HARALD NIEDERREITER

Rearrangement theorems for sequences

Astérisque, tome 24-25 (1975), p. 243-261
<http://www.numdam.org/item?id=AST_1975__24-25__243_0>

Société Mathématique de France
Astérisque 24-25 (1975)

REARRANGEMENT THEOREMS FOR SEQUENCES

by

Harald NIEDERREITER *

-:-:-:-

It is a well-known principle in the theory of uniform distribution that, under suitable conditions on the underlying space, a sequence satisfying certain denseness properties can be rearranged to produce a uniformly distributed sequence in that space. The first result in that direction is due to von Neumann [11] who showed that any everywhere dense sequence in $\mathbb{R}/\mathbb{Z}$ can be arranged so as to yield a sequence that is uniformly distributed in $\mathbb{R}/\mathbb{Z}$ (or, equivalently, uniformly distributed mod 1). Hlawka [6] established a quantitative version of this statement. Evidently, the everywhere denseness of the sequence is also a necessary condition for the theorem to hold. For proofs of these results, one may also consult [7, ch. 2, Sect. 4]. A number of other rearrangement theorems for $\mathbb{R}/\mathbb{Z}$ were shown by van der Corput [10].

* This research was supported by National Science Foundation grant GP-36418X1.

In the more general case where the underlying space is a compact Hausdorff space with countable base, the following basic result is essentially due to Hlawka [5], the present formulation being that of Descovich [4]. See also [7, Ch. 3, Sect. 2] for a proof.

THEOREM 1. - Let X be a compact Hausdorff space with countable base, and suppose μ is a nonnegative normed Borel measure on X. Then a sequence (x_n), $n = 1, 2, \dots$ in X has a μ-uniformly distributed rearrangement if and only if all open neighborhoods of points in the support of μ contain infinitely many terms of the sequence (x_n).

For the sake of completeness, we recall that a sequence (x_n) is μ-uniformly distributed in X if

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} f(x_n) = \int_X f d\mu$$

holds for all complex-valued (real-valued) continuous functions f on X.

We shall generalize Theorem 1 considerably. First of all, the result can be put in a quantitative form. Moreover, we will make it clear that the second axiom of countability is not an essential feature of rearrangement theorems ; rather, it is the existence of certain equicontinuous families of functions that is important. Also, the use of such equicontinuous families leads to surprisingly simple proofs.

Before we enunciate our results, we recall the following notions. For an arbitrary nonempty set M, for a sequence (x_n), $n = 1, 2, \dots$, in M, and for a complex-valued function f on M, we introduce the oscillations

$$S(f\ ;\ (x_n)) = \sup_{m,n=1,2,\ldots} |f(x_m) - f(x_n)|$$

and

$$S(f\ ;\ M) = \sup_{x,y \in M} |f(x) - f(y)|$$

(see [3, ch. IX, §. 2, n° 3]). Of course, these numbers need not be finite. Furthermore, if X is a topological space and $\{f_i\}_{i\in I}$ is a family of complex-valued functions on X, where I is an arbitrary index set, then the family $\{f_i\}_{i\in I}$ is called equicontinuous at $x \in X$ if for every $\varepsilon > 0$ there exists a neighborhood $V = V(\varepsilon)$ of x such that

$$|f_i(y) - f_i(x)| < \varepsilon$$

for all $y \in V$ and for all $i \in I$. The family $\{f_i\}_{i\in I}$ is called equicontinuous on X if it is equicontinuous at each $x \in X$.

THEOREM 2. - *Let* X *be an arbitrary topological space, and let* (y_k), $k = 1, 2, \ldots$, *be a sequence in* X. *Suppose* (x_n), $n = 1, 2, \ldots$, *is a sequence in* X *satisfying the following condition* :

(*) *Every open neighborhood of each* y_k *contains infinitely many terms of the sequence* (x_n).

Let $\{f_i\}_{i\in I}$ *be a family of complex-valued functions on* X *that is equicontinuous at each* y_k, *and such that* $\sup_{i\in I} S(f_i\ ;(x_n)) < \infty$. *Let* $a_1, a_2, \ldots, a_N, \ldots$ *be an increasing sequence of positive real numbers with* $\lim_{N\to\infty} a_N = \infty$. *Then the sequence* (x_n) *can be rearranged into a sequence* (u_k), $k = 1, 2, \ldots$, *with*

$$\text{(1)} \qquad \sum_{k=1}^{N} |f_i(y_k) - f_i(u_k)| < a_N$$

for all $N \geq 1$ *and for all* $i \in I$.

Proof. - Choose a sequence (ε_k), $k=1,2,\ldots$, of positive real numbers with $\varepsilon_1 \le \frac{1}{2} a_1$ and $\varepsilon_k \le \frac{1}{2}(a_k - a_{k-1})$ for $k \ge 2$. For each $k \ge 1$, the set

$$E_k = \{x \in X : |f_i(y_k) - f_i(x)| < \varepsilon_k \quad \text{for all } i \in I\}$$

is a neighborhood of y_k by the equicontinuity of the family $\{f_i\}_{i \in I}$ at y_k. Using condition (*), we can construct a sequence $n_1, n_2, \ldots, n_k, \ldots$ of distinct positive integers such that $x_{n_k} \in E_k$ for all $k \ge 1$. We note that for each $i \in I$ we have

$$(2) \qquad \sum_{k=1}^{N} |f_i(y_k) - f_i(x_{n_k})| < \sum_{k=1}^{N} \varepsilon_k \le \tfrac{1}{2} a_N \qquad \text{for all } N \ge 1 .$$

Set $S = \sup_{i \in I} S(f_i ; (x_n))$. Suppose first $S > 0$. We enumerate those positive integers n that are either no n_k or else are n_k with $k \ge 2$ and $[a_{k-1}/2S] < [a_k/2S]$, in an arbitrary fashion : $m_1, m_2, \ldots$. The sequence (u_k) is then defined as follows : $u_k = x_{n_k}$ if $k = 1$ or $[a_{k-1}/2S] = [a_k/2S]$; $u_k = x_{m_p}$ if $[a_{k-1}/2S] < [a_k/2S]$ and $k \ge 2$ is the p-th subscript for which this happens. It is easily seen that (u_k) is a rearrangement of (x_n). For each $i \in I$ and all $N \ge 1$ we have

$$\sum_{k=1}^{N} |f_i(x_{n_k}) - f_i(u_k)| = \sum_{\substack{k=2 \\ [a_{k-1}/2S] < [a_k/2S]}}^{N} |f_i(x_{n_k}) - f_i(u_k)|$$

$$\le S(f_i ; (x_n)) \times (\text{number of } k , \ 2 \le k \le N , \text{ such that } [\tfrac{a_{k-1}}{2S}] < [\tfrac{a_k}{2S}])$$

$$\le S(f_i ; (x_n)) ([\tfrac{a_N}{2S}] - [\tfrac{a_1}{2S}]) \le S(f_i ; (x_n)) \frac{a_N}{2S} ,$$

and so

$$(3) \qquad \sum_{k=1}^{N} |f_i(x_{n_k}) - f_i(u_k)| \le \frac{a_N}{2}$$

for each $i \in I$ and all $N \ge 1$ The inequality (3) is trivial for $S = 0$. By combining (2) and (3), we arrive at the desired result.

Remark 1. - Instead of considering complex-valued functions, one may consider a family $\{f_i\}_{i\in I}$ of functions on X with values in a metric space. Then theorem 2 remains true with obvious modifications.

Remark 2. - For applications to uniform distribution, theorem 2 is only of interest if the sequence (a_N), $N = 1, 2, \dots$, satisfies $\lim\limits_{N\to\infty} a_N/N = 0$.

Remark 3. - The condition (*) in theorem 2 cannot be relaxed. For suppose X has at least two points and contains an isolated point z, and let $y_k = z$ for all $k\geq 1$. Let (x_n) be a sequence that contains z only finitely many times. Let the family $\{f_i\}_{i\in I}$ consist only of the characteristic function f of the singleton $\{z\}$. Then, whatever the rearrangement (u_k) of (x_n), we have $\lim\limits_{N\to\infty} \frac{1}{N}\sum\limits_{k=1}^{N} |f(y_k)-f(u_k)| = 1$. If theorem 2 were to hold, this would result in a contradiction for a sequence (a_N), $N = 1,2,\dots$, with $\underline{\lim}_{N\to\infty} a_N/N < 1$.

Remark 4. - The equicontinuity condition in theorem 2 cannot be relaxed. Let $X = [0,1]$ with the usual topology, let $I = \mathbb{Z}^+$ (= the set of positive integers), and let $f_i(x) = x^i$, $0\leq x\leq 1$, for all $i\in I$. Then the family $\{f_i\}_{i\in I}$ is not equicontinuous at the point 1. Now let $y_k = 1$ for all $k\geq 1$. Consider a sequence (x_n) in $[0,1)$ that has 1 as a limit point. Let (u_k) be any rearrangement of this sequence. For fixed $N\geq 1$, set $t_N = \max\limits_{1\leq k\leq N} u_k$. Then, for every $\varepsilon>0$, there exists $i\in I$ such that $t_N^i\leq\varepsilon$. For this i, we get

$$\sum_{k=1}^{N} |f_i(y_k) - f_i(u_k)| \geq N(1-t_N^i) \geq N(1-\varepsilon) ,$$

and so

$$\sup_{i\in I} \sum_{k=1}^{N} | f_i(y_k)-f_i(u_k)| \geq N .$$

Hence, if theorem 2 were to hold in this case, this would lead to a contradiction for a sequence (a_N) with $a_N < N$ for some N.

Remark 5. - The condition $\lim_{N\to\infty} a_N = \infty$ cannot be relaxed. Let the space X, the sequence (y_k), and the family $\{f_i\}_{i\in I}$ be as in Remark 3. Take a sequence (x_n) in X that contains z infinitely often, but for which also $x_n \neq z$ for infinitely many n. Then, for any rearrangement (u_k) of (x_n), we have $\lim_{N\to\infty} \sum_{k=1}^{N} |f(y_k) - f(u_k)| = \infty$, so that, in this situation, theorem 2 can only hold for sequences (a_N) with $\lim_{N\to\infty} a_N = \infty$.

Remark 6. - If I is infinite, if each f_i is continuous on X with $S(f_i; X)$ finite, and if $S(f_i; X)$ tends to 0 along the Fréchet filter of I (i. e., along the filter consisting of the complements of the finite subsets of I), then the family $\{f_i\}_{i\in I}$ is automatically equicontinuous on X and $\sup_{i\in I} S(f_i; X) < \infty$. The proof is immediate. Since $S(f; X) \leq 2\,\|f\|$, where $\|.\|$ denotes the supremum norm, it follows that theorem 2 includes [7, ch. 3, theorem 2. 6] as a special case.

Remark 7. - The sufficiency part of theorem 1 can be deduced from theorem 2 as follows. By [7, ch. 3, theorems 1. 3. and 2. 2], there exists a μ-uniformly distributed sequence (y_k) contained in the support of μ. By [7, ch. 3, theorem 2.1], there is a countable convergence-determining class $\{g_i\}_{i\in\mathbb{Z}^+}$ with respect to μ, each g_i being a real-valued continuous function on X. By replacing those g_i with $\|g_i\| > 0$ by $(i\|g_i\|)^{-1} g_i$, we obtain a convergence-determining class $\{f_i\}_{i\in\mathbb{Z}^+}$ with respect to μ satisfying $\lim_{i\to\infty} \|f_i\| = 0$. According to remark 6, the conditions in theorem 2 hold for the family $\{f_i\}_{i\in\mathbb{Z}^+}$. Since each y_k is in the support of μ, the given sequence (x_n) in theorem 1 satisfies the condition

(*) in theorem 2. Choose the sequence (a_N) in theorem 2 in such a way that $\lim_{N\to\infty} a_N/N = 0$. Then (1) holds for a suitable rearrangement (u_k) of (x_n). Consequently, we get

$$\left|\frac{1}{N}\sum_{k=1}^{N} f_i(u_k) - \int_X f_i\, d\mu\right| \leq \left|\frac{1}{N}\sum_{k=1}^{N} f_i(y_k) - \int_X f_i\, d\mu\right| + \frac{a_N}{N}$$

for all $N \geq 1$ and all $i \in \mathbb{Z}^+$, and so

$$\lim_{N\to\infty} \frac{1}{N}\sum_{k=1}^{N} f_i(u_k) = \int_X f_i\, d\mu \quad \text{for all } i \in \mathbb{Z}^+ ,$$

i. e., the sequence (u_k) is μ-uniformly distributed in X.

For a compact Hausdorff space X, let $C(X)$ be the Banach space of real-valued continuous functions on X with the supremum norm and let $\mathcal{M}(X)$ be its dual space, which can be identified with the space of signed regular Borel measures on X.

THEOREM 3. - Let X be a compact Hausdorff space, and let $\mu \in \mathcal{M}(X)$ be non-negative and normed. Furthermore, suppose that μ satisfies the following conditions :

(i) μ lies in the weak* sequential closure of the convex hull of normed point measures in $\mathcal{M}(X)$;

(ii) there exists a relatively compact $\mathcal{a} \subseteq C(X)$ which is convergence-determining with respect to μ.

Then a sequence (x_n) in X has a μ-uniformly distributed rearrangement if and only if all open neighborhoods of points in the support of μ contain infinitely many terms of the sequence (x_n).

Proof. - The necessity of the condition is clear (compare with [7, ch. 3, exercises 1. 5. and 1. 6]). As to the sufficiency, we note first that condition (i) and a result in [9] imply the existence of a μ-uniformly distributed sequence (w_k) in X. Let $\mathfrak{a} = \{f_i\}_{i \in I}$. Since a relatively compact subset of $C(X)$ is norm-bounded, we have $\sup_{i \in I} S(f_i ; X) < \infty$ by an inequality in remark 6. Moreover, the Arzelà-Ascoli theorem yields that $\{f_i\}_{i \in I}$ is equicontinuous on X. Therefore, the family $\{f_i\}_{i \in I}$ satisfies all the conditions in theorem 2.

Let K be the support of μ. We shall construct a μ-uniformly distributed sequence (y_k), $k = 1, 2, \ldots$, that is contained in K. First some preliminary remarks. For $x, y \in X$, we set

$$p(x, y) = \sup_{i \in I} |f_i(x) - f_i(y)| \ .$$

Then $p(. \ , .)$ is a finite pseudometric on X in the sense of [3, ch. IX, §. 1, n°1] (however, the topology on X induced by this pseudometric need not be identical with the original topology on X). If $x \in X$ is fixed, then $p(a, x)$ is a continuous function of $a \in X$: for given $a \in X$ and $\varepsilon > 0$, there exists, by the equicontinuity of $\{f_i\}_{i \in I}$, a neighborhood $V(a)$ of a such that $|f_i(a) - f_i(b)| < \varepsilon$ for all $b \in V(a)$ and for all $i \in I$. But then $p(a, b) \leq \varepsilon$ for all $b \in V(a)$, and since we have $|p(a, x) - p(b, x)| \leq p(a, b)$, it follows that $|p(a, x) - p(b, x)| \leq \varepsilon$ for all $b \in V(a)$. Next we define $p(K, x) = \inf_{a \in K} p(a, x)$ for $x \in X$. Let $x, y \in X$ be arbitrary ; since K is compact and $p(. \, , .)$ is a continuous function of its first coordinate, there exists $a \in K$ such that $p(a, x) = p(K, x)$. Then, $p(a, y) \leq p(a, x) + p(x, y) = p(K, x) + p(x, y)$, and so $p(K, y) \leq p(K, x) + p(x, y)$. By interchanging the roles of x and y, we arrive at the inequality

$$|p(K, x) - p(K, y)| \leq p(x, y) \quad \text{for all} \quad x, y \in X \ .$$

From this and the equicontinuity of $\{f_i\}_{i \in I}$, it follows in a similar manner as above that $p(K, x)$ is a continuous function of $x \in X$.

The construction of the sequence (y_k) is now carried out as follows. For each $k = 1, 2, \dots$, choose a $y_k \in K$ such that $p(y_k, w_k) = p(K, w_k)$. Then, for every $i \in I$ we have

$$\left|\frac{1}{N}\sum_{k=1}^{N}(f_i(y_k)-f_i(w_k))\right| \leq \frac{1}{N}\sum_{k=1}^{N}|f_i(y_k)-f_i(w_k)|$$

$$\leq \frac{1}{N}\sum_{k=1}^{N} p(y_k, w_k) = \frac{1}{N}\sum_{k=1}^{N} p(K, w_k) \quad \text{for all } N \geq 1 .$$

However, since $p(K, .)$ is continuous and (w_k) is μ-uniformly distributed, we have

$$\lim_{N\to\infty} \frac{1}{N}\sum_{k=1}^{N} p(K, w_k) = \int_X p(K, x)\, d\mu = 0 ,$$

where the last identity holds because of $p(K, x) = 0$ for $x \in K$. Therefore,

$$\lim_{N\to\infty} \frac{1}{N}\sum_{k=1}^{N} f_i(y_k) = \lim_{N\to\infty} \frac{1}{N}\sum_{k=1}^{N} f_i(w_k) = \int_X f_i\, d\mu$$

for all $i \in I$. But $\{f_i\}_{i \in I}$ is convergence-determining with respect to μ, and so the sequence (y_k) is μ-uniformly distributed in X.

We can now apply theorem 2. Condition (*) in this theorem is satisfied because of the given property of (x_n) and the fact that each y_k is in the support of μ. The sequence (a_N) in theorem 2 is chosen in such a way that $\lim_{N\to\infty} a_N/N = 0$. It follows then as in remark 7 that for a suitable rearrangement (u_k) of (x_n) we have

$$\lim_{N\to\infty} \frac{1}{N}\sum_{k=1}^{N} f_i(u_k) = \int_X f_i\, d\mu \quad \text{for all } i \in I .$$

Since $\{f_i\}_{i \in I}$ is convergence-determining with respect to μ, the sequence (u_k) is μ-uniformly distributed in X.

Remark 8. - The condition (i) in theorem 3 is certainly necessary, since it is even necessary for the existence of a μ-uniformly distributed sequence in X

(see [9]).

Rearrangement theorems for sequences can be applied in various ways. One area of applications is the construction of sequences in locally compact abelian groups with prescribed asymptotic distribution with respect to various subgroups of compact index. Consider, for instance, the discrete additive group $\mathbb{Z}$ of integers. For every integer $m \geq 2$, let μ_m be a nonnegative normed Borel measure on $\mathbb{Z}/m\mathbb{Z}$. These measures should be compatible in the sense that whenever m divides n and $\Phi : \mathbb{Z}/n\mathbb{Z} \to \mathbb{Z}/m\mathbb{Z}$ is the canonical homomorphism, then $\mu_m(E) = \mu_n(\Phi^{-1}(E))$ for every subset E of $\mathbb{Z}/m\mathbb{Z}$.

THEOREM 4. - Under the above compatibility condition, there exists a sequence (a_n), $n = 1, 2, \ldots$, in $\mathbb{Z}$ such that $(a_n + m\mathbb{Z})$, $n = 1, 2, \ldots$, is μ_m-uniformly distributed in $\mathbb{Z}/m\mathbb{Z}$ for every $m \geq 2$.

Proof. - For every prime p, let $\mathbb{Z}_p$ be the additive group of p-adic integers in the p-adic topology. Let $p_1, p_2, \ldots$ be the sequence of primes, and let $G = \prod_{i=1}^{\infty} \mathbb{Z}_{p_i}$ be the direct product of the topological groups $\mathbb{Z}_{p_i}$, so that G is a compact abelian group with countable base. We construct a nonnegative normed Borel measure μ on G as follows. Consider first an open box D in G of the form

$$D = (b_1 + p_1^{r_1} \mathbb{Z}_{p_1}) \times \ldots \times (b_k + p_k^{r_k} \mathbb{Z}_{p_k}) \times \mathbb{Z}_{p_{k+1}} \times \ldots \tag{4}$$

with $b_1, \ldots, b_k \in \mathbb{Z}$ and nonnegative $r_1, \ldots, r_k \in \mathbb{Z}$, not all r_i being zero. Let b be an integer with $b \equiv b_i \pmod{p_i^{r_i}}$ for $i = 1, 2, \ldots, k$. Then define $\mu(D) = \mu_m(\{b + m\mathbb{Z}\})$, where $m = p_1^{r_1} \ldots p_k^{r_k}$. Extend μ by additivity to the algebra of finite unions of boxes of the form (4). The compatibility condition on the μ_m shows that the resulting σ-additive set function on this algebra is well defined. By the usual procedure, μ can then be extended to a nonnegative normed Borel measure on G, which we shall also denote by μ.

We note that $\mathbb{Z}$ can be embedded into G via the mapping $\psi : a \in \mathbb{Z} \to (a, a, \dots) \in G$. The Chinese remainder theorem shows that $\psi(\mathbb{Z})$ is dense in G. Arrange the elements of $\psi(\mathbb{Z})$ into a sequence in an arbitrary way. Then, according to theorem 1, this sequence can be rearranged into a sequence (y_n), say, that is μ-uniformly distributed in G. For every $n \geq 1$, let $a_n \in \mathbb{Z}$ be such that $\psi(a_n) = y_n$. We claim that the sequence (a_n) satisfies the desired property.

For this purpose, let $m \geq 2$ be fixed and let $m = p_1^{r_1} \dots p_k^{r_k}$ be its prime factorization, where some of the r_i are possibly zero. For given $b \in \mathbb{Z}$, we note that $a_n + m\mathbb{Z} = b + m\mathbb{Z}$ if and only if $y_n = (a_n, a_n, \dots)$ is contained in the box

$$B = (b + p_1^{r_1} \mathbb{Z}_{p_1}) \times \dots \times (b + p_k^{r_k} \mathbb{Z}_{p_k}) \times \mathbb{Z}_{p_{k+1}} \times \dots \,.$$

Therefore, for every $N \geq 1$, we have the identity $A(\{b+m\mathbb{Z}\} ; N ; (a_n + m\mathbb{Z})) = A(B ; N ; (y_n))$ between the counting functions corresponding to the sequences $(a_n + m\mathbb{Z})$ and (y_n), respectively. Since B, being both open and closed in G, is evidently a μ-continuity set, we obtain

$$\lim_{N\to\infty} \frac{A(\{b+m\mathbb{Z}\} ; N ; (a_n + m\mathbb{Z}))}{N} = \lim_{N\to\infty} \frac{A(B ; N ; (y_n))}{N} = \mu(B)$$
$$= \mu_m(\{b+m\mathbb{Z}\}),$$

and so $(a_n + m\mathbb{Z})$ is μ_m-uniformly distributed in $\mathbb{Z}/m\mathbb{Z}$.

<u>Remark 9.</u> - We have even shown that the sequence (a_n) attains every integral value exactly once.

If μ_{m_i} , $i=1,2,\ldots$, is a nonnegative normed Borel measure on $\mathbb{Z}/m_i\mathbb{Z}$ with $m_1, m_2, \ldots$ pairwise relatively prime, then one may find a sequence (a_n) in $\mathbb{Z}$ such that $(a_n + m_i\mathbb{Z})$, $n=1,2,\ldots$, is μ_{m_i}-uniformly distributed in $\mathbb{Z}/m_i\mathbb{Z}$ for all $i = 1,2,\ldots$ in the following alternative way. One simply considers the direct product $X = \prod_{i=1}^{\infty} (\mathbb{Z}/m_i\mathbb{Z})$ furnished with the product measure $\nu = \mu_{m_1} \times \mu_{m_2} \times \ldots$, notes that the image of $\tau : a \in \mathbb{Z} \to (a+m_1\mathbb{Z}, a+m_2\mathbb{Z}, \ldots) \in X$ is dense in X , and arranges $\tau(\mathbb{Z})$ into a ν-uniformly distributed sequence in X. The corresponding sequence in $\mathbb{Z}$ satisfies then the required property. If, however, the moduli $m_1, m_2, \ldots$ are such that m_i divides m_{i+1} for $i=1,2,\ldots$ and the μ_{m_i} satisfy a compatibility condition analogous to the one enunciated in the paragraph preceding theorem 4, one may proceed by a similar argument, with X now being the inverse limit of the topological groups $\mathbb{Z}/m_i\mathbb{Z}$ (in the sense of [3, ch. III, 3e éd., §.7, n°2]) and ν the measure on X induced by the μ_{m_i} . We remark that, for the two special cases considered above, the existence of sequences (a_n) with the desired properties was shown in [8] by means of explicit constructions.

Let now G be an arbitrary locally compact abelian group, with the operation being written multiplicatively. All topological groups considered will be Hausdorff. A closed subgroup H of G is called a subgroup of compact index if G/H is compact. A character χ of G is called periodic if its kernel is a subgroup of compact index. For more on these notions, see [7, ch. 4, Sect. 5]. Let $\mathcal{H}$ be the family of all subgroups of G of compact index. For each $H \in \mathcal{H}$, let λ_H be a regular Borel measure on G/H . According to [2], the family $\{\lambda_H\}_{H\in\mathcal{H}}$ is called a density on G if the total variations of the λ_H are uniformly bounded and if λ_{H_1} and λ_{H_2} satisfy the obvious compatibility condition whenever $H_1 \subseteq H_2$. We shall speak of a nonnegative normed density in case all the λ_H are nonnegative and normed.

THEOREM 5. - Let G be a locally compact abelian group for which the periodic characters form a countable subgroup of the dual group $\hat{G}$. Then, for every non-negative normed density $\{\lambda_H\}_{H\in\mathcal{H}}$ on G there exists a sequence (x_n) in G such that $(x_n H)$ is λ_H-uniformly distributed in G/H for every $H\in\mathcal{H}$.

Proof. - Let first G be an arbitrary locally compact abelian group (not necessarily satisfying the condition of the theorem), and let Γ be the subgroup of $\hat{G}$ generated (algebraically) by the periodic characters of G. If Γ is furnished with the discrete topology, then $\hat{\Gamma}$ is a compact abelian group, the so called periodic compactification $\bar{G}^p$ of G (see [1]). A natural continuous homomorphism $\varphi : G \mapsto \bar{G}^p$ can be defined such that $\varphi(G)$ is dense in $\bar{G}^p$: for $x\in G$, let $\hat{x}$ be the character of $\hat{G}$ given by $\hat{x}(\chi) = \chi(x)$ for $\chi\in\hat{G}$; the restriction of $\hat{x}$ to Γ, which we denote by $\tilde{x}$, is then a character of Γ ; we define $\varphi(x) = \tilde{x}$. For a proof of the indicated properties of φ, see [7, ch. 4, lemma 5.3].

Let now H be a subgroup of G of compact index. Then there is a natural continuous homomorphism α_H from $\bar{G}^p$ onto G/H. This is constructed as follows. Let $\Phi_H = (G/H)^\wedge$, considered as a subgroup of Γ (thus, a character χ of G/H will always be identified with the character χ^* of G given by $\chi^*(x) = \chi(xH)$ for $x\in G$). Take $\tau\in\bar{G}^p$, so that τ is a character of Γ. The restriction of τ to Φ_H is a character of Φ_H. By Pontryagin duality, there exists a unique $a_\tau\in G/H$ such that $\tau(\chi) = \chi(a_\tau)$ for all $\chi\in\Phi_H$. We set $\alpha_H(\tau) = a_\tau$. It is clear that α_H is a surjective homomorphism. Therefore, it suffices to prove continuity at the identity element of G/H. Thus, let V be a neighborhood of the identity element of G/H. Again by Pontryagin duality, we may view G/H as $(G/H)^{\wedge\wedge}$, so that from the compactness of G/H and the definition of the compact-open topology it follows that there exist finitely many $\chi_1, \ldots, \chi_m \in \Phi_H$ and an $\varepsilon > 0$ such that $\{b \in G/H : |\chi_i(b) - 1| < \varepsilon$ for $1 \leq i \leq m\}$ is contained in V. Let U be the neighborhood of the identity element of $\bar{G}^p$ given by $U = \{\tau \in \bar{G}^p : |\tau(\chi_i) - 1| < \varepsilon$ for $1 \leq i \leq m\}$. Then it follows immediately that $\alpha_H(U) \subseteq V$.

As mentioned in [2, pp. 96-97], every regular Borel measure λ on $\bar{G}^p$ induces a density on G. Explicitly, the induced density $\{\lambda_H\}_{H \in \mathcal{K}}$ is obtained by choosing, for each $H \in \mathcal{K}$, the measure λ_H on G/H as the one that is induced by λ via the mapping $\alpha_H : \bar{G}^p \mapsto G/H$. If λ is nonnegative and normed, then the induced density is nonnegative and normed. The following important theorem was shown in [2, theorems 2.10 and 3.10]. Every density on G is induced in the above way by a regular Borel measure on $\bar{G}^p$ if and only if G satisfies one of the two subsequent conditions : (A) G is totally disconnected ; (B) each discrete quotient of G is of bounded order and, in the decomposition according to the structure theorem for locally compact abelian groups, G has no $\mathbb{R}^n$ part (i. e., $n = 0$).

Let now G be a locally compact abelian group satisfying the condition of theorem 5. Since, by assumption, the periodic characters of G form a subgroup of $\hat{G}$, it follows from a result in [1] that G is either totally disconnected or else each discrete quotient of G is of bounded order. If G had a nontrivial $\mathbb{R}^n$ part, one could show easily that G would have uncountably many periodic characters (by using the fact that every character of $\mathbb{R}$ is periodic), and thus would arrive at a contradiction. Hence, G satisfies one of the conditions (A) and (B) from above, so that the given density $\{\lambda_H\}_{H \in \mathcal{K}}$ on G is induced by a nonnegative normed regular Borel measure λ on $\bar{G}^p$. Since, in our case, Γ is countable, $\bar{G}^p$ is a compact abelian group with countable base. Furthermore, since $\varphi(G)$ is dense in $\bar{G}^p$, there exists a sequence of elements of $\varphi(G)$ such that every nonvoid open set in $\bar{G}^p$ contains infinitely many terms of the sequence. By theorem 1, this sequence can be rearranged to produce a λ-uniformly distributed sequence in $\bar{G}^p$. In other words, there is a sequence (x_n) in G such that $(\varphi(x_n))$ is λ-uniformly distributed in $\bar{G}^p$.

We claim that the sequence (x_n) has the desired property. For given $H \in \mathcal{H}$, it suffices to show that

$$(5) \qquad \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \chi(x_n) = \int_{G/H} \chi \, d\lambda_H$$

for all $\chi \in \Phi_H$. For given $\chi \in \Phi_H$, let $\hat{\chi}$ be the character of $\bar{G}^p$ defined by $\hat{\chi}(\tau) = \tau(\chi)$ for all $\tau \in \bar{G}^p$. Then, using the notation from the first paragraph of this proof, we get $\chi(x_n) = \hat{x}_n(\chi) = \tilde{x}_n(\chi) = \hat{\chi}(\tilde{x}_n) = \hat{\chi}(\varphi(x_n))$ for all $n \geq 1$. From the λ-uniform distribution of $(\varphi(x_n))$ in $\bar{G}^p$, we obtain

$$(6) \qquad \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \chi(x_n) = \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \hat{\chi}(\varphi(x_n)) = \int_{\bar{G}^p} \hat{\chi} \, d\lambda \ .$$

It remains to show that the integrals in (5) and (6) are equal. But since λ_H is the measure on G/H induced by λ via the mapping α_H, we have

$$\int_{G/H} \chi \, d\lambda_H = \int_{\bar{G}^p} (\chi \circ \alpha_H) \, d\lambda \ ,$$

and from the definition of α_H we see immediately that $\chi \circ \alpha_H = \hat{\chi}$. This completes the proof of theorem 5.

Remark 10. - In fact, we have shown a somewhat stronger statement, since the desired sequences (x_n) arise by suitable rearrangement from any sequence (y_n) in G for which every nonvoid open set in $\bar{G}^p$ contains infinitely many terms of the sequence $(\varphi(y_n))$. Here, "every nonvoid open set" can even be replaced by "every open neighborhood of points in the support of λ ".

One can establish a result analogous to theorem 5 for any locally compact abelian group if one restricts the attention to certain families of subgroups of compact index. The ensuing theorem is related to a result in [12].

THEOREM 6. - Let G be a locally compact abelian group and let $\{H_j\}_{j\in J}$ be a countable family of subgroups of G of compact index which is closed under finite intersections, and such that for each $j\in J$ the quotient G/H_j has a countable base. For each $j\in J$, let λ_j be a nonnegative normed Borel measure on G/H_j such that the family $\{\lambda_j\}_{j\in J}$ is compatible in the sense that whenever H is a closed subgroup of G with $H \supseteq H_i \cup H_j$ for some $i, j\in J$, then λ_i and λ_j induce the same measure on G/H . Then, there exists a sequence (x_n) in G such that $(x_n H_j)$, $n = 1, 2, \ldots$, is λ_j-uniformly distributed in G/H_j for every $j\in J$.

Proof. - As in the proof of theorem 5, the crucial step is to verify that the "partial density" $\{\lambda_j\}_{j\in J}$ is induced by a measure on a suitable compactification of G . This can be achieved by adapting arguments from [2, §. 3]. For each $j\in J$, we consider again $(G/H_j)^\wedge$ as a subgroup of $\hat{G}$. Put $\Psi = \bigcup_{j\in J} (G/H_j)^\wedge$. Since $\{H_j\}_{j\in J}$ is closed under finite intersections, Ψ is a subgroup of $\hat{G}$. Moreover, Ψ is countable as the countable union of countable sets. Furnish Ψ with the discrete topology, and let K be its dual. A continuous homomorphism $\varphi : G \mapsto K$ making $\varphi(G)$ dense in K is constructed as in the proof of theorem 5.

We consider now various function spaces on G . First, let P be the set of all complex-valued continuous functions on G that are periodic modulo some H_j (i. e., that are constant on the cosets of some H_j). From the fact that $\{H_j\}_{j \in J}$ is closed under finite intersections, it follows that P is a (complex) vector space. Let $sp(\Psi)$ be the linear subspace of P generated by Ψ. Let A be the set of complex-valued continuous functions g on G of the form $g = f \circ \varphi$ for some complex-valued continuous function f on K. The elements of A may be called the K-almost periodic functions on G. Clearly, A is a Banach space under the supremum norm. We note that $P \subseteq A$, for if $g \in P$ (let us say, g is periodic modulo H_j), then $g = (g_j \circ \alpha_j) \circ \varphi$, where g_j is the continuous function on G/H_j induced by g and $\alpha_j = \alpha_{H_j}$ is the natural continuous homomorphism from K onto G/H_j constructed in the same way as in the proof of theorem 5. We remark also that, by Pontryagin duality, the characters of K are obtained by taking a $\chi \in \Psi$ and defining $\hat{\chi}(\tau) = \tau(\chi)$ for all $\tau \in K$. We have $\hat{\chi} \circ \varphi = \chi$ for all $\chi \in \Psi$ (compare with the last part of the proof of theorem 5). It follows now from the Peter-Weyl theorem, applied to the compact abelian group K, that $sp(\Psi)$ is dense in A.

We define a linear functional L on P as follows. For $g \in P$ (say, g periodic modulo H_j), we set

$$L(g) = \int_{G/H_j} g_j \, d\lambda_j \quad ,$$

where g_j is defined as above. Then L is well defined by the compatibility condition on the family $\{\lambda_j\}_{j \in J}$ (compare with [7, ch. 4, lemma 5. 1]), and is evidently a nonnegative normed functional on P. Now P is dense in A because of $sp(\Psi) \subseteq P \subseteq A$, and so L can be extended uniquely to a nonnegative normed functional on A which we also denote by L. For every complex-valued continuous function f on K, set $L_1(f) = L(f \circ \varphi)$. Then L_1 is a nonnegative normed functional on the Banach space of complex-valued continuous functions on K under the supremum norm, and so, by the Riesz representation theorem, L_1 corresponds to a unique nonnegative normed regular Borel measure λ on K. This measure λ induces the "partial density" $\{\lambda_j\}_{j \in J}$ via the mappings α_j (compare with the proof of theorem 5).

By noting that K is a compact abelian group with countable base and that $\varphi(G)$ is dense in K, one can now complete the argument in exactly the same way as in the proof of the preceding theorem.

-:-:-:-

REFERENCES

[1] I. D. BERG, M. RAJAGOPALAN, and L. A. RUBEL. - Uniform distribution in locally compact abelian groups. Trans. Amer. Math. Soc. 133, 435-446 (1968).

[2] I. D. BERG and L. A. RUBEL. - Densities on locally compact abelian groups. Ann. Inst. Fourier 19, fasc. 1, 81-107 (1969) ; Erratum, ibid. 19, fasc. 2, 533 (1969).

[3] N. BOURBAKI. - Topologie générale. Hermann, Paris, 1948, 1960.

[4] J. DESCOVICH. - Zur Theorie der Gleichverteilung auf kompakten Räumen. Sitzgsber. Österr. Akad. Wiss., Math.-naturw. Kl., Abt. II, 178, 263-283 (1969).

[5] E. HLAWKA. - Folgen auf kompakten Räumen. Abh. Math. Sem. Hamburg 20, 223-241 (1956).

[6] E. HLAWKA. - Interpolation analytischer Funktionen auf dem Einheitskreis. Number Theory and Analysis (edited by P. Turán), pp. 97-118, Plenum, New York, 1969.

[7] L. KUIPERS and H. NIEDERREITER. - Uniform Distribution of Sequences. Wiley-Interscience, New York, 1974.

[8] M. B. NATHANSON. - Asymptotic distribution and asymptotic independence of sequences of integers. (to appear).

[9] H. NIEDERREITER. - On the existence of uniformly distributed sequences in compact spaces. Compositio Math. 25, 93-99 (1972).

[10] J. G. van der CORPUT. - Verteilungsfunktionen I-VIII. Proc. Akad. Amsterdam, 38, 813-821, 1058-1066 (1935) ; 39, 10-19, 19-26, 149-153, 339-344, 489-494, 579-590 (1936).

[11] J. von NEUMANN. - Egyenletesen sürü számsorozatok. Mat. Fiz. Lapok 32, 32-40 (1925).

[12] A. ZAME. - On a problem of Narkiewicz concerning uniform distributions of sequences of integers. Colloq. Math. 24, 271-273 (1972).

-:-:-:-

Harald NIEDERREITER
The Institute for Advanced Study
School of Mathematics
PRINCETON
NEW JERSEY 08540