PIERRE MICHEL

Minimal sets generated by a substitution of non constant length

Astérisque, tome 40 (1976), p. 143-146
<http://www.numdam.org/item?id=AST_1976__40__143_0>

Société Mathématique de France
Astérisque 40 (1976) p.143-146

MINIMAL SETS GENERATED BY A SUBSTITUTION OF NON CONSTANT LENGTH

Pierre MICHEL

(Rennes)

Let P denote the alphabet $\{0,1\}$ and let

$$\Omega = P^{\mathbb{Z}} = \{\omega = \ldots\omega_{-1}\ \omega_0\ \omega_1\ \omega_2\ \ldots \mid \omega_i \in P\}$$

A substitution θ is a map from P to $\bigcup_{n\geqslant 2} P^n$. If card $\theta(0)$ = card $\theta(1)$ the substitution is said of constant length. If not, it is said of non constant length.

For suitable i and j we can generate (1) a bisequence by the following:

$$\omega[-\text{card }\theta^{2n}(i),\ \text{card }\theta^{2n}(i)+\text{card }\theta^{2n}(j)] = \theta^{2n}(i)\ \theta^{2n}(j)\ ,\forall n \in \mathbb{N}$$

where $\omega(a,k)$ is the k-block of ω which begin at ω_a.

Under a condition of non-degeneracy, the dynamical system (O_ω,T) where O_ω is the orbit closure of ω, and T the shift, is minimal (1) and strictly ergodic (2).

We can prove that the dynamical system associated with the substitution θ_1 defined by

$$\begin{cases} \theta_1(0) = 0^{n+1-p}\ 1\ 0^p \\ \theta_1(1) = 1\ \ 0^n \end{cases} \qquad \text{for } 0 \leqslant p \leqslant n\ ,$$

has a purely discrete spectrum. (i^k, $i \in P$, $k \in \mathbb{N}$ means $\underbrace{i\ i \ldots i}_{n \text{ times}}$).

(1) GOTTSCHALK. Substitution minimal sets. Trans. Amer. Math. Soc. 109 (1963)

(2) MICHEL. Stricte ergodicité d'ensembles minimaux de substitution. CRAS Paris, 278, (1974).

For this, we introduce the notion of "coïncidence-value of the substitution θ" by the following :

$$\text{if} \quad \theta^{\infty}(0) = \lim_{n\to\infty} \theta^{2n}(0) = \omega_1^0\,\omega_2^0 \ldots \omega_n^0 \ldots$$

$$\theta^{\infty}(1) = \lim_{n\to\infty} \theta^{2n}(1) = \omega_1^1\,\omega_2^1 \ldots \omega_n^1 \ldots ,$$

we say that n is a "coïncidence value of θ" if and only if

$$\omega_n^0 = \omega_n^1 .$$

If we denote the density of the coïncidence values of θ in $\mathbb{N}$ by $\delta_c(\theta)$, we can prove that $\delta_c(\theta_1)$ is equal to one.

Conversely if we consider the substitution θ_2 :

$$\begin{cases} \theta_2(0) = 0\ 1 \\ \theta_2(1) = 1\ 1\ 0\ 0 \end{cases}$$

we have $\delta_c(\theta_2) = \frac{1}{3}$ and we can prove that there is a continuous part in the spectrum of θ_2.

Moreover, if we consider the two closely related substitutions θ_2' and θ_2'' defined by

$$\theta_2'(0) = \theta_2''(0) = 0\ 1$$

and

$$\begin{cases} \theta_2'(1) = 1\ 0\ 1\ 0 \\ \theta_2''(1) = 1\ 0\ 0\ 1 \end{cases}$$

we can prove that θ_2' and θ_2'' have purely discrete spectra and that

$$\delta_c(\theta_2') = \delta_c(\theta_2'') = 1 .$$

More generally, we conjecture that the substitution θ_n :

$$\theta_n(0) = 0^n\ 1^n = \underbrace{0\ldots0}_{n \text{ times}}\ \underbrace{11\ldots1}_{n \text{ times}}$$

$$\theta_n(1) = 1^{2n}\ 0^{2n} = \underbrace{11\ldots1}_{2n \text{ times}}\ \underbrace{00\ldots0}_{2n \text{ times}} ,$$

has :

1) a purely discrete spectrum if n is even (then $\delta_c=1$)

2) a mixed spectrum (partly continuous and partly discrete) if n is odd (then $\delta_c=\frac{1}{3}$).

At last, if we consider the constant-length case (3), we know that the so-called

- regular Toeplitz substitution generates a dynamical system with discrete spectrum. In this case $\delta_c = 1$.
- generalized Morse substitution generates a dynamical system with mixed spectrum and in this case $\delta_c = 0$.

Thus we can draw the following table, and we state the question

Is the following true :

$$\delta_c = 1 \Longrightarrow \text{discrete spectrum}$$
$$\delta_c < 1 \Longrightarrow \text{mixed spectrum ?}$$

(see the following page)

---

(3) COVEN and KEANE. The structure of substitution minimal sets. (1971)

SUBSTITUTION			δ_c	SPECTRUM
constant length	Regular Toeplitz substitution ($\theta(0) \neq \widetilde{\theta(1)}$) (1)		1	Discrete
	Generalized Morse substitution ($\theta(0) = \widetilde{\theta(1)}$) (1)		0	Mixed
non constant length	$\begin{cases} \theta(0) = 0^{n+1-p}\ 1\ 0^{p} \\ \theta(1) = 1\ 0^{n} \end{cases}$ $0 \leq p \leq n$		1	Discrete
	$\begin{cases} \theta(0)=01 \\ \theta(1)=1010 \end{cases}$ and $\begin{cases} \theta(0)=01 \\ \theta(1)=1001 \end{cases}$		1	Discrete
	$\begin{cases} \theta(0) = 0\ 1 \\ \theta(1) = 1\ 1\ 0\ 0 \end{cases}$		$\frac{1}{3}$	Mixed
	$\begin{cases} \theta(0) = 0^{n}\ 1^{n} \\ \theta(1) = 1^{2n}\ 0^{2n} \end{cases}$	n even	1	Discrete
		n odd	$\frac{1}{3}$	Mixed
	other cases		?	?

(1) with $\widetilde{\omega}_k = 1 - \omega_k$