi(\alpha)} m(b) : \pi : \alpha \longrightarrow \beta$ injective$\}$.

3.2. Definition.

Given two covers α,β I shall write $\alpha \underset{\varepsilon}{\leq} \beta$, if there is a cover α' satisfying $d(\alpha,\alpha') < \varepsilon$ and $\alpha' \le \beta$.

3.3. Definition.

A cover β is called a generator, if for every cover α and every $\varepsilon > 0$ there is $n \in \mathbb{N}$ such that $\alpha \underset{\varepsilon}{\leq} \beta^n$. A subset W of V is called generating, if for every cover α and every $\varepsilon > 0$ there is a cover $\beta \subseteq W$ such that $\alpha \underset{\varepsilon}{\leq} \beta$.

With these notions we can prove a generalized version of the well-known Kolmogoroff-Sinai theorem (along the lines of [7], see especially Lemma 5.8) [5].

3.4. Theorem.

Let (V,m,f) *be a dynamical lattice,* V *a Boolean algebra,* m *monotone* $(a \le b \Rightarrow m(a) \le m(b))$ *and subadditive*

$(m(a\ b) \leq m(a)+m(b))$ *and* $m \circ f=m$, *then*

a) $h(f,\beta) = h(V,m,f)$ *for every generator* β .

b) $h(V,m,f) = \sup\{h(f,\beta) : \beta \in \tilde{V}, \beta \subseteq W\}$ *for every generating* $W \subseteq V$.

4. Goodwyn's theorem.

4.1.

Finally I will sketch a new proof of Goodwyn's theorem [1]: Given a topological dynamical system (X,ϕ) and a ϕ-invariant regular Borel measure μ on X, the topological entropy h_t of ϕ is $\geq$ the measure entropy h_μ of ϕ with respect to μ .

According to 2.3. the topological entropy h_t is $h(V,m_1,f)$, where $V = \{\text{open sets in } X\}$ and $f = \phi^{-1}$, and the measure entropy is $h(\Sigma,\mu,f)$ where Σ denotes the σ-algebra of Borel-sets. Since μ is regular, V is a generating subset of Σ . Therefore we have (3.4.b):

(*) $h(\Sigma,\mu,f)=\sup\{h(f,\alpha):\alpha \in \tilde{\Sigma},\ \alpha \subseteq V\}=\sup\{h(f,\alpha):\alpha \text{ open cover of } X\}$.

If α is an open cover of X , clearly $h^*(\alpha)$ computed for (V,m_1,f) is $\log N(\alpha)$, which is $\geq h^*(\alpha)$ computed for (Σ,μ,f) .

Therefore $h(f,\alpha)$ computed for (V,m_1,f) is $\geq h(f,\alpha)$ computed for (Σ,μ,f) (according to definition 2.1.).

So we can continue (*):

$$h(\Sigma,\mu,f)=\sup\{h(f,\alpha):\alpha \in \tilde{\Sigma},\ \alpha \subseteq V\} \leq \sup\{h(f,\alpha):\alpha \in \tilde{V}\} = h(V,m_1,f) .$$

With the same ideas the following generalization of Goodwyn's theorem can be proved [5]:

4.2. Theorem.

Let X *be a compact Hausdorff space and* (E,u,T) *an abstract dynamical system satisfying:*

a) C(X) *is a dense* T*-invariant sublattice of* E .

b) *The norm of* E *is order-continuous.*

c) u *is the function* $1 \in C(X)$.

d) T *is an isometry.*

Then $T|_{C(X)}$ *corresponds to a homeomorphism* $\phi : X \longrightarrow X$ *by means of* $Tf = f \circ \phi$ *, and the topological entropy of* ϕ *is* $\geq$ *the entropy of* (E,u,T) .

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