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by
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1. - THE WEIGHTED SIEVE

As usual, let P_k denote any integer having at most k prime divisors, equal or distinct. Let a , h be non-zero integers such that

$$a = \pm 1 \ , \quad 2 | h$$

and let x be a large enough positive number [(1)]. We shall prove :

THEOREM (Chen [1,2]).- There exists an absolute constant x_o such that, if $x \geq x_o$,

$$|\{p : 0 < ap + h < x \ , \ ap + h = P_2\}| >$$

$$(0.689) \prod_{p>2} \left(1 - \frac{1}{(p-1)^2}\right) \prod_{2<p|h} \frac{p-1}{p-2} \frac{x}{\log^2 x} .$$

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(1) If $a=1$, let h be a fixed integer. If $a = -1$, let $h = x = N$, a large enough even integer.

The special cases (i) $a = -1$, $h = x = N$; (ii) $a = 1$, $h = 2$, give the closest known approximations to the Goldbach and prime twins conjectures respectively.

Let $\mathfrak{P} = \{ p : (p, h) = 1 \}$, $P(z) = \prod_{\substack{p<z \\ p\in\mathfrak{P}}} p$, and consider the weighted sum

$$\sum_{\substack{0<ap+h<x \\ (ap+h,\, P(x^{1/10}))=1}}{}' \Big\{ 1 - \frac{1}{2} \sum_{\substack{x^{1/10}\le p_1<x^{1/3} \\ p_1 \mid ap+h\,,\ p_1\in\mathfrak{P}}} 1 - \frac{1}{2} \sum_{\substack{x^{1/10}\le p_1<x^{1/3} \\ p_1 \mid ap+h\,,\ p_1\in\mathfrak{P}}} \ \sum_{\substack{x^{1/3}\le p_2<(x/p_1)^{1/2} \\ p_2 \mid ap+h\,,\ p_2\in\mathfrak{P} \\ ap+h=p_1p_2p_3}} 1 \Big\},$$

where Σ' signifies that the summation extends only over those p's for which $(p, ab) = 1$ and $ap+b$ is squarefree relative to all pairs of primes p_1, p_2 of $\mathfrak{P}$ appearing in the inner sums.

Although the expression appears complicated, its arithmetical significance is not difficult to perceive : the only primes $p\le x$ counted in the sum that have a positive weight (i. e. the expression in parentheses) attached to them are precisely those for which $ap+h = P_2$. To see this, observe that the only positive values taken by the weight are 1 and $\frac{1}{2}$. Now if the weight is 1, both the inner sums must be empty, so that in fact $(ap+h\ ,\ P(x^{\frac{1}{3}})) = 1$; and this is possible only if $ap+h = P_2$. Suppose now that the weight is $\frac{1}{2}$. Then $ap+h$ has no prime factor $< x^{1/10}$ and precisely one between $x^{1/10}$ (inclusive) and $x^{\frac{1}{3}}$ (exclusive). Hence $ap+h = p_1 m$, where $(m, P(x^{\frac{1}{3}})) = 1$. Thus m has at most two prime factors. If m has precisely two prime factors, then

$$ap + h = p_1p_2p_3 \quad , \quad x^{\frac{1}{3}} \le p_2 < p_3 \quad ;$$

clearly we must have $p_2 < (x/p_1)^{\frac{1}{2}}$, since otherwise $ap+h = p_1p_2p_3 > p_1(x/p_1) = x$, which is impossible. But in such a case the weight attached to $ap+h$ receives a further contribution of $-\frac{1}{2}$ from the double sum in parentheses, and so is reduced to 0. Hence, if the weight is to be $\frac{1}{2}$, m has at most one prime factor and $ap+b$ is again a P_2. Hence

$$X := |\{p : 0 < ap+h < x \ , \ ap+h = P_2\}| \tag{1.1}$$

is an upper bound of the weighted sum, and it remains to derive for this sum the lower bound indicated in the statement of the theorem.

2. - THE LOWER BOUND SIEVE

Removing the restriction implied by Σ' induces, as is easy to confirm, an error of magnitude $\ll x^{9/10}$.

For any integer sequence $\mathcal{S}$, let $\mathcal{S}_q = \{s : s \in \mathcal{S} \ , \ q|s\}$ and let

$$S(\mathcal{S} ; \mathfrak{P}, z) = |\{s : s \in \mathcal{S} \ , \ (s, P(z)) = 1\}| .$$

Then we have proved in section 1 that if $\mathcal{C} = \{ap+h : 0 < ap+h < x\}$, then

$$\begin{aligned} X \geq \; & S(\mathcal{C} ; \mathfrak{P}, x^{1/10}) - \frac{1}{2} \sum_{\substack{x^{1/10} \leq p_1 < x^{\frac{1}{3}} \\ p_1 \in \mathfrak{P}}} S(\mathcal{C}_{p_1} ; \mathfrak{P}, x^{1/10}) \\ & - \frac{1}{2} \sum_{\substack{x^{1/10} \leq p_1 < x^{\frac{1}{3}} \leq p_2 < (x/p_1)^{\frac{1}{2}} \\ p_1, p_2 \in \mathfrak{P}}} |\{p_3 : p_3 < x/(p_1p_2) , (p_1p_2)p_3 - h = ap\}| \\ & + O(x^{9/10}) . \end{aligned} \tag{2.1}$$

The first two expressions on the right are part of the Selberg theory as it has been developed by Jurkat-Richert [3] and Halberstam-Jurkat-Richert [4], and we give no details (for these, see the forthcoming book [1] on "Sieve methods" by Halberstam and Richert, Academic Press, 1974).
Their contribution is at least (including also the error term $0(x^{9/10})$)

$$X_o := 2.653 \prod_{p>2}\left(1-\frac{1}{(p-1)^2}\right)\prod_{2<p|h}\frac{p-1}{p-2}\,\frac{x}{\log^2 x} \qquad (x \geq x_o)\ ; \tag{2.2}$$

and our sole concern from now on is with the third expression on the right of (2.1): we require an <u>upper</u> bound for

$$Y_o := \sum_{q\in Q} |\{p' : p' < x/q\ ,\ |qp'-h| = p\}|\ , \tag{2.3}$$

where

$$Q = \{p_1 p_2 : x^{1/10} \leq p_1 < x^{1/3} \leq p_2 < (x/p_1)^{1/2}\ ;\ p_1, p_2 \in \mathfrak{P}\}\ . \tag{2.4}$$

So far we have proved that X, the expression defined by (1.1), satisfies

$$X \geq X_o - \frac{1}{2} Y_o\ , \tag{2.5}$$

where X_o is given by (2.2) and Y_o by (2.3). Note that

$$|Q| \ll x^{2/3}\ , \tag{2.6}$$

and that, since h is even, $|qp'-h| \geq 1$ for every $q \in Q$.

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(1) This book gives a full proof of Chen's theorem.

3. - THE EXPRESSION Y_o : APPLICATION OF THE SELBERG UPPER BOUND SIEVE. - Let :

$$\mathcal{A} = \mathcal{A}(q) = \{qp - h : p < x/q\} .$$

Then

$$Y_o = \sum_{q \in Q} |\{n : n \in \mathcal{A} , |n| \text{ is a prime}\}| ;$$

if z is any number satisfying

$$2 \le z \le x^{\frac{1}{4}} , \tag{3.1}$$

it is certainly true that

$$Y_o \le \sum_{q \in Q} \{S(\mathcal{A}; \mathfrak{P}, z) + z\}$$

$$\le \sum_{q \in Q} S(\mathcal{A} ; \mathfrak{P}, z) + x^{11/12} . \tag{3.2}$$

We tackle the upper bound for $S(\mathcal{A} ; \mathfrak{P}, z)$ in the classical Selberg manner : for any set fof real numbers λ_d satisfying $\lambda_1 = 1$ and $\lambda_d = 0$ if $d \ge z$, defined on the set of squarefree integers coprime with h , we have

$$S(\mathcal{A} ; \mathfrak{P} , z) = |\{qp-h : p < x/q , (qp - h , P(z)) = 1\}|$$

$$\le \sum_{p<x/q} \Big(\sum_{\substack{d|P(z) \\ d|qp-h}} \lambda_d \Big)^2 = \sum_{d_1, d_2 | P(z)} \lambda_{d_1} \lambda_{d_2} \sum_{\substack{p<x/q \\ qp \equiv h \bmod D}} 1$$

where we use the notation $D = \text{L. C. M.} [d_1, d_2]$. For each q , the appropriate choice of the λ's is

$$\lambda_d = \frac{\mu(d)}{\prod_{p|d} (1 - \frac{1}{p-1})} \frac{G_d(z/d)}{G(z)} \qquad (\mu(d) \neq 0 , (d, h) = 1)$$

where, if $g(p) = 1/(p-2)$ $(p > 2)$, and g is multiplicative,

$$G_d(\xi) = \sum_{\substack{m<\xi \\ (m, dh)=1}} \mu^2(m) g(m) , \quad G(\xi) = G_1(\xi) ;$$

with this choice, it is routine to check that

$$(3.3)\qquad \lambda_1 = 1\ ,\quad \lambda_d = 0 \ \text{ if } \ d \geq z\ ,\quad |\lambda_d| \leq 1 \ \text{ if } \ d|P(z)$$

and

$$(3.4)\qquad \sum_{d_1, d_2 | P(z)} \frac{\lambda_{d_1} \lambda_{d_2}}{\Phi(D)} = \frac{1}{G(z)}\ .$$

Writing

$$(3.5)\qquad b(n) = \sum_{\substack{pq=n \\ q \in Q}} 1\ ,$$

it follows from (3.2) that

$$Y_o \leq \sum_{d_1, d_2 | P(z)} \lambda_{d_1} \lambda_{d_2} \sum_{\substack{n<x \\ n \equiv h \bmod D}} b(n)$$

$$= \frac{1}{G(z)} \sum_{n<x} b(n) + \sum_{d_1, d_2 | P(z)} \lambda_{d_1} \lambda_{d_2} \{ E(x; D, h) + \frac{1}{\Phi(D)} \sum_{\substack{n<x \\ (n,D)>1}} b(n) \}$$

by (3.4), where

$$E(x; D, h) = \sum_{\substack{n<x \\ n \equiv h \bmod D}} b(n) - \frac{1}{\Phi(D)} \sum_{\substack{n<x \\ (n,D)=1}} b(n)\ .$$

Hence [(1)], by (3.3)

$$(3.6)\qquad Y_o \leq \frac{1}{G(z)} \sum_{n<x} b(n) + \sum_{\substack{D<z^2 \\ D|P(z)}} \frac{3^{\nu(D)}}{\Phi(D)} \sum_{\substack{n<x \\ (n,D)>1}} b(n)$$

$$+ \sum_{\substack{D<z^2 \\ D|P(z)}} 3^{\nu(D)} |E(x; D, h)|$$

$$= Y_1 + Y_2 + Y_3\ ,$$

(1) $\nu(D)$ denotes the number of distinct prime factors of D .

say. It can be shown by standard methods (e. g. Halberstam-Richert [5]) that

$$\frac{1}{G(z)} \leq 2 \prod_{p>2} \left(1 - \frac{1}{(p-1)^2}\right) \prod_{2<p|h} \frac{p-1}{p-2} \cdot \frac{1}{\log z} \{ 1+0 \left(\frac{1}{\log z}\right) \},$$

and also that

$$\sum_{n<x} b(n) < 0.490996 \frac{x}{\log x} \quad (x \geq x_o) .$$

Choosing

$$(3.7) \qquad z = z^{\frac{1}{4} - \varepsilon} , \quad 0 < \varepsilon < \frac{1}{100} ,$$

it follows that

$$(3.8) \qquad Y_1 < (3.928) \prod_{p>2} \left(1 - \frac{1}{(p-1)^2}\right) \prod_{2<p|h} \frac{p-1}{p-2} \frac{x}{\log^2 x} \quad (x \geq x_o) .$$

LEMMA 1. - *If* $x \geq x_o$, $Y_2 \ll x^{10/11}$.

Proof. - Remember that if $q \in Q$, $q = p_1 p_2$ where $p_2 \geq x^{\frac{1}{3}} > z$. Hence

$$\sum_{\substack{n<x \\ (n,D)>1}} b(n) \leq \sum_{q \in Q} \sum_{\substack{p<x/q \\ p|D}} 1 + \sum_{p<x} \sum_{\substack{p_1 p_2 \in Q \\ p_1 p_2 < x/p \\ p_1 | D}} 1$$

$$\leq \nu(D)|Q| + x \sum_{p<x} \frac{1}{p} \sum_{x^{1/10} \leq p_1 | D} \frac{1}{p_1}$$

$$\leq \nu(D)(|Q| + x^{9/10} \sum_{p<x} \frac{1}{p}) , \ll x^{9/10} (\log \log x) \nu(D)$$

by (2. 6). Hence

$$Y_2 \ll x^{9/10} (\log\log x) \sum_{D < z^2} \frac{\mu^2(D)\,\nu(D)\,3^{\nu(D)}}{\Phi(D)}$$
$$\ll x^{9/10} \log^4 x . \log\log x \ ,$$

and this proves the lemma.

It follows from (3. 6), (3. 8), lemma 1, (2. 5) and (2. 2) that

(3. 9) $$X \geq (0.\,689) \prod_{p>2} \left(1 - \frac{1}{(p-1)^2}\right) \prod_{2<p|h} \frac{p-1}{p-2} \frac{x}{\log^2 x} - \frac{1}{2} Y_3 + 0(x^{10/11}) \ ,$$

and it remains to deal with Y_3 .

4. - <u>THE EXPRESSION</u> Y_3 : <u>TRANSITION TO PRIMITIVE CHARACTER SUMS</u>

We have by Cauchy's inequality that

(4. 1) $$Y_3^2 \leq \sum_{\substack{D < z^2 \\ D | P(z)}} 9^{\nu(D)} \, |E(x ; D, h)| \, . \sum_{D < z^2} |E(x ; D, h)|$$
$$\ll x(\log x)^9 \, Y_4 \ ,$$

where

(4. 2) $$Y_4 = \sum_{D < z^2} |E(x ; D, h)| \quad ;$$

here we have estimated the first factor on the right of (4. 1) by applying the Brun-Titchmarsh inequality to the first expression in $E(x ; D, h)$ and so arriving directly at the bound $E(x ; D, h) \ll x/\Phi(D)$. We are left now with the "Bombieri sum" Y_4 .

Clearly

$$Y_4 = \sum_{D<z^2} \frac{1}{\Phi(D)} \left| \sum_{\substack{\chi \bmod D \\ \chi \neq \chi_o}} \bar{\chi}(h) B_1(x, \chi) \right|$$

where

$$(4.3) \qquad B_\ell(x, \chi) = \sum_{\substack{n<x \\ (n,\ell)=1}} \chi(n)\, b(n) \ .$$

Each non-principal character $\chi \bmod D$ is induced by a unique primitive character $\chi^* \bmod d$, $1 < d | D$; and $\chi(n) = \chi^*(n)$ whenever $(n, D) = 1$, so that $B_1(x, \chi) = B_{D/d}(x, \chi^*)$. Hence

$$Y_4 \le \sum_{D<z^2} \frac{1}{\Phi(D)} \sum_{1<d|D} \left| \sideset{}{^*}\sum_{\chi \bmod d} \bar{\chi}(h) B_{D/d}(x, \chi) \right|$$

where $\sum^*$ indicates summation over primitive characters only. Two interchanges of summation bring us to

$$Y_4 \le (\log x) \max_{\ell < z^2} \sum_{1<d<z^2} \frac{1}{\Phi(d)} \sideset{}{^*}\sum_{\chi \bmod d} |B_\ell(x, \chi)|$$

$$(4.4) \qquad = (\log x) \max_{\ell<z^2} Y_5(\ell) \ ,$$

say. Now

$$B_\ell(x, \chi) = \sum_{\substack{q \in Q \\ (q,\ell)=1}} \chi(q) \sum_{\substack{p<x/q \\ p \nmid \ell}} \chi(p) \ ,$$

and by the Siegel-Walfisz theorem the inner sum is (remember that if $q \in Q$, then $q \ll x^{2/3}$) at most of order

$$\frac{x}{q} \exp(-c\sqrt{\log x}) + \nu(\ell) \ ,$$

where c is an absolute positive constant, if $d \le \log^{100} x$. Hence, for $\ell \le z^2$,

$$B_\ell(x, \chi) \ll x \exp(-c\sqrt{\log x}) + |Q| \log x \ ,$$

whence, using (2.6) ,

$$Y_5(\ell) \ll \sum_{\log^{100} x < d < z^2} \frac{1}{\Phi(d)} \sum_\chi{}^* |B_\ell(x,\chi)| + x(\log x)^{100} \exp(-c\sqrt{\log x})$$

$$\ll Y_6(\ell) + \frac{x}{\log^{110} x}, \quad \ell < z^2,$$

say, with

$$(4.5) \qquad Y_6(\ell) = \sum_{\log^{100} x < d < z^2} \frac{1}{\Phi(d)} \sum_\chi{}^* |B_\ell(x,\chi)|, \quad \ell < z^2.$$

Summing up,

$$(4.6) \qquad Y_3^2 \ll x(\log x)^{10} \max_{\ell < z^2} Y_6(\ell) + \frac{x^2}{\log^{100} x},$$

where $Y_6(\ell)$ is given by (4.5).

Let $\theta = 1 + \frac{1}{\log x}$ and $T = x^{10}$. By Perron's formula

$$B_\ell(x,\chi) = \frac{1}{2\pi i} \int_{\theta - iT}^{\theta + iT} \Big(\sum_{\substack{n=1 \\ (n,\ell)=1}}^{\infty} \frac{\chi(n)\, b(n)}{n^s} \Big) \frac{x^s}{s} + O\Big(\frac{x \log x}{T}\Big),$$

so that

$$(4.7) \qquad Y_6(\ell) \ll \sum_{\log^{100} x < d < z^2} \frac{1}{\Phi(d)} \sum_\chi{}^* \Big| \int_{\theta - iT}^{\theta + iT} \Big(\sum_{\substack{n=1 \\ (n,\ell)=1}}^{\infty} \frac{\chi(n)\, b(n)}{n^s} \Big) \frac{x^s}{s} \Big| + \frac{x z^2 \log x}{T}.$$

We shall prove in the next section that if $\xi \le x$, then

$$(4.8) \qquad \sum_{d \le \xi} \frac{1}{\Phi(d)} \sum_\chi{}^* \Big| \int_{\theta - iT}^{\theta + iT} \Big(\sum_{\substack{n=1 \\ (n,\ell)=1}}^{\infty} \frac{\chi(n)\, b(n)}{n^s} \Big) \frac{x^s}{s}\, ds \Big|$$

$$\ll (x + x^{5/6}\xi + \xi^2) \log^{3/2} x + \xi^3 x^{-9}.$$

From this Abel summation applied to the first expression on the right of (4.7) shows at once that

$$Y_6(\ell) \ll \frac{x}{\log^{98} x} \quad , \qquad \ell < z^2 \ ;$$

then, from (4. 6) ,

$$Y_3^2 \ll \frac{x^2}{\log^{88} x} \ ,$$

and, in view of (3. 9), this completes the proof of the theorem of Chen.

5. - THE LARGE SIEVE : PROOF OF (4. 8)

We require the following large sieve inequality : For any complex numbers a_n ,

$$\sum_{d \leq \xi} \frac{d}{\Phi(d)} \sum_{\chi}{}^{*} \int_{-T}^{T} \Big| \sum_{n=M+1}^{M+N} a_n \chi(n) n^{-it} \Big|^2 \frac{dt}{1+|t|} \ll \sum_{n=M+1}^{M+N} (n+\xi^2 \log T)|a_n|^2 \ ,$$

and this result remains valid for $N = \infty$ provided that $|a_n| \ll n^{-1-\varepsilon}$.

For any function $f(x, \chi)$, $s = \sigma + it$, write

$$\mathcal{B}(f \ ; \sigma, \xi) = \sum_{d \leq \xi} \frac{d}{\Phi(d)} \sum_{\chi}{}^{*} \int_{-T}^{T} |f(\sigma+it, \chi)| \frac{dt}{1+|t|} \ ;$$

then, by the inequalities of Cauchy and Schwarz,

$$(5.1) \qquad \mathcal{B}(f_1 f_2 \ ; \sigma, \xi) \leq \mathcal{B}^{\frac{1}{2}}(f_1^2 \ ; \sigma, \xi)\, \mathcal{B}^{\frac{1}{2}}(f_2^2 \ ; \sigma, \xi) \ .$$

We now turn to (4. 8). Remember that $\xi \leq x$. First of all, since $b(n)$ is an arithmetical convolution, we have

$$\sum_{(n,\ell)=1} \chi(n) b(n) n^{-s} = \Big(\sum_{\substack{q \in Q \\ (q,\ell)=1}} \chi(q) q^{-s} \Big) \Big(\sum_{p \nmid \ell} \chi(p) p^{-s} \Big) \ .$$

We write

$$F(s,\chi) = \sum_{\substack{q\in Q\\(q,\ell)=1}} \chi(q)\, q^{-s} ,$$

a finite sum, and

$$G(s,\chi) = \sum_{\substack{p\le \xi^2\\p\nmid \ell}} \chi(p)p^{-s} , \quad H(s,\chi) = \sum_{\substack{p> \xi^2\\p\nmid \ell}} \chi(p)\, p^{-s} ,$$

a finite sum and the "tail-end" of an infinite series respectively. Hence the integral on the left of (4.8) is

$$\int_{\theta-iT}^{\theta+iT} F(s,\chi)\, G(s,\chi)\, \frac{x^s}{s}\, ds + \int_{\theta-iT}^{\theta+iT} F(s,\chi)\, H(s,\chi)\, \frac{x^s}{s}\, ds ;$$

since FG is the product of two finite sums we may move the line of integration to $\sigma = \frac{1}{2}$ at the cost of a small contribution from the horizontal parts of the rectangle of integration. In fact, the expression above is equal to

$$\int_{\theta-iT}^{\theta+iT} (FH)(s,\chi)\, \frac{x^s}{s} ds + \int_{\frac{1}{2}-iT}^{\frac{1}{2}+iT} (FG)(s,\chi)\, \frac{x^s}{s}\, ds + 0\left(\frac{\xi\, x^{\frac{1}{3}}}{T}\right) ,$$

so that the expression on the left of (4.8) is at most of order

$$x\, \mathcal{B}(FH ; \theta, \xi) + x^{\frac{1}{2}} \mathcal{B}(FG ; \tfrac{1}{2}, \xi) + \frac{\xi^3 x^{\frac{1}{3}}}{T} .$$

We apply (5.1) and the large sieve inequality. We have, first of all,

$$\begin{aligned}\mathcal{B}(FH ; \theta, \xi) &\ll \Big(\sum_{q\in Q} (q+\xi^2 \log T)q^{-2}\Big)^{\frac{1}{2}} \Big(\sum_{p>\xi^2} (p+\xi^2 \log T)p^{-2\theta}\Big)^{\frac{1}{2}}\\ &\ll \left((1+\xi^2 x^{-13/30}) \log x\right)^{\frac{1}{2}} (\log x)^{\frac{1}{2}}\\ &\ll (1+\xi\, x^{-13/60}) \log x ;\end{aligned}$$

and then

$$\mathcal{B}(FG\ ;\ \tfrac{1}{2},\xi) \ll \Big(\sum_{q\in Q}(q+\xi^2\log T)q^{-1}\Big)^{\frac{1}{2}}\Big(\sum_{p\leq\xi^2}(p+\xi^2\log T)p^{-1}\Big)^{\frac{1}{2}}$$
$$\ll (x^{2/3}+\xi^2\log x)^{\frac{1}{2}}(\xi^2\log^2 x)^{\frac{1}{2}}$$
$$\ll x^{\frac{1}{3}}\xi\log x+\xi^2\log^{3/2}x\ .$$

We conclude that the expression on the left of (4. 8) is

$$\ll (x+x^{5/6}\xi+\xi^2)\log^{3/2}x+\frac{\xi^3 x^{\frac{1}{3}}}{T}\ ,$$

and this proves (4. 8).

In conclusion, let me remark that this proof is in essence Chen's argument, but it incorporates substantial technical simplifications shown to me by Dr. R. C. Vaughan, in the use of the analytic and large sieve methods. The proof of Mr Ross is simples still ; it will appear in the Journal of the London Mathematical Society.

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