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VALUES OF ZETA AND L-FUNCTIONS AT ZERO

by

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Let F be an algebraic number field, and ζ_F its Dedekind zeta-function. Then it is a classical result that ζ_F has a simple pole at $s = 1$, with residue equal to $\dfrac{2^{r_1}(2\pi)^{r_2} hR}{w\sqrt{|d|}}$, where r_1 and r_2 are the number of real and imaginary places of F, h is the class number of F, R is the regulator of F, w is the number of roots of unity of F, and d is the discriminant of F. We will be interested in generalizing this result to Artin L-functions.

By using the functional equation of the zeta-function, we may translate the description of the behavior of the zeta-function at $s = 1$ to a description of its behavior at $s = 0$, and we find :

PROPOSITION 1. - (i) $\zeta_F(s)$ <u>has a zero of order</u> $r_1 + r_2 - 1$ <u>at</u> $s = 0$

(ii) $\lim\limits_{s \to 0} \zeta_F(s)\, s^{-(r_1+r_2-1)} = -hR/w$.

It is this statement that we will attempt to generalize to L-functions.

Let K be a finite Galois extension of F with Galois group $G = G(K/F)$, and let ρ be a representation of G which is realizable over $\mathbb{Q}$. Then in fact ρ is obtained from a finitely-generated G-module M by tensoring with $\mathbb{C}$, and our goal will be to express the behavior of the Artin-L-function $L(\rho, s)$ at $s = 0$ in terms of cohomological invariants of M. We write $\rho = \rho_M$ and define $L(M, s)$ to be $L(\rho_M, s)$.

We may regard M as a module for $G_F = G(\overline{F}/F)$, since the choice of the extension K through which the representation factors does not affect the L-function associated to M.

Let $\mathcal{O}_F$ be the ring of integers of F, $X = \operatorname{Spec} \mathcal{O}_F$, X_∞ = the set of infinite places of F, and $\tilde{X} = X \cup X_\infty$. We give $\tilde{X}$ a Grothendieck topology by choosing as objects in our category complements of finite sets of primes (both finite and infinite) in finite extensions K of F, such that each prime of K is unramified over its restriction to F. (Recall that an infinite prime is unramified over another one unless the first is complex and the second real). The coverings are finite families of these sets the union of whose restriction to $\tilde{X}$ is all of $\tilde{X}$. This topology was first considered, in a more general setting, by Artin and Verdier in [1].

Let j be the natural inclusion of $\operatorname{Spec} F$ in $\tilde{X}$, and $\varphi_!$ be the functor which takes a sheaf on X and "extends it by zero" to $\tilde{X}$. We note that $\varphi_!$ is the left adjoint of the restriction functor φ^*. Since M is a G_F-module, we may regard M as a sheaf for the etale topology on $\operatorname{Spec} F$, and take its direct image $j_* F$ on X. We would like to determine the cohomology groups $H^i(X, \varphi_! j_* M)$.

For the sake of simplicity of statement, we will henceforth assume that F is totally imaginary, although pratically everything we say will be either true or true up to a finite 2-torsion group in the general case.

THEOREM 1. - $H^o(\widetilde{X}, \varphi_! j_* M) = 0$.

$H^1(\widetilde{X}, \varphi_! j_* M)$ is a finitely-generated abelian group.

$H^2(\widetilde{X}, \varphi_! j_* M)$ is a finite abelian group.

$H^3(\widetilde{X}, \varphi_! j_* M)$ is the dual of the finitely-generated abelian group $\mathrm{Hom}_{\widetilde{X}}(\varphi_! j_* M, \varphi_! G_{m,X})$, via the natural pairing into $H^3(\widetilde{X}, \varphi_! G_{m,X})$ which is naturally isomorphic to $\mathbb{Q}/\mathbb{Z}$.

$H^q(\widetilde{X}, \varphi_! j_* M) = 0$ for $q > 3$.

We remark that $\mathrm{Hom}_{\widetilde{X}}(\varphi_! j_* M, \varphi_! G_{m,X})$ is easily seen to be just $\mathrm{Hom}_{G_F}(M, U(F))$, where $U(K)$ denotes the units of K .

The zeta-function itself corresponds to the G_F-module $\mathbb{Z}$ with trivial action, and it may be helpful to see what the non-zero cohomology groups are in this case.

PROPOSITION 2. - $H^1(\widetilde{X}, \varphi_! j_* \mathbb{Z}) = (\coprod_v \mathbb{Z}_v)/\mathbb{Z}$, where $\mathbb{Z}_v = \mathbb{Z}$, the sum is taken over all infinite places v , and $\mathbb{Z}$ is regarded as a subgroup of $\coprod_v \mathbb{Z}_v$ via the diagonal embedding.

$H^2(\widetilde{X}, \varphi_! \mathbb{Z})$ is the dual of the class group of F .

$H^3(\widetilde{X}, \varphi_! \mathbb{Z})$ is the dual of the units of F .

Going back now to the general case, we have :

THEOREM 2. - <u>The order</u> r_M <u>of the zero of</u> $L(M,s)$ <u>at</u> $s=0$ <u>is equal to the rank of the finitely generated abelian group</u> $H^1(\tilde{X}, \varphi_! j_* M)$ <u>and to the rank of</u> $\mathrm{Hom}_{\tilde{X}}(\varphi_! j_* M , \varphi_! G_{m,X}) = \mathrm{Hom}_{G_F}(M, U(\overline{F}))$.

The next step is to define the regulator of the module M. In order to do this, it is necessary to compute $H^1(\tilde{X}, \varphi_! G_{m,X})$.

PROPOSITION 3. - $H^1(X, \varphi_! G_{m,X})$ <u>is isomorphic to</u> $(\coprod_{v\in X} \mathbb{Z}_v \oplus \coprod_{w\in X_\infty} \tilde{F}_w^*)/F^*$, <u>where</u> $\mathbb{Z}_v = \mathbb{Z}$ <u>and</u> $\tilde{F}_w$ <u>is the algebraic closure of</u> F <u>in the completion</u> F_w. (F^* <u>is regarded as a submodule of</u> $\coprod \mathbb{Z}_v \oplus \tilde{F}_w^*$ <u>by sending</u> $\alpha \in F^*$ <u>to</u> $\{v(\alpha), \alpha_w\}$. <u>At a finite prime</u> v, $v(\alpha)$ <u>denotes the discrete valuation determined by the prime</u> v <u>and</u> α_w <u>is the image of</u> α <u>in</u> $\tilde{F}_w$.)

DEFINITION 1. - We define the absolute value $\| \ \|$ of an element γ of $H^1(\tilde{X}, \varphi_! G_{m,X})$ represented by $\{n_v, \beta_w\}$ by $\|\gamma\| = (\prod_{v\in X} q_v^{-n_v})(\prod_{w\in X_\infty} |\beta_w|)$. Here q_v is the number of elements in the residue field of v, and $\|\alpha\| = 1$ for $\alpha \in F^*$ by the product formula for F^*.

We remark that $H^1(\tilde{X}, \varphi_! G_{m,X})$ is a dense subgroup of the idèles of F, modulo principal idèles and idèles which are units at finite primes and 1 at infinite primes, and the absolute value of $H^1(\tilde{X}, \varphi_! G_{m,X})$ is induced from the usual absolute value on the idèles.

Now let $< , >$ denote the obvious pairing from

$$H^1(\tilde{X}, \varphi_! j \ M) \times \mathrm{Hom}_{\tilde{X}}(\varphi_! j_* M , \varphi_! G_{m,X}) \text{ to } H^1(\tilde{X}, \varphi_! G_{m,X}).$$

Let $\alpha_1, \dots, \alpha_r$ be a basis for $H^1(\tilde{X}, \varphi_! j_* M)$ modulo torsion, and $\beta_1, \dots, \beta_b$ be a basis for $\mathrm{Hom}_X(\varphi_! j_* M , \varphi_! G_{m,X})$ modulo torsion.

DEFINITION 2. - We define the regulator of M, $R(M)$, to be $|\det(\log \|\langle \alpha_i , \beta_j \rangle \|)|$. This is as usual independent of the choice of the bases.

It is not difficult to check that $R(M)$ reduces to the usual regulator when $M = \mathbb{Z}$.

THEOREM 3. - $\lim_{s \to 0} L(M, s) s^{-r_M} = C_M$, *where*

$$C_M = \pm \frac{\# H^2(\tilde{X} , \varphi_! j_* M) R(M)}{\# H^1(\tilde{X} , \varphi_! j_* M)_{tor} \# H^3(\tilde{X} , \varphi_! j_* M)_{cotor}} .$$

(*Here* $\#(A)$ *is the order of a finite group* A , B_{tor} *is the torsion subgroup of an abelian group* B , *and* C_{cotor} *is* C *divided by the maximal divisible subgroup of* C .)

We observe that for $M = \mathbb{Z}$, theorem 3 merely gives Proposition 1, up to sign.

The details of the proofs of these theorems will appear elsewhere [2] ; we here only remark that by using results of Swan [4] as they are used by Ono in [3], it suffices to check theorem 3 when $M = \mathbb{Z}$, to show that C_M is a function on the Grothendieck group of finitely-generated G_F-modules M , and to show the compatibility of theorem 3 with induced representations.

COROLLARY 1. - *If* $r_M = 0$, *then* $L(M, 0)$ *is equal, up to sign, to the Euler-Poincaré characteristic of the sheaf* $\varphi_! j_* M$ *on* $\tilde{X}$.

We remark that this is in particular the case when M is finite, in which case $L(M, 0) = 1$. In this case, the theorem is equivalent to theorem 2.2. of Tate in [5], which is the key lemma in proving that the Birch-Swinnerton-Dyer-Tate conjecture is compatible with isogenies.

We have restricted ourselves throughout to the case of representations realizable over $\mathbb{Q}$. It is possible to state a generalization of theorem 3 to the case of any representation, but the Grothendieck-group methods of Swan and Ono fail to provide a proof.

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