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Deformations of algebraic varieties with G_m action

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Table of Contents

1. Introduction

This thesis* is devoted to a study of the deformations of an isolated singularity of an algebraic variety admitting a multiplicative group action. In this section we give a summary of the techniques used and the results obtained.

(1.1) First we set up some notation and review the relevant elements of deformation theory. Details can be found in [43] and [45]. We fix once and for all an algebraically closed field k.

Let B be a local k-algebra of finite type. Let $\mathcal{C}$ be the category of local artin k-algebras, $\hat{\mathcal{C}}$ that of complete noetherian local k-algebras: hence if $A \in \hat{\mathcal{C}}$ with maximal ideal m, then $A/m^i \in \mathcal{C}$ for all i.

Definition (1.2) An infinitesimal deformation of B to $A \in \mathcal{C}$ is a cartesian diagram

$$\begin{array}{ccc} B' & \longrightarrow & B = B' \otimes_A k \\ \uparrow & & \uparrow \\ A & \xrightarrow{\text{res}} & k \end{array}$$

where $A \longrightarrow B'$ is flat.

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* A slightly different version of this work was submitted to Harvard University in partial fulfillment of the Ph.D. requirements in May 1974.

<u>Definition</u> (1.3) D, the <u>local deformation functor</u> from $\mathcal{C}$ to {sets} takes $A \in \mathcal{C}$ to isomorphism classes of deformations of B to A, isomorphism being defined in the obvious way. We extend D to $\hat{\mathcal{C}}$ by taking inverse limits to get the notion of <u>formal deformation</u>.

We will usually be interested in minimal "complete" families of formal deformations, in the following sense:

<u>Definition</u> (1.4) A formal deformation $\zeta \in D(R)$, $R \in \hat{\mathcal{C}}$ is <u>versal</u> if

(i) any formal deformation $\zeta' \in D(S)$ may be deduced from ζ by a base change $R \longrightarrow S$.

(ii) the Zariski tangent space of R is minimal for (i).

R (or Spec R) is then called the <u>formal moduli space</u> of B; its tangent space (the k-vector space of first order deformations) is denoted T^1_B, or just T^1 if no confusion is possible.

Lichtenbaum-Schlessinger [28] have studied T^1 using the cotangent complex. Schlessinger's theorem [43] implies that when Spec B has an isolated singularity at its closed point, then B has a versal deformation.

Finally note that for simplicity we often speak of deforming Spec B when we mean we are deforming B, and write down the obvious dual diagrams.

(1.5) We use the standard notation $\mathbb{G}_m$ to refer to the group of units k^* of k under multiplication.

Assume now that $\mathbb{G}_m$ acts on B. The first chapter of this work is devoted to a general study of the deformations of B. Our first result is that we can construct a versal deformation of B:

$$\begin{array}{ccc} B' & \longrightarrow & B = B' \otimes_R k \\ \uparrow & & \uparrow \\ R & \longrightarrow & k \end{array}$$

so that $\mathbb{G}_m$ also acts on R and B' in such a way that all the maps are "equivariant"(i.e., they respect the group action). This is proved in section 2. The first step towards proving this result is to show that T^1 has a $\mathbb{G}_m$ action; therefore decomposing T^1 into eigenspaces we can write

$$T^1 = \sum_{\nu} T^1(\nu)$$

where ν ranges from $-\infty$ to $+\infty$. In this work we will concern ourselves exclusively with deformations derived from the non positive part of the grading and our main theme will be that these deformations lift to deformations of certain projective varieties.

(1.6) Schlessinger [45] has studied the grading of T^1 in the following special case: let X be a smooth, projectively normal projective variety, and let C_X be the affine cone over X. Projective normality (sometimes called arithmetic normality) just means the vertex of C_X is normal. The local ring B of C_X at the vertex obviously has a $\mathbb{G}_m$ action. Schlessinger shows the eigenspaces $T^1(\nu)$ can be interpreted in terms of cohomology groups on X. In section 3 we sketch his results which we later use.

(1.7) Now suppose that B is the local ring of the vertex of the cone C_X over a smooth projective variety X, no longer assumed to be projectively normal. If $T^1_B(\nu) = 0$ for all $\nu > 0$ we say that B, or C_X, has <u>negative grading</u>. In order to state our main result, we make some definitions.

Let $\overline{C}_X$ be the projective cone over X. If $X \subset \mathbb{P}^n$, then $\overline{C}_X \subset \mathbb{P}^{n+1}$. Let Hilb be the local Hilbert functor of $\overline{C}_X$ in $\mathbb{P}^{n+1}$, defined in section 4, and let φ denote the morphism of functors: $\text{Hilb} \longrightarrow D$ obtained by "forgetting what happens at infinity".

Theorem (1.8) Let X, C_X, B, etc. be as in (1.7), and suppose C_X has negative grading. Then we can construct a section to φ. Furthermore if X is projectively normal then φ is a smooth morphism.

For details, see sections 4 and 5 where this theorem is proved.

In chapters II and III we use (1.8) to study the question of existence of smooth deformations of a cone C_X.

Definition (1.9) Let Spec B be an isolated singularity, and let Spec B' $\longrightarrow$ Spec R be the versal deformation of Spec B. We say Spec B has smooth deformations if a generic fibre of Spec B' $\longrightarrow$ Spec R is smooth.

Chapter II deals with cones over projectively normal curves X. When X has dimension ≥ 2 Schlessinger [44,45] shows that C_X gives many examples of singularities without smooth deformations. Mumford [31] extends this to curves: if X is a smooth nonhyperelliptic curve of genus ≥ 3, sufficiently amply embedded, then C_X has no smooth deformations. Our main result is the following

<u>Theorem</u> (1.10) Let X be a smooth curve of genus $g \geq 2$ (resp. $g = 1$) embedded by a complete linear system of degree $d \geq 4g+5$ (resp. $D \geq 10$). Then the vertex of the cone C_X has no smooth deformations.

This is proved in section 7. Note the explicit bound for d is new. In sections 8 and 9 we analyze in detail the deformations of cones over curves of genus 0 and 1. Results are stated at the beginning of each section.

In Chapter III we study cones over 0-dimensional varieties: X is a finite set of points, so that C_X is a collection of lines all passing through one point. Using (1.8) again we obtain in section 11:

Theorem (1.11) The general cone consisting of $g+n+1$ lines in general position in $\mathbb{A}^{n+1}$, $3 \le g \le n$, does not have smooth deformations for $n > 4 + \frac{6}{g-2}$. (The condition $g \le n$ implies negative grading).

This gives the first simple, explicit example of a curve singularity without a smooth deformation. Of course Mumford [30] had already shown that the "generic" curve singularity has no smooth deformations, but it was not known how complicated an explicit example would be.

(1.12) Finally in Chapter IV we study irreducible curve singularities with $\mathbb{G}_m$ action.[1] Such singularities can be described parametrically as follows. Let H be any additive semigroup of non negative integers. B_H, the semi-group ring of H, is the subring of $k[t]$ generated by t^h, $h \in H$. B_H obviously has a $\mathbb{G}_m$ action. Unfortunately the formal moduli space R of B_H does not necessarily have negative grading. Let R' be the quotient of R obtained by setting equal to zero the variables corresponding to $T^1(\nu)$, $\nu > 0$. (See section 13 for a precise definition). We have now

[1] See (1.20)

reduced ourselves to a negative grading situation. This means that R' and the restriction of the versal family above it B" are graded in positive degrees (the change in sign occurs because the variables of R' are dual to the corresponding elements in T^1, hence have opposite weight).

(1.13) Putting aside the above situation for a second, note that semigroups of positive integers occur in another area of the theory of algebraic curves.

Definition (1.14). Let X be a smooth proper curve over k, P a point of X. The semi-group H_P of P is the set of non negative integers h such that

$$\dim H^o(X,\mathcal{O}(hP)) - \dim H^o(X,\mathcal{O}((h-1)P) = 1.$$

Our main result (proved in section 13) relates the two semigroups:

Theorem (1.15) With the notation of (1.12) and (1.13), there is an open set $\bar{U}$ in Proj R', which classifies the set of pairs consisting of a smooth and proper algebraic curve X together with a point $P \in X$ with semigroup $H_P = H$.

We do not claim that $\overline{U}$ is non-void. We discuss some cases of this in section 14. If $\overline{U}$ is non void, then we have constructed directly a compactification of a "moduli space" for curves with points of semi group H. In section 14 we also discuss how to compute the dimension of Proj R' using a result of Deligne [10] stated in section 10.

(1.16) To explain this construction we illustrate it in the simplest case: we let H be the semigroup generated by 2 and 3. $B_H = k[t^2,t^3]$ can then be written as $k[X_1,X_2]/(f)$, $f = X_1^3-X_2^2$. This is the ordinary cusp. $\mathbb{G}_m$ acts with weight 2 on X_1, weight 3 on X_2. It is well known that the versal deformation of this singularity is given by $F = X_1^3 - X_2^2 + t_1X_1 + t_2$ over $R = k[t_1,t_2]$. To make this equation homogeneous for the $\mathbb{G}_m$ action we must assign to t_1 and t_2 the weights 4 and 6. Therefore in this special case we are directly in a situation of negative grading. We now modify F in the following way: in F replace t_i by $t_iX_0^{\text{weight } t_i}$, X_0 a new variable. We obtain

$$\overline{F} = X_1^3 - X_2^2 + t_1X_0^4X_1 + t_2X_0^6 \in R[X_0,X_1,X_2] .$$

Note that $\overline{F}$ is homogeneous in X_0, X_1, X_2 alone if we assign to them the weights 1,2,3. Considering $R[X_0,X_1,X_2]/\overline{F}$ to be graded in the X_i's only, look at

$$Y = \mathrm{Proj}(R[X_0,X_1,X_2]/(\overline{F})) \longrightarrow \mathrm{Spec}\ R.$$

$\mathbb{G}_m$ acts on R as before: it is easy to see that all fibres above points of the <u>same</u> $\mathbb{G}_m$ orbit in Spec R are isomorphic. All the fibres of $Y \longrightarrow \mathrm{Spec}\ R$ are smooth elliptic curves except

1) the original cusp above (t_1,t_2)

2) a family of rational curves with a single node above <u>one</u> orbit of $\mathbb{G}_m$ in Spec R.

We have a natural section obtained by setting $X_0 = 0$. It picks out a point on the smooth fibres of semigroup H, not a very surprising fact in this case since all points on an elliptic curve do!

Since the isomorphism class of the fibres is constant on $\mathbb{G}_m$ orbits of Spec R, it is natural to quotient out by the action, i.e., take Proj R. In the case at hand we obtain the usual compactification of the coarse moduli space for elliptic curves with section. $\overline{U}$ in this case is Proj R $-$ $\{$image of orbit corresponding to node$\}$.

All the features of this special case generalize, except that, as we have already mentioned, it is not known in the general case whether or not U is void.

This construction seems to raise more questions than it answers. Some of them are discussed in sections 13 and 14.

(1.17) To conclude, let me mention some results that are biproducts of our main constructions.

1) In section 8 we show that the formal moduli space of the cone over the rational curve of degree 4 in $\mathbb{P}^4$ has 2 positive-dimensional components. By a result of Artin [4] only one of the components is obtained by simultaneous resolution, therefore giving an example of a rational surface singularity with deformations that cannot be obtained in that manner. (8.3)

2) In (12.8) we give an example of an obstructed Gorenstein curve singularity.

(1.18) It is with great pleasure and gratitude that I thank my advisor, David Mumford, who stimulated my interest in this topic and proposed several of the problems treated here. His contributions to this thesis in both form and content are too numerous to be mentioned. Needless to say, the remaining errors are my own. I would also like to thank Mike Artin, David Eisenbud, Heisuke Hironaka and George Kempf for many interesting discussions, and Laura Schlesinger for her rapid and precise typing job.

(1.19) <u>Notation</u>. We collect here some notation we will use without further warning.

Spec will be used to designate both ordinary Spec and formal Spec.

$\Omega_{B/A}$ = module of differentials of $A \longrightarrow B$([1], VI)

Θ_X = tangent bundle of the variety X.

N_X = normal bundle of X for its embedding in projective space.

$\chi(X,F)$ = Euler characteristic of the sheaf F on X.

$h^i(X,F) = \dim H^i(X,F)$.

Finally when we speak of the affine cone of a projective variety $X \subset \mathbb{P}^n$ we mean Spec R, where R is the unique graded ring such that Proj R = X and R_+ (= maximal ideal of R) is not associated to R. In other words there is no embedded component at the vertex.

(1.20) In chapter 4 we must make the restriction characteristic k = 0. We actually only need the characteristic to be large compared to g, the number of gaps of the semigroup.

CHAPTER I. Generalities on deformations of varieties with $\mathbb{G}_m$ action.

2. Deformations of singularities with $\mathbb{G}_m$ action

In this section C will be an affine variety defined over k with an isolated singular point p and a $\mathbb{G}_m$ action. In a suitable coordinate system we have:

(2.1) C can be written as Spec(P/I), where $P = k[X_1,\cdots,X_n]$ and I an ideal of P containing $p = (X_1,\ldots,X_n)$. $g \in \mathbb{G}_m$ acts by $X_i \longmapsto g^{c_i}X_i$. $c_i \in \mathbb{Z}$ is called the weight of X_i. I is invariant under $\mathbb{G}_m$. $B = P/I$.

Such singularities have been studied extensively by Orlik and Wagreich in the case of surfaces [34] and Herzog and Kunz in the case of curves [24]. This generalizes the situation where C is the cone over a smooth projective variety, in which case $c_i = 1$ for all i. Using elementary techniques we prove the following results:

Proposition (2.2). The tangent space T^1 of the deformation functor D of p in C has a natural grading $T^1 = \Sigma T^1(\nu)$ with ν ranging from $-\infty$ to ∞.

<u>Proposition</u> (2.3) A formal versal deformation of C

$$\begin{tikzcd} Y \arrow[d] & C \arrow[l, hook'] \arrow[d] \\ V & \operatorname{Spec} k \arrow[l, hook'] \end{tikzcd}$$

can be constructed so that $\mathbb{G}_m$ acts on Y and on V compatibly in such a way that the action on Y extends the action of $\mathbb{G}_m$ on C.

<u>Remark</u> (2.4) These gradings are of course the same obtained by Schlessinger on the projectively normal case, and described in section 3, since in both cases they are eigenspaces of the $\mathbb{G}_m$ action.

(2.5) The proof of (2.2) is purely formal. Given an infinitesimal deformation of Spec B to A:

$$\begin{tikzcd} \operatorname{Spec} B' \arrow[d] & \operatorname{Spec} B \arrow[l] \arrow[d] \\ \operatorname{Spec} A & \operatorname{Spec} k \arrow[l] \end{tikzcd}$$

as in (1.2); if g is any automorphism of Spec B then we get a new deformation of Spec B to A as follows:

$$\begin{array}{ccccc} \mathrm{Spec}\ B' & \longleftarrow & \mathrm{Spec}\ B & \xrightarrow[\sim]{g} & \mathrm{Spec}\ B \\ \downarrow & & \downarrow & & \downarrow \\ \mathrm{Spec}\ A & \longleftarrow & \mathrm{Spec}\ k & \approx & \mathrm{Spec}\ k \end{array} .$$

Therefore g also acts on $D(A)$, for any $A \in \mathcal{C}$. In particular g acts on $T^1 = D(k[\epsilon])$. It is not hard to show g is a vector space automorphism of T^1. When the group of automorphisms is $\mathbb{G}_m$, T^1 decomposes into eigenspaces which give (2.2).

(2.6) Let us see what this means when $g \in \mathbb{G}_m$. We keep the notation of (2.1). Choose a basis $\{f_i\}_{i=1,\dots,N}$ of I, so that $g \in \mathbb{G}_m$ takes f_i to $g^{d_i} f_i$ for some $d_i \in \mathbb{Z}$ called the weight of f_i. Any relation between the f_i decomposes into a sum of weighted relations, since the f_i are weighted. We use vector notation and write F^0 for the column vector (f_i). Then we can find a basis (as a $B = P/I$ module) for the relations, $\{R_j^0\}_{1 \le j \le J}$, where R_j^0 is a row vector with entries r_{ij} of weight $N_j - d_i$ for some integer N_j.

We use the exact sequence:

$$\mathrm{Hom}_B(\Omega_P \otimes B, B) \longrightarrow \mathrm{Hom}_B(I/I^2, B) \longrightarrow T^1 \longrightarrow 0 .$$

Now $\mathrm{Hom}_B(I/I^2,B) \simeq \{$N-tuples $b = (b_i)$, $b_i \in B \,|\, R_j^0 \cdot b = 0$ on B, $\forall j\}$

Definition (2.7) We grade column (resp. row) vectors of elements of B as follows: a column (resp. row) vector is homogeneous of degree ν if its i-th entry is weighted with weight $d_i+\nu$ (resp. $\nu-d_i$) in B.

Hence F^0 has degree 0, and R_j^0 degree N_j.

This gives a grading on $\mathrm{Hom}_B(I/I^2,B)$. Since $\partial/\partial X_k \in \mathrm{Hom}_B(\Omega_P \otimes B, B)$ maps to $(F^0 \longrightarrow \frac{\partial f_i}{\partial X_k} \quad i = 1,\cdots,N)$ $\in \mathrm{Hom}_B(I/I^2,B)$, which by (8.7) is homogeneous of degree $-c_k$, the cokernel T^1 is also graded. We denote the ν-the graded piece $T^1(\nu)$. Of course since the singularity of C is assumed isolated, only finitely many of the $T^1(\nu)$ are non-zero [43].

(2.8) We proceed to the proof of (2.3). It is not clear to us how formal this result is. Ideally one would like to have it for any sufficiently nice group G. If D also satisfies condition H_4 of Schlessinger [43], hence is prorepresentable, the result is well known and obvious using the uniqueness of the maps on the base. The standard example is the (global) deformation functor of a proper smooth curve with finite automorphism group G. However if D does not satisfy H_4, there seems to be a problem in extending the group action to the total space of the deformations. Therefore we give the proof only in the special case of (2.3), continuing the explicit description of (2.6).

(2.9) Choose a homogeneous basis F_k^1 of T^1, $1 \leq k \leq r$, $F_k^1 \in T^1(e_k)$. Let $S = k[[t_1,\cdots,t_r]]$, m the maximal ideal of S. We know there is a $\xi_2 \in D\ (S/m^2)$ inducing a bijection of $\mathrm{Hom}(S/m^2,k[\epsilon])$ with T^1; choose it so that the "dual" of t_k maps to F_k^1. Hence t_k has weight $-e_k$ under the natural dual action of $\mathbb{C}_m$. The general first order lifting of F^0 is then written $F^1 = F^0 + \Sigma t_k F_k^1$. The extension of the action of $\mathbb{C}_m$ to S/m^2 by duality makes F^1 homogeneous of degree 0 by (8.7).

Recall that to get the versal deformation of C, i.e., the hull of D, Schlessinger constructs by induction on q an ideal $J_q \subset S$, $J_2 = m^2$, and a $\xi_q \in D(S/J_q)$ such that J_q is the minimal ideal J satisfying the following two conditions:

i) $mJ_{q-1} \subset J \subset J_{q-1}$

ii) there exists a $\xi \in D(S/J)$ inducing ξ_{q-1}.

Then pick any $\xi_q \in D(S/J_q)$ inducing ξ_{q-1}. Finally set $J = \bigcap J_q$, $R = S/J$.

(2.10) We now prove by simultaneous induction on q the following two statements (by this we mean a_q depends on both a_{q-1} and b_{q-1}, etc.).

a_q: J_q is invariant under the $\mathbb{G}_m$ action on S.

b_q: ξ_q can be chosen so that the liftings F^q of F^0 and R_j^q of R_j^0 are homogeneous of degree 0 and N_j respectively.

Proof. (2.9) shows that both a_2 and b_2 hold. First we prove a_q using a_{q-1} and b_{q-1}. Apply $g \in \mathbb{G}_m$ to any $\xi \in D(S/J_q)$ inducing ξ_{q-1}. We obtain $g(\xi)$ flat over $S/g(J_q)$. Since C is invariant under g, $g(\xi) \in D(S/g(J_q))$. Furthermore $g(\xi)$ induces ξ_{q-1} by b_{q-1}, so that by versality $J_q \subset g(J_q)$. Since this is true for all $g \in \mathbb{G}_m$ we get a_q.

We now prove b_q. Let us remark here that everything we have done up to now can be done by general nonsense as in (2.5). It is not clear that the same holds for b_q. To prove b_q recall that $R_j^q \cdot F^q \equiv 0 \bmod J_q$ for any lifting $\xi \in D(S/J_q)$. Replace R_j^q and F^q by their degree 0 and N_j parts respectively. Call the polynomials so obtained $\overline{R}^q$ and $\overline{F}^q$. By b_{q-1} this only affects terms of $R_j^q - R_j^{q-1}$ and $F^q - F^{q-1}$. Notice this operation replaces $R_j^q \cdot F^q$ by its weighted part of weight N_j. Therefore since J_q is weighted (by a_q) we have $\overline{R}_j^q \cdot \overline{F}^q \equiv 0 \bmod J_q$. Hence we have replaced ξ by a new element $\xi_q \in D\,(S/J_q)$ satisfying b_q, and inducing ξ_{q-1}.

This completes the proof of (2.3). In the next section we specialize to the situation where C is a cone. However in section 13 we will apply (2.3) to varieties with $\mathbb{G}_m$ action which are not cones.

3. A Sketch of Schlessinger's results for cones over projectively normal varieties

(3.1) Let X be a smooth and proper algebraic variety, L a very ample, projectively normal invertible sheaf on X. The embedding we obtain: $X \xrightarrow{|L|} \mathbb{P}^n$ will be fixed throughout. Let C_X (or just C if no confusion is possible) be the cone over X in this embedding. $C_X \hookrightarrow \mathbb{A}^{n+1}$. Projective normality means the vertex p of C is normal. For simplicity we write V for $\mathbb{A}^{n+1}$ and $\mathbb{P}(V)$ or just $\mathbb{P}$ for $\mathbb{P}^n$. C_X has obvious $\mathbb{G}_m$ action.

(3.2) In this section we sketch some results of Schlessinger which in the situation described in (3.1) interpret the eigenspaces $T^1(\nu)$ of T^1, the tangent space of the deformations of the vertex of C_X, in terms of cohomology groups on X:

$$(3.3)\quad T^1(\nu) = \operatorname{coker}(H^0(X,\mathcal{O}_X(\nu+1)\otimes V) \longrightarrow H^0(X,N_X(\nu))) .$$

Here $\mathcal{O}_X(\nu) = \mathcal{O}_X(1)^{\otimes\nu}$ where $\mathcal{O}_X(1) = L$ is the canonical sheaf for the projective embedding $X \longrightarrow \mathbb{P}^n$, N_X is the normal bundle for the embedding. For any sheaf F on X,

$F(\nu) = F \otimes \mathcal{O}_X(\nu)$.

The proof of (3.3) is in two parts. First we transfer information from C to E = C-p, using projective normality. This is Schlessinger's comparison theorem ([44] or [45]) which can be applied to a much greater class of singularities. Then we use $E = \operatorname{Spec} \sum_{\nu=-\infty}^{\infty} \mathcal{O}_X(\nu)$ over X to compute in terms of X.

<u>Step 1</u>. Since p is a normal point on C, its <u>depth</u> in C is ≥ 2. ([1], VII.2.12). Now for any coherent sheaf F on C of depth ≥ 2 at p, $H^0(C,F) \approx H^0(C-p,F)$ by the long exact sequence of local cohomology [16].

<u>Definition</u> (3.4) X an algebraic variety. A sheaf F on X is <u>reflexive</u> if $F = \underline{\mathrm{Hom}}_{\mathcal{O}_X}(G,\mathcal{O}_X)$ for some coherent sheaf G on X.

The following observation is elementary but important:

<u>Lemma</u> 3.5 ([44], lemma 1). All reflexive sheaves F on a variety X with a normal point p have depth ≥ 2 at p, hence $H^0(X,F) = H^0(X-p,F)$.

We return to the situation described in (3.1). From the exact sequence

$$I/I^2 \longrightarrow \Omega_V \otimes \mathcal{O}_C \longrightarrow \Omega_C \longrightarrow 0$$

(see 1.18) where I is the ideal sheaf on C in V, the results of [28] give after applying $\underline{\mathrm{Hom}}(\cdot,\mathcal{O}_C)$ the exact sequence:

$$\begin{array}{ccccccccc} 0 \to & \underline{\mathrm{Hom}}(\Omega_C,\mathcal{O}_C) & \to & \underline{\mathrm{Hom}}(\Omega_V\otimes\mathcal{O}_C,\mathcal{O}_C) & \to & \underline{\mathrm{Hom}}(I/I^2,\mathcal{O}_C) & \to & T^1 & \to 0 \\ & \|\text{def.} & & \|\text{def.} & & \|\text{def.} & & & \\ & \Theta_C & & \Theta_V|_C & & N_C & & & \end{array}$$

Taking global sections, since C is affine:

$$\begin{array}{ccccccccc} 0 \to & H^0(C,\Theta_C) & \to & H^0(C,\Theta_V|_C) & \to & H^0(C,N_C) & \to & T^1 & \to 0 \\ & \| & & \| & & \| & & & \\ & H^0(E,\Theta_C) & & H^0(E,\Theta_V|_C) & & H^0(E,N_C) & \text{by (3.5).} & & \end{array}$$

We will work mainly with this exact sequence.

<u>Note</u> (3.6) T^1 is supported at p, since X is smooth, so that the sequence of sheaves

$$0 \longrightarrow \Theta_C|_E \longrightarrow \Theta_V|_E \longrightarrow N_C|_E \longrightarrow 0$$

is exact. Its cohomology sequence (E is not affine) combined with the previous exact sequence shows that $T^1 \hookrightarrow H^1(E,\Theta_E)$. Similarly one can show $T^2 \hookrightarrow H^1(E,N_E)$, [45].

Step 2. $E \cong \operatorname{Spec} \sum_{\nu=-\infty}^{\infty} \mathcal{O}_X(\nu)$, hence is affine over X with structure map $\pi: E \longrightarrow X$. Therefore we get:

$$H^o(X, \pi_* \Theta_{V|_E}) \longrightarrow H^o(X, \pi_* N_{C|_E}) \longrightarrow T^1 \longrightarrow 0$$

$\mathbb{G}_m$ acts on both $\pi_*(\Theta_{V|_E})$ and $\pi_*(N_{C|_E})$. The invariants under $\mathbb{G}_m$ are $\mathcal{O}_X(1)\otimes V$ and N_X respectively, and the map on the invariants is the obvious one:

$$\begin{tikzcd}
H^o(X, \mathcal{O}_X(1)\otimes V) \arrow[rr] \arrow[dr] & & H^o(X, N_X) \\
& H^o(X, \Theta_{\mathbb{P}|X}) \arrow[ur] &
\end{tikzcd}$$

(3.8) follows, since

$$H^o(X, \pi_* \Theta_{V|_E}) \cong \sum_{\nu=-\infty}^{\infty} H^o(X, \mathcal{O}_X(\nu+1)\otimes V)$$

$$H^o(X, \pi_* N_{C|_E}) \simeq \sum_{\nu=-\infty}^{\infty} H^o(X, N_X(\nu))$$

because π is affine.

Similarly $T^2(\nu) = H^1(X, N_X(\nu)) \cap T^2$ gives a grading on T^2.

4. A theorem on negatively graded cones

We keep the hypotheses of (3.1) except we no longer assume that X is projectively normal.

Definition (4.1) We say that C_X has negative grading if $T^1_C(\nu) = 0$ for all $\nu > 0$.

Theorem (4.2) If C_X has negative grading then any infinitesimal deformation of C_X lifts to an infinitesimal deformation of the projective cone $\overline{C}_X \subset \mathbb{P}^{n+1}$. More precisely we construct a section to the natural morphism φ from the Hilbert functor of $\overline{C}_X$ in $\mathbb{P}^{n+1}$ to the deformation functor D of the vertex of C_X.

Definition (4.3) Hilb, the local Hilbert functor of $\overline{C}_X$ in $\mathbb{P}^{n+1}$, is the functor $\mathcal{C}$ to {sets} which to the artin ring $A \in \mathcal{C}$ associates the cartesian diagrams:

$$\begin{tikzcd} \mathbb{P}^{n+1}_A \supset Y \arrow[d] & & \bar{C} = Y \otimes_k A \arrow[ll, hook'] \arrow[d] \\ \operatorname{Spec} A & & \operatorname{Spec} k \arrow[ll, hook'] \end{tikzcd}$$

with Y flat over Spec A. The Hilbert functor was introduced by Grothendieck [17] and has been studied by Artin [3].

(4.4) The natural morphism φ: Hilb $\longrightarrow$ D of the theorem is just restriction to an affine containing C_X.

Proof of (4.2). Let $\begin{tikzcd} Y \arrow[d] & C \subset \mathbb{A}^{n+1} \arrow[l] \arrow[d] \\ V & \operatorname{Spec} k \arrow[l] \end{tikzcd}$ with coordinates $X_1, \cdots, X_n$ as in (2.1)

be the versal deformation of C constructed in (2.3). We construct a projective family $\bar{Y}$ over V as follows. Let $F^\infty = \sum_{i=0}^{\infty} F^i$ be the lifting of F^0 constructed in the proof of (2.3). Similarly let R_j^∞ be the liftings of R_j^0. In these power series substitute formally $t_k \longrightarrow X_{n+1}^{-e_k} t_k$ and call the ensuing power series $\bar{F}^\infty$ and $\bar{R}_j^\infty$. By (2.3) $\bar{F}^\infty$ and $\bar{R}_j^\infty$ are now homogeneous under the $\mathbb{G}_m$ action in the variables $X_0, X_1, \cdots, X_{n+1}$ alone. By hypothesis $e_k \leq 0$, hence by this substitution we introduce no denominator.

Note the $\overline{F}^{\infty}$ define a family over V, since the ideal defining V is weighted. We have

$$\begin{array}{ccc} Y & \hookrightarrow & \overline{Y} \subset \mathbb{P}^{n+1}_V \\ & \searrow \quad \swarrow & \\ & V & \end{array}$$

To finish the proof we must show $\overline{Y} \longrightarrow V$ is flat. If $\overline{Y} = \mathrm{Proj}(A)$ and $V = \mathrm{Spec}(S)$ it suffices to show that A is flat over S. This is equivalent to being able to lift the homogeneous relations among the generators f_i of the ideal of C to homogeneous relations among the $\overline{f}_i^{\infty}$, where $\overline{f}_i^{\infty}$ are the components of the vector $\overline{F}^{\infty}$. But we know the homogeneous relations among the f_i; they are generated by R_j^o. To lift them we need only take $\overline{R}_j^{\infty}$.

5. <u>The theorem on negative grading when X is projectively normal</u>

We keep the notation and hypothesis of (3.1). In particular X is projectively normal. With this extra condition we can improve (4.2):

<u>Theorem</u> (5.1) If X is also projectively normal, then the morphism φ: Hilb $\longrightarrow$ D is smooth.

Hilb and φ are defined in (4.4) and (4.5). "Smooth" means that if B $\longrightarrow$ A is a surjection in $\mathcal{C}$, then the obvious map $\mathrm{Hilb}(B) \longrightarrow \mathrm{Hilb}(A) \times_{D(A)} D(B)$ is a surjection ([43],§2). Thus if H prorepresents Hilb, then H is a formal power series ring over R, the formal moduli space of the vertex of C (cf. (1.4)). Hence we get the algebraizability of R directly, since H is algebraizable ([17]; also [3]) without appealing to Elkik's more general result [12].

By similar techniques we show that C_X has no locally trivial deformation (5.3).

Theorem (5.1) of course is an improvement over (4.2). As it turns out however, (4.2) is sufficient for much of our work. We only need (5.1) in the fine analysis of the deformations of cones over rational and elliptic curves in sections 8 and 9.

<u>Proof</u>. Let B $\longrightarrow$ A be a surjection in $\mathcal{C}$. We may assume that J = kernel(B $\longrightarrow$ A) has square zero, and that $J \cdot m_B = 0$, where m_B is the maximal ideal of B. This makes J

a k-vector space.

By standard obstruction theory we need only show

(a) the map of tangent spaces $\mathrm{Hilb}(k[\epsilon]) \longrightarrow D(k[\epsilon])$ is surjective.

(b) Hilb is "less obstructed" than D. We make this precise below.

<u>Proof of (a)</u>: By Grothendieck [17], $\mathrm{Hilb}(k[\epsilon]) \approx H^0(\overline{C}, N_{\overline{C}})$, where $N_{\overline{C}}$ is the normal sheaf to $\overline{C}$ in $\mathbb{P}_k^{n+1}$. Since the vertex p has depth ≥ 2 (X is projectively normal), and since $N_{\overline{C}}$ is reflexive, by (3.5)

$$H^0(\overline{C}, N_{\overline{C}}) \approx H^0(\overline{E}, N_{\overline{C}}) \approx H^0(\overline{E}, N_{\overline{E}})$$

where $\overline{E} = \overline{C}-p$ ("cone projectif épointé": see [15], §8) and $N_{\overline{E}}$ is its normal sheaf. Now $\overline{E} = V(\mathcal{O}_X(-1))$, so we have a natural affine structure morphism $\pi\colon \overline{E} \longrightarrow X$. It is known that $N_{\overline{E}} \approx \pi^* N_X$, so that

$$H^0(\overline{E}, N_{\overline{E}}) \approx H^0(\overline{E}, \pi^* N_X) \approx H^0(X, \pi_* \pi^* N_X) \approx \sum_{\nu=-\infty}^{0} H^0(X, N_X(\nu)).$$

Since we are assuming $T^1(\nu) = 0$, for ν positive, (3.3) induces a surjection $\mathrm{Hilb}(k[\epsilon]) \approx \sum_{\nu=-\infty}^{0} H^0(X, N_X(\nu)) \longrightarrow T^1 = D(k[\epsilon])$

which is easily seem to be the tangent space map of φ.

Proof of (b): Let $B \longrightarrow A$ be as before. Take $\xi_o \in \mathrm{Hilb}(A)$, $\zeta_o = \varphi(\xi_o) \in D(A)$, and suppose that D is not obstructed on $B \longrightarrow A$, so that there is a $\zeta \in D(B)$ mapping to ζ_o. We must show that Hilb is not obstructed on $B \longrightarrow A$, i.e., there exists a $\xi \in \mathrm{Hilb}(B)$ mapping to ξ_o such that $\varphi(\xi) = \zeta$.

Notice that since we are assuming that ζ_o lifts to ζ, there is no local obstruction to lifting ξ_o: indeed $\overline{C}$ is smooth outside of p, so there is no problem lifting except at p (see [18], exposé III), and ζ gives a lifting there.

By [17], the global obstruction lies in $H^1(\mathbb{P}_B^{n+1}, \mathcal{A})$, where

$$\mathcal{A} = \underline{\mathrm{Hom}}_{\mathcal{O}_{\mathbb{P}_A^{n+1}}}(I(\xi_o), \mathcal{O}_{\xi_o} \otimes_A J) .$$

Here $I(\xi_o)$ is the ideal of ξ_o in $\mathbb{P}_A^{n+1}$ and $\mathcal{O}_{\xi_o}$ is its structure sheaf. $\mathcal{O}_{\xi_o} \otimes_A J \approx \mathcal{O}_{\overline{C}} \otimes_k J$ since $J \cdot m_A = 0$, so $\mathcal{A} = \underline{\mathrm{Hom}}_{\mathcal{O}_{\overline{C}}}(I/I^2, \mathcal{O}_{\overline{C}} \otimes_k J)$ since $\mathcal{O}_{\overline{C}} \otimes_k J$ is annihilated by $m_A \cdot \mathcal{O}_{\xi_o}$. ($I$ is the ideal of $\overline{C}$ in $\mathbb{P}_k^{n+1}$)

We will apply the following lemma to show that $H^1(\mathbb{P}_B^{n+1}, \mathcal{A}) = 0$, thus completing the proof.

Lemma 5.2 Consider the diagram

$$\begin{array}{ccc} \mathbb{P}^{n+1} - p & \xrightarrow{\quad i \quad} & \mathbb{P}^{n+1} \\ \Big\downarrow{\scriptstyle \pi} & & \\ \mathbb{P}^n & & \end{array}$$

where $\mathbb{P}^n$ is the projective space of lines through p and π is the map taking a point $x \in \mathbb{P}^{n+1}-p$ to the line through p and x. Let F be any coherent sheaf on $\mathbb{P}^n$. Then

$$H^1(\mathbb{P}^{n+1}, i_*\pi^*F) = 0.$$

When $\overline{C}$ is the cone over a projectively normal variety X the lemma applies to $\mathcal{O}_{\overline{C}}$, $N_{\overline{C}}$ and $\Theta_{\overline{C}}$ since they all have depth ≥ 2 at p, so that they can be written as $i_*\pi^*F$. To apply the lemma to $\mathcal{A}$ note that $\mathcal{A} = \underline{\mathrm{Hom}}(I/I^2, \mathcal{O}_{\overline{C}} \otimes J)$ has depth ≥ 2 at p by (3.5), so that $H^1(\overline{C}, \mathcal{A}) \hookrightarrow H^1(\overline{E}, \mathcal{A})$. But on $\overline{E}$ $\mathcal{A} = N_{\overline{E}} \otimes J$, so the result follows from that for $N_{\overline{C}}$.

Proof of (5.2). The exact sequence of low degree terms of the Leray spectral sequence for i shows that $H^1(\mathbb{P}^{n+1}, i_*\pi^*F) \hookrightarrow H^1(\mathbb{P}^{n+1}-p, \pi^*F)$. On the other hand since π is affine

$$H^1(\mathbb{P}^{n+1}-p,\pi^*F) = \sum_{\nu=-\infty}^{0} H^1(\mathbb{P}^n, F\otimes\mathcal{O}_{\mathbb{P}^n}(\nu)) .$$

Let σ be the section of π: $\mathbb{P}^{n+1}-p \longrightarrow \mathbb{P}^n$ "at infinity". As we have already seen in the first part of the proof of (5.1), $H^1(\mathbb{P}^{n+1}-p-\sigma(\mathbb{P}^n),\pi^*F) = \sum_{\nu=-\infty}^{\infty} H^1(\mathbb{P}^n,F(\nu))$, so that

combining with the maps above we get an injection:

$$H^1(\mathbb{P}^{n+1},i_*\pi^*F) \hookrightarrow \sum_{\nu=-\infty}^{\infty} H^1(\mathbb{P}^n, F(\nu)).$$

But this map is induced by the inclusion j: $\mathbb{P}^{n+1}-p-\sigma(\mathbb{P}^n) \longrightarrow \mathbb{P}^{n+1}$; since j factors through the affine $\mathbb{P}^{n+1}-\sigma(\mathbb{P}^n)$ the map induced of H^1 must be the zero map. Therefore $H^1(\mathbb{P}^{n+1},i_*\pi^*F) = 0$, proving (5.2).

This completes the proof of (5.1). The proof presented here is a slight rearrangement of that appearing in [35].

Remarks (5.3) We have noticed that (5.2) shows that $H^1(\overline{C},\Theta_{\overline{C}}) = 0$, which means that $\overline{C}$ has no locally trivial deformations. When C has negative grading we can use (5.1) to show that the local and the global deformation functors of $\overline{C}$ are identical, since there is no extra global obstruction. This could conceivably fail in the general case: all we know is that the obstruction in $H^2(\overline{C}, \Theta_{\overline{C}})$ maps to 0 in $H^2(\overline{E}, \Theta_{\overline{E}})$.

(5.4) Our interpretation of the grading on T^1, given in section 2 (which does not rely on projective normality) shows that deformations with "tangent vector" in $\sum_{\nu>0} T^1(\nu)$ cannot be lifted to projective deformations.

(5.5) If $T^1(0) = 0$ we get some extra information. Write $\mathrm{Hilb}_{\overline{C}}$ for Hilb and let Hilb_X be the Hilbert functor of X in $\mathbb{P}^n$. Let $\psi: \mathrm{Hilb}_{\overline{C}} \longrightarrow \mathrm{Hilb}_X$ be the natural morphism of functors obtained by "restricting to the hyperplane at ∞". (4.2) gives a section $\varphi': D \longrightarrow \mathrm{Hilb}_{\overline{C}}$ of φ, so by composing we get $\psi \cdot \varphi': D \longrightarrow \mathrm{Hilb}_X$. When $T^1(0) = 0$ $\psi \cdot \varphi'$ is constant, i.e., all deformations of the cone map to closed point of Hilb_X. We say that "the intersection with the hyperplane at infinity does not move".

CHAPTER II. Deformations of cones over projectively normal curves.

6. Mumford's study of T^1

As always the notation and hypotheses of (3.1) are in force.

(6.1) When dim $X = d \geq 2$, Schlessinger [45] shows that if L is sufficiently ample, $T^1(\nu) = 0$ for all $\nu \neq 0$: just note that $T^1 \hookrightarrow H^1(X, \pi_* \Theta_E)$ by (3.6) and use Serre duality. The point is that $d-1 > 0$. On the other hand he establishes that cones with only $T^1(0) \neq 0$ have only conical deformations, i.e., deformations that arise by taking cones over projective deformations of X in $\mathbb{P}^n$. Therefore such cones have no non-singular deformations.

(6.2) Mumford [31] develops a technique for studying the case where X is a curve. He shows

1) if X is $\mathbb{P}^1$ embedded by $\Theta_{\mathbb{P}^1}(n)$, then $T^1(\nu) = 0$ for all $\nu \neq -1$, and dim $T^1(-1) = 2n-4$.

2) if X has genus ≥ 3, is not hyperelliptic and is embedded by a sufficiently ample invertible sheaf L, then $T^1(\nu) = 0$ for all $\nu \neq 0$. HEnce C_X has no smooth

deformations. Unfortunately the proof does not yield an explicit bound on degree L for this to be true: compare with (7.5).

(6.3) In this section we outline a proof of 1) and then we apply Mumford's techniques to treat the elliptic curve case: $T^1(-1)$ is always $\neq 0$, so we have some non conical deformations. (However the results of section 9 show that for an embedding of degree ≥ 10, all the deformations coming from $T^1(-1)$ are obstructed.) Before that we give a more elementary proof of some of the easier parts of his results which are of special interest to us.

But first we state here, for lack of a better place, some relevant theorems on invertible sheaves on (smooth) curves.

<u>Theorem</u> (6.4) (M. Noether, see [41]) If X is a non-hyperelliptic curve of genus ≥ 3, then Ω_X is (very ample and) projectively normal.

<u>Theorem</u> (6.5) (Mumford [32]) If X is any curve of genus g, then all invertible sheaves of degree $\geq$ 2g+1 are (very ample

and) projectively normal.

We now give an elementary proof of a result contained in [31]:

<u>Lemma</u> (6.6) Let X be a (smooth) curve of genus g embedded by an invertible sheaf L of degree $d > 4g-4$. Then $T^1_C(\nu) = 0$ for all $\nu > 0$, i.e., C_X has negative grading. Also if X is rational $T^1(0) = 0$.

<u>Proof</u>. Consider the following diagram of exact sequences:

$$\begin{tikzcd}
 & & & H^1(X,\Theta_X(\nu)) & \\
H^0(X,\mathcal{O}_X(\nu+1)\otimes V) \arrow[rr, "\alpha"] \arrow[dr] & & H^0(X,N_X(\nu)) \arrow[r] \arrow[ur] & T^1(\nu) \arrow[r] & 0 \\
 & H^0(X,\Theta_{\mathbb{P}|X}\otimes\mathcal{O}_X(\nu)) \arrow[ur] \arrow[drr] & & & \\
H^0(X,\Theta_X(\nu)) \arrow[ur] & & & H^1(X,\mathcal{O}_X(\nu)) &
\end{tikzcd}$$

If $\deg L > 4g-4$, $H^1(X,\Theta_X(\nu)) = 0$ for $\nu \geq 1$

$$H^1(X,\mathcal{O}_X(\nu)) = 0 \quad \text{for} \quad \nu \geq 1 \ .$$

Therefore α is surjective and $T^1(\nu) = 0$ for $\nu \geq 1$. The last statement of the lemma is proved in the same way.

(6.7) We now turn to Mumford's paper. We use * to denote $\underline{Hom}_{\mathcal{O}_X}(\cdot,\mathcal{O}_X)$. Apply * to $\mathcal{O}_X \otimes_k V \longrightarrow \Theta_{\mathbb{P}}(-1) \longrightarrow N_X(-1)$ to obtain $N_X^*(1) \longrightarrow \Theta^*_{\mathbb{P}}(1)/X \longrightarrow \mathcal{O}_X \otimes V^* \cong \mathcal{O}_X \otimes_k \Gamma(X,L)$.

On the other hand consider the product: $X \times X$ with its two projections p_1 and p_2 to X. Let Δ be the diagonal. The sheaf $p_{2,*}[p_1^*L(-2\Delta)] \longrightarrow p_{2,*}[p_1^*L] = \mathcal{O}_X \otimes_k \Gamma(X,L)$ is clearly locally free. Mumford then shows that the two subsheaves $N_X^*(1)$ and $p_{2,*}[p_1^*L(-2\Delta)]$ of $\mathcal{O}_X \otimes \Gamma(X,L)$ are equal by showing that their fibres at the point x are both $\Gamma(Y,m_x^2 \cdot L) \hookrightarrow \Gamma(Y,L)$.

<u>Example</u> (6.8) Following Mumford we compute the normal bundle of the rational curve $X = \mathbb{P}^1$ embedded by $L = \mathcal{O}_{\mathbb{P}^1}(n)$. Indeed in this case $\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(-2\Delta) = p_1^* \mathcal{O}_{\mathbb{P}^1}(-2) \otimes p_2^* \mathcal{O}_{\mathbb{P}^1}(-2)$, so

$$\begin{aligned} p_{2,*}[p_1^*L(-2\Delta)] &= p_{2,*}[p_1^*\mathcal{O}_{\mathbb{P}^1}(n-2) \otimes p_2^*\mathcal{O}_{\mathbb{P}^1}(-2)] \\ &= p_{2,*}[p_1^*\mathcal{O}_{\mathbb{P}^1}(n-2)] \otimes \mathcal{O}_{\mathbb{P}^1}(-2) \\ &= \Gamma(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(n-2)) \otimes \mathcal{O}_{\mathbb{P}^1}(-2). \end{aligned}$$

Hence $N_X \simeq \Gamma(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(n-2))^* \otimes \mathcal{O}_{\mathbb{P}^1}(n+2)$.

We can now compute $T^1_C(\nu)$ for all ν. We already know $T^1(\nu) = 0$ for all $\nu \geq 0$. For $\nu \leq -2$, $\Gamma(X, N_X \otimes L^\nu) = 0$ so $T^1(\nu) = 0$ also in this case. Finally

$$T^1(-1) = \operatorname{coker}[\Gamma(X, \mathcal{O}_X \otimes_k V) \xrightarrow{\alpha} \Gamma(X, N_X \otimes L^{-1})]$$

can be computed, noting that α is injective,

$$\dim \Gamma(X, N_X \otimes L^{-1}) = 3(n-1)$$

$$\dim \Gamma(X, \mathcal{O}_X \otimes V) = n+1,$$

so that

$$\dim T^1(-1) = 2n-4.$$

(6.9) We return to the general case. Once we have $N_X^*(1) = p_{2,*}[p_1^* L(-2\Delta)]$, by manipulations involving Serre duality and the Leray spectral sequence for the morphism p_2, we see $T^1(\nu)$ is the cokernel of the following map (at least when $\deg L > 2g$):

$$H^1(X \times X,\ p_1^*(\Omega_X \otimes L^{-1}) \otimes p_2^* L^{\nu+1})) \longrightarrow$$

$$H^1(X \times X, p_1^*(\Omega_X \otimes L^{-1}) \otimes p_2^* L^{\nu+1} \otimes \mathcal{O}(2\Delta).$$

If X is a non hyperelliptic curve of genus ≥ 2, Mumford shows this map is surjective for all $\nu \neq 0$ [31].

Example (6.10) Now suppose X is elliptic. We already know that $T^1(\nu) = 0$ for $\nu > 0$. It is easy to see that $T^1(0) = 1$ and that if $\deg L = d \geq 5$, $T^1(\nu) = 0$ for $\nu \leq -2$ (the cone over X is then not a complete intersection and is defined by quadrics). The only interesting case is $\nu = -1$. We claim: $\dim T^1(-1) = d$.

Note that $\Delta^2 = 0$, and $\Omega_X = \mathcal{O}_X$, so we want to examine the cokernel of the map α:

$$\alpha: \quad H^1(X\times X, p_1^*(L^{-1})) \longrightarrow H^1(X\times X, p_1^*(L^{-1}) \otimes \mathcal{O}_{X\times X}(2\Delta)).$$

Since the normal bundle of Δ in $X\times X$ is trivial, and factoring:

$$H^1(X\times X, p_1^*(L^{-1})) \longrightarrow H^1(X\times X, p_1^*(L^{-1}) \otimes \mathcal{O}_{X\times X}(\Delta)) \longrightarrow H^1(X\times X, p_1^*(L^{-1}) \otimes \mathcal{O}(2\Delta))$$

it is easy to see α is injective. Now

$$h^1(X\times X, p_1^*(L^{-1}) \otimes \mathcal{O}(2\Delta)) = -\chi(p_1^*(L^{-1}) \otimes \mathcal{O}(2\Delta)$$

(use Serre duality to show both H^0 and H^2 are 0).

Therefore since $\chi(p_1^*(L^{-1}) \otimes \mathcal{O}(2\Delta)) = 2d$ by Riemann-Roch,

$$\underset{\text{dimension } d}{H^1(X\times X, p_1^*(L^{-1}))} \xrightarrow{\alpha} \underset{\text{dimension } 2d}{H^1(X\times X, p_1^*(L^{-1})\otimes\mathcal{O}_{X\times X}(2\Delta))} \longrightarrow \underset{\Rightarrow \text{dimension } d}{T^1(-1)} \longrightarrow 0$$

(6.11) Similarly for hyperelliptic curves in general $T^1(-1) \neq 0$.

7. Smooth deformations of cones over curves

(7.1) The main result of this section is that cones over sufficiently amply embedded curves of positive genus do not have smooth deformations. In outline the proof runs as follows: X is a curve of genus g, $\deg L = d$. By theorem (4.2) since C_X has negative grading (6.6) such a deformation would lift to a smooth deformation of $\bar{C}_X$ in $\mathbb{P}^{n+1}$. We then show that the general fibre would be a rational surface containing a smooth curve of genus g and degree d. But for $d > 4g+4$ if $g \geq 2$ and $d > 9$ if $g = 1$ this contradicts Hartshorne's theorem [20].

Proposition (7.2). 1) Let X be a smooth curve of genus g, embedded by a very ample, projectively normal invertible sheaf $\bar{L}$ of degree $d \geq 2g-1$. Hence $X \xrightarrow{|\bar{L}|} \mathbb{P}^{d-g}$. Assume the projective cone $\bar{C}_X$ in $\mathbb{P}^{d-g+1}$ can be infinitesimally projectively deformed to a smooth surface Y. Then Y is embedded by a projectively normal invertible sheaf L and is rational.

2) If X is non hyperelliptic, and $\bar{L}$ is replaced by the canonical sheaf Ω_X, then Y is a projectively normal K-3 surface (cf. [29] and [40]).

(7.3) Proof of 1). Let X' denote a smooth hyperplane section of Y. X' has genus g and degree d and is projectively normal. A. Mayer [29] has shown that a projective variety with projectively normal hyperplane section is projectively normal. So we only have to show Y is rational.

Let K be a canonical divisor on Y. By the genus formula $(K+X')\cdot X' = 2g-2$ and $(X')^2 = d$, so that $K\cdot X' < 0$. Similarly $nK\cdot X' < 0$ for all n. Therefore $P_n = h^0(Y,\mathcal{O}(nK))=0$ for all n. On the other hand by upper semicontinuity $h^1(Y,\mathcal{O}_Y) \leq h^1(\bar{C},\mathcal{O}_{\bar{C}})$ and $h^1(\bar{C},\mathcal{O}_{\bar{C}}) = 0$ by (5.2). Therefore Castelnuovo's theorem [46] proves the rationality of Y.

<u>Proof of 2)</u>. Let X' be the smooth hyperplane section of Y at infinity (in the notation of section 4, $X_{n+1} = 0$). X' is a projective deformation of X and is therefore also embedded by its canonical bundle. We use the exact sequence

$$(*)_\nu \qquad 0 \longrightarrow L^\nu \longrightarrow L^{\nu+1} \longrightarrow \Omega_{X'}^{\nu+1} \longrightarrow 0$$

where L is of course $\mathcal{O}_Y(1)$. As before $h^1(Y, \mathcal{O}_Y) = 0$, and of course $h^1(X', \Omega_{X'}) = 1$. $(*)_\nu$ shows that $h^2(Y, L^\nu) = h^2(Y, L^{\nu+1})$ for all $\nu \geq 1$; since $h^2(Y, L^\nu) = 0$ for large ν since L is ample, $h^2(Y,L) = 0$. Therefore by $(*)_0$ we see that $h^2(Y, \mathcal{O}_Y) = 0$ or 1. If $h^2(Y, \mathcal{O}_Y) = 0$ then $h^1(Y,L) = 1$. As before, this implies $h^1(Y, L^\nu) = 1$ for large ν, a contradiction since L is ample. Hence $h^1(Y, \mathcal{O}_Y) = 1$, so K is effective. Since $K \cdot X' = 0$, K must be the trivial divisor. Therefore Y is a K-3 surface.

<u>Theorem</u> (7.4) (Hartshorne [20], Theorem 3.5). Let X be a nonsingular curve of genus $g \geq 0$ on a rational surface Y. Then either

a) $g = 0$ and the embedding $X \longrightarrow Y$ is equivalent to a section of a geometrically ruled rational surface; or

b) $g = 1$ and the embedding $X \longrightarrow Y$ is equivalent to a nonsingular cubic curve in $\mathbb{P}^2$, in which case $X^2 = 9$; or

c) $X^2 \leq 4g+4$.

Note: The cases $g = 0$ and $g = 1$ were obtained by Nagata [33].

Combining this theorem with Theorem (4.2), we obtain

Theorem (7.5). Let X be a smooth curve of genus $g \geq 2$ (resp. $g = 1$) embedded by a line bundle of degree $d \geq 4g+5$ (resp. $d \geq 10$). Then the vertex of the cone C_X over X has no smooth deformations.

Proof. By Lemma 6.6 we see C has negative grading, hence by Theorem 4.2 its local deformations can be lifted to projective deformations of $\overline{C}$. Such a projective deformation Y, if it is smooth, must be a rational surface by (7.2). This contradicts Hartshorne's theorem.

Remarks (7.6) i) We did not use (5.1): we only used the existence of a section to φ. We will use the smoothness of φ in our detailed study of genus 0 and 1 (sections 8 and 9).

ii) For non hyperelliptic curves of genus ≥ 3, the only improvement over Mumford's result ([31],d')) is the explicit bound for d. In the elliptic and hyperelliptic case the theorem answers (negatively) a question posed by Mumford ([31], remark B): we do not have nonsingular deformations even though $T^1(-1) \neq 0$.

iii) Suppose that a smooth curve X is the hyperplane section of a smooth surface Y on $\mathbb{P}^m$ so that $\mathcal{O}_Y(1) \otimes \mathcal{O}_X$ is projectively normal. As we have seen already, $\mathcal{O}_Y(1)$ is projectively normal. In that case $\overline{C}_X$ (and hence C_X) has a smooth deformation with generic fibre isomorphic to Y, constructed as follows. Let H be the hyperplane in $\mathbb{P}^m$ cutting out X on Y. Consider $\overline{C}_Y$ in $\mathbb{P}^{m+1}$. $H \hookrightarrow \mathbb{P}^m \hookrightarrow \mathbb{P}^{m+1}$. Let H_t be the pencil of hyperplanes of $\mathbb{P}^{m+1}$ containing H, parametrized so that H_0 passes through the vertex of $\overline{C}_Y$. Then $H_t \cap \overline{C}_Y$ is a flat family of projective surfaces over Spec $k[t]$, with fibre $\overline{C}_X$ above 0 and all other fibres isomorphic to Y. The family if flat because $\overline{C}_Y$ is normal. We call this construction "sweeping out the cone with hyperplane sections".

(7.7) To conclude we examine the "canonical embedding" case, i.e., X non hyperelliptic of genus g, $\overline{L} = \Omega_X$.

Unfortunately in this case $T^1(1)$ is not zero; indeed has dimension at least g.

A simple counting argument (Mayer [29], p. 9) shows that for $g \geq 12$ only a proper subspace of the non hyperelliptic curves of genus g can occur as hyperplane section of a K-3 surface. Take a curve X that does, i.e., X is the hyperplane section of a K-3 surface Y. Then by sweeping out the cone C_Y by hyperplane sections as above, we obtain a curve in the formal moduli space R of C_X with tangent vector $\eta_1 \in \Sigma T^1(\nu)$, $-\infty < \nu < 0$. Another way of deforming C_X is to deform X into curves X' not appearing as hyperplane sections of a K-3 surface. This will give us another curve R with tangent vector η_2 in $T^1(0)$. Clearly the plane spanned by η_1 and η_2 in T^1 must be obstructed (although η_1 and η_2 are not) since otherwise X' would be the hyperplane section of a K-3 surface. Hence the deformation spaces of C_X and $C_{X'}$ will look entirely different.

8. The cone over the rational curve of degree n in $\mathbb{P}^n$

(8.1) We embed $\mathbb{P}^1$ in $\mathbb{P}^n$ by $\mathcal{O}_{\mathbb{P}^1}(n)$. The image X is projectively normal by (6.5). One easily sees that the homogeneous prime ideal I of X is generated by the 2×2 minors of the matrix

$$1)\quad \begin{pmatrix} x_0 & x_1 & x_2 & \cdots\cdots\cdots & x_{n-1} \\ x_1 & x_2 & \cdots\cdots\cdots & & x_n \end{pmatrix}$$

By (6.8) we know $T^1_C(\nu) = 0$ for $\nu \neq -1$, so we can apply Theorem (5.1), and study only the projective deformations of $\overline{C}$, a surface of degree n in $\mathbb{P}^{n+1}$. They are surfaces of the same degree, and such surfaces have a simple classification: the only singular one is $\overline{C}$, the others are rational ruled surfaces, and, if $n = 4$, the Veronese surface in $\mathbb{P}^5$ (i.e., the image of $\mathbb{P}^2$ by $\mathcal{O}_{\mathbb{P}^2}(2)$). (We exclude the trivial cases $n \leq 3$). This classification is a result of Del Pezzo. A modern reference is Nagata [33]. Using this classification we determine here the versal deformation of the cone C_X, by studying the Hilbert scheme at the ruled surfaces and the Veronese.

(8.2) When $n = 4$, the formal moduli space is reduced with two components, of dimensions 3 and 1 meeting transversally, the fibres of which are the rational ruled and the Veronese surface, respectively. When $n \geq 5$ it is a nonsingular $n-1$ dimensional space with a 0-dimensional embedded component at the origin. Thus the reduced formal moduli space is smooth for all n except $n = 4$.

(8.3) The deformations have simple matrix representations. The 2×2 minors of the matrix

$$\begin{pmatrix} x_0 & x_1 & \cdots\cdots\cdots & x_{n-2} & x_{n-1} \\ x_1 - t_1 & x_2 - t_2 & & x_{n-1} - t_{n-1} & x_n \end{pmatrix} \tag{2}$$

describe, for all $n \geq 2$, the deformations, which, after homogenization in an extra variable x_{n+1}, are rational ruled surfaces. These deformations were known previously (Tjurina [47] who states, however, that there are no obstructions), and they form the component of the deformation space that can be "simultaneously resolved" (Brieskorn [9]; also see Artin [4]). When $n = 4$ we can also write $\bar{C}$ as the locus of 2×2 minors of the matrix:

$$3)\qquad \begin{pmatrix} x_0 & x_1 & x_2 \\ x_1 & x_2 & x_3 \\ x_2 & x_3 & x_4 \end{pmatrix}$$

The component of the deformation space corresponding to the Veronese is a line k[t], and the fibre is the locus of 2×2 minors of the matrix

$$4)\qquad \begin{pmatrix} x_0 & x_1 & x_2-t \\ x_1 & x_2 & x_3 \\ x_2-t & x_3 & x_4 \end{pmatrix}$$

which after homogenization is indeed a Veronese surface. This deformation does not admit a simultaneous resolution (Artin [4]) and is the first example of a rational singularity with such a phenomenon (for another example see [37]).

At the end of the section we exhibit the direct computation of the versal deformation using generators and relations, in the case n = 4. It is of some independent interest, since it can serve as a model for all such cone computations. For $n = 5$ we give the equations of the formal moduli space of the singularity.

Proofs.

(8.4) As in Theorem 5.2, H prorepresents $\mathrm{Hilb}_{\overline{C}}$ and R is the formal moduli space of D. By that theorem, $H \cong R[[Z_1,\cdots,Z_N]]$. We first compute N. We have the exact sequence

$$\begin{array}{c} \sum_{\nu=-\infty}^{0} H^0(X,N_X(\nu)) \longrightarrow T_C^1 \longrightarrow 0 \\ \| \\ \text{Tangent space of Hilb.} \end{array}$$

Using (6.8) we know $\dim T_C^1 = 2n-4$ and compute easily that

$$\sum_{\nu=-\infty}^{0} H^0(X,N_X(\nu)) = (n-1)(n+3) + 3(n-1)$$

$$= n^2 + 5n - 6$$

so that $N = n^2 + 3n-2$.

We now use the notation Hilb for the global Hilbert functor of subschemes of $\mathbb{P}^{n+1}$ and Hilb_F for the local Hilbert functor at the subscheme F. Let $f\colon W \longrightarrow \operatorname{Spec} H$ be the total space of $\mathrm{Hilb}_{\overline{C}}$. We consider Spec R as a subscheme of Spec H via the section φ' of φ given in 4.2, and examine the fibres of f above points of Spec R. By the classification theory of surfaces of degree n in $\mathbb{P}^{n+1}$ ([33]) the fibres of f are rational ruled surfaces or cones over rational curves, and, if n = 4, Veronese surfaces. However, by (5.5), there are no cones in the fibres above Spec S except the original $\overline{C}$ above the origin, since $T^1(0) = 0$ (6.6). By [45], §4.3, this is equivalent to the well known fact that all rational curves of degree n in $\mathbb{P}^n$ are projectively equivalent. Therefore the other fibres above Spec R are rational ruled surfaces and, if n = 4, Veronese surfaces.

If we study Hilb_F for a rational ruled surface F in $\mathbb{P}^{n+1}$, we find:

(a) $h^1(F,N_F) = 0$, so that Hilb_F is smooth.

(b) $h^0(F,N_F) = n^2 + 4n - 3$ = (by smoothness) dimension of Hilb at the corresponding point. We defer these standard computations.

Since Hilb_F is smooth, and since Hilb is smooth over D by the theorem, we see that Spec R is smooth at the points with rational ruled surfaces as fibres, of dimension = $h^0(F,N_F)-N = n-1$.

If $n \geq 5$ this shows that Spec R is smooth outside the origin, of dimension n-1. Furthermore there can be only one component through the origin: were there two or more, since they lie in a 2n-4 dimensional space they would intersect outside the origin (they all are of dimension n-1), which would contradict the smoothness of Spec R outside the origin. So for $n \geq 5$ Spec R is irreducible, except perhaps for a 0-dimensional component at the origin. The positive dimensional component is that found by Tjurina [47] and given by Eq. 2; it is obviously nonsingular at the origin. Therefore, since $n-1 < 2n-4 = \dim T^1$, Spec R does have an embedded component of dimension 0. This settles the case $n \geq 5$.

When $n = 4$ we must also study Hilb_V, for the Veronese surface V in $\mathbb{P}^5$. Since V is the image of $\mathbb{P}^2$ by the linear system $\mathcal{O}_{\mathbb{P}^2}(2)$, we get easily:

(a') $h^1(V,N_V) = 0$

(b') $h^0(V,N_V) = 27.$

Since for $n = 4$ $N = 26$, we see Spec R is smooth and has dimension 1 at points corresponding to Veronese surfaces. However we cannot conclude from this analysis that there is only one component corresponding to Veronese surfaces; we prove this by direct computation in (8.6).

We now prove (a) and (b).

Proof of (a). Using the standard exact sequences

$$0 \longrightarrow \Theta_F \longrightarrow \Theta_{\mathbb{P}^{n+1}}|_F \longrightarrow N_F \longrightarrow 0$$

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^{n+1}} \longrightarrow \bigoplus_{n+2} \mathcal{O}_{\mathbb{P}^{n+2}}(1) \longrightarrow \Theta_{\mathbb{P}^{n+1}} \longrightarrow 0$$

and $h^2(F,\mathcal{O}_F) = 0$ since F is rational, it is enough to prove that $h^2(F,\Theta_F) = 0$. Since this is true for $\mathbb{P}^2$, the following lemma completes the proof:

Lemma (6.5) $h^2(F,\Theta_F)$ is a birational invariant of smooth surfaces F.

Proof. By Serre duality $h^2(F,\Theta_F) = h^0(F,\Omega_F^1 \otimes \Omega_F^2)$, Ω_F^1 and Ω_F^2 the bundles of 1 and 2 differential forms. To show this is a birational invariant we proceed as in the case of the plurigenera (see [46], p. 5). The proof works for any bundle Ω of differential forms. Since we are dealing with surfaces, we need only show that $h^0(\Omega)$ is invariant under the blow up π of one point: $F' \longrightarrow F$. Global sections of Ω_F pull back to global sections of $\Omega_{F'}$. Conversely, given a section of $\Omega_{F'}$, by restricting to the open set on which π is an isomorphism, we get a section on $\pi(U)$ of Ω_F. Since $F-\pi(U)$ has codimension 2, this section extends to a global section of Ω_F. This sets up an isomorphism between $H^0(F',\Omega_{F'})$ and $H^0(F,\Omega_F)$, as desired. (6.5) is proved.

<u>Proof of (b)</u>. By the same exact sequences and (a), we see that $h^0(F,N_F) = h^0(F,\Theta_{\mathbb{P}^{n+1}}|_F) - \chi(F,\Theta_F)$. By Riemann-Roch (see [48], p. 127, formula (***))

$$\chi(\Theta_F) = 2(K^2) - 10(\chi(\mathcal{O}_F))$$

where K is the canonical divisor of F. Since F is a rational ruled surface $\chi(\mathcal{O}_F) = 1$; the explicit formula for the canonical divisor on F (given for example in [40], §1) shows that $K^2 = 8$. Hence $\chi(\Theta_F) = 6$ and

$$h^0(F,N_F) = (n+2)^2 - 1 - 6 = n^2 + 4n - 3$$

as claimed.

(8.6) The direct computation.

X is $\mathbb{P}^1$ embedded in $\mathbb{P}^n$ by $\mathcal{O}_{\mathbb{P}^1}(n)$ as before. Using the surjectivity of the map

$$\Gamma(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(\nu)) \longrightarrow \Gamma(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(\nu n) \quad \text{(projective normality)}$$

we obtain without difficulty the following information concerning X. Its ideal I in $P = k[X_0, \cdots, X_n]$ is generated by $f_{ij} = X_i X_j - X_{i+1} X_{j-1}$, $0 \leq i \leq n-2$, $i+2 \leq j \leq n$. There are $N = n(n-1)/2$ such generators. The relations between the f_{ij} are given by

$$R^k_{ij} = X_k f_{ij} - X_i f_{kj} + X_{j-1} f_{k,i+1}, \qquad 0 \leq k < i < j-1,\ j \leq n$$

$$S^k_{ij} = X_{k+1} f_{i-1,j} - X_i f_{kj} + X_j f_{ki}, \qquad k+1 < i < j \leq n,\ 0 \leq k.$$

We want to give an explicit basis for T^1. The well known exact sequence

$$I/I^2 \longrightarrow \Omega_{P/k} \otimes_P B \longrightarrow \Omega_{B/k} \longrightarrow 0$$

with $B = P/I$, gives after dualizing

$$\mathrm{Hom}_B(\Omega_{P/k} \otimes B, B) \longrightarrow \mathrm{Hom}_B(I/I^2, B) \longrightarrow T^1 \longrightarrow 0$$

(see [28]). So we first exhibit a set of elements of $\mathrm{Hom}_B(I/I^2,B)$ from which we later select a basis for T^1. Elements of $\mathrm{Hom}_B(I/I^2,B)$ are N-tuples of elements of B, which we write as column matrices A, such that $R \cdot A = 0$ (matrix multiplication), where R is the $M \times N$ matrix of relations of the f_{ij}, i.e., the entries of R are the coefficients of R^k_{ij} and S^k_{ij}.

<u>Note</u>: M is the number of relations. For convenience we index $(1,2,\cdots,N)$ by a double index (i,j) with $0 \le i \le n-2$ and $i-2 \le j \le n$. (i,j) corresponds to f_{ij} of course. If A is an N-vector, we denote by $A(i,j)$ the corresponding entry. It is easy, albeit tedious, to check that the vectors below are in $\mathrm{Hom}_B(I/I^2,B)$. We list only the nonzero entries.

A_α, $0 \le \alpha \le n-1$. $A_\alpha(j,\alpha+1)=X_j$, $j \le \alpha-1$; $A_\alpha(\alpha,j) = -X_{j-1}$, $\alpha \le j-2$.

B_α, $0 \le \alpha \le n-1$. $B_\alpha(j,\alpha+1)=X_{j+1}$, $j \le \alpha-1$; $B_\alpha(\alpha,j) = -X_j$, $\alpha \le j-2$.

C_0 . $C_0(0,j) = -X_{j-2}$, $2 \le j \le n$.

C_{n-1} . $C_{n-1}(j,n)=X_{j+2}$, $0 \le j \le n-2$.

And if $n \geq 6$,

D_α, $0 \leq \alpha \leq n-6$. $D_\alpha(j,k) = X_{j+k-\alpha-3}$, $0 \leq j \leq \alpha+3$ and $\alpha+5 \leq k \leq n$.

For $n \geq 5$ it is easy to show they are linearly independent (over k); for $n = 4$ we have one relation $C_0 + A_1 + B_2 + C_3 = 0$.

On the other hand an elementary computation shows that the image of $\mathrm{Hom}_B(\Omega_{P/k} \otimes B, B)$ in $\mathrm{Hom}_B(I/I^2, B)$ is generated over B by B_0, $A_\alpha - B_{\alpha+1}$ $(0 \leq \alpha \leq n-1)$, and A_{n-1}.

<u>Claim:</u> A_α $(0 \leq \alpha \leq n-2)$, C_0, C_{n-1} and D_α $(0 \leq \alpha \leq n-6)$ are a k-basis for T^1 if $n \geq 5$. If $n = 4$ delete C_3 to get a basis.

Indeed, these give us $2n-4$ linearly independent elements in T^1.

This completely determines the first order deformations. To get the higher order ones we "lift generators and relations". (For a description of this process see [14], §3.1). We now write F^0 for the column vector (f_{ij}), and F^1 for the first order lifting of F^0 which we have just computed: F^1 is the column vector

$$(6) \qquad \sum_{\alpha=0}^{n-2} t_{\alpha+1}A_\alpha + t_n C_0 + t_{n+1}C_{n-1} + \sum_{\alpha=0}^{n-6} s_{\alpha+1}D_\alpha ,$$

where the t_α and the s_α are parameters.

We lift the matrix of relations R^0 (previously denoted R) to first order; call the first order lifting R^1. Since we must have $R^1 \cdot F^0 + R^0 \cdot F^1 = 0$ (matrix multiplication) and since R^0 (resp. F^0, resp. F^1) is homogeneous of degree 1 (resp. 2, resp. 1) in the X_i, R^1 must be homogeneous of degree 0 in the X_i.

Next we want to lift to second order. By this we mean we want to find a vector F^2 and a matrix R^2, of degree 2 in the parameters t_α and s_α, so that to solve the equation $R^2 \cdot F^0 + R^1 \cdot F^1 + R^0 \cdot F^2 = 0$ we introduce a minimum number of relations among the parameters. This is what is meant by <u>versality</u>. Now $R^1 \cdot F^1$ has degree 1 in the X_i, and thus cannot cancel against $R^2 \cdot F^0$ which has degree at least 2. Therefore we will be able to find a versal lifting with $R^2 = 0$. Then F^2 is homogeneous of degree 0 in the X_i.

Finally the only higher order cross term is $R^1 \cdot F^2$, which has degree 0; it cannot cancel against any of the other third order terms, so it must be identically zero. All the higher R^j and F^j may be taken to be 0.

These homogeneity considerations make it possible to carry out the computation explicitly for small n. When $n = 4$ and t_1, t_2, t_3, t_4 are the parameters corresponding to A_0, A_1, A_2, C_0 respectively, then the formal versal deformation is $k[[t_1, \cdots, t_4]]/J$, with $J = (t_1t_4, t_3t_4, t_4(t_4-t_2))$. Therefore we have 2 components: the hyperplane $t_4 = 0$ and the line $t_1 = t_3 = t_4-t_2 = 0$. The fibres of the deformation space above each of these components are given the 2×2 minors of the matrices (2) and (4) respectively.

We now give the details of the actual computation for $n = 4$. We will only lift the relations R^k_{ij} since the S^k_{ij} can be obtained by symmetry.

As we have seen, F^1 is the column vector

$$\begin{array}{cc}
\left(\begin{array}{llll}
-t_1X_1 & + t_2X_0 & & - t_4X_0 \\
-t_1X_2 & & + t_3X_0 & - t_4X_1 \\
-t_1X_3 & & & - t_4X_2 \\
 & - t_2X_2 & + t_3X_1 & \\
 & - t_2X_3 & & \\
 & & - t_3X_3 &
\end{array}\right)
&
\begin{array}{c}
02 \\ 03 \\ 04 \\ 13 \\ 14 \\ 24
\end{array}
\end{array}$$

The matrix R^0 is (the (i,j,k) row corresponds to R^k_{ij}):

$$
\begin{array}{c}
\\
(0,1,3)\\
(0,1,4)\\
(0,2,4)\\
(1,2,4)
\end{array}
\begin{array}{c}
\begin{array}{cccccc}
02 & 03 & 04 & 13 & 14 & 24
\end{array}\\
\begin{pmatrix}
x_2 & -x_1 & & x_0 & & \\
x_3 & & -x_1 & & x_0 & \\
 & x_3 & -x_2 & & & x_0\\
 & & & x_3 & -x_2 & x_1
\end{pmatrix}
\end{array}
$$

Now

$$
R^0 \cdot F^1 = \begin{pmatrix} -t_4 f_{02}\\ -t_4 f_{03}\\ -t_4 f_{13}\\ 0 \end{pmatrix}
$$

so that R^1 is

$$
\begin{array}{c}
(0,1,3)\\
(0,1,4)\\
(0,2,4)\\
(1,2,4)
\end{array}
\begin{pmatrix}
t_4 & & & & & \\
 & t_4 & & & & \\
 & & & t_4 & & \\
 & & & & &
\end{pmatrix}
$$

Therefore

$$R^1 \cdot F^1 = \begin{pmatrix} -t_4^2X_0 - t_1t_4X_1 + t_2t_4X_0 \\ -t_4^2X_1 - t_1t_4X_2 + t_3t_4X_0 \\ -t_2t_4X_2 + t_3t_4X_1 \\ 0 \end{pmatrix}$$

It is easy to see that to lift to second order we must set $t_1t_4 = t_3t_4 = t_4(t_4 - t_2) = 0$. Then

$$F^2 = \begin{pmatrix} 0 \\ 0 \\ -t_4^2 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

and we are done.

An easy computation shows the formal moduli space is rigid. This also follows from a general result of R. Sheets, Thesis, Brandeis University.

(8.7) We also have computed the versal deformation when $n = 5$. We give only the ideal of the formal moduli space:

$$(t_5t_1,\ t_5t_3,\ t_6t_2,\ t_6t_4,\ t_5^2 - t_6t_1,\ t_5t_4 - t_3t_6,\ t_5t_2 - t_6t_1,\ t_5t_4 - t_6^2)$$

t_5 and t_6 are nilpotent: $t_5^3 = t_6^3 = 0$. Mod (t_5,t_6) we get a polynomial ring on $t_1,\cdots,t_4$, checking our general result.

9. Cones over elliptic curves

(9.1) X is an elliptic curve embedded in $\mathbb{P}^{d-1}$ by a complete linear system of degree $d \geq 5$, and $C_X \subset \mathbb{A}^d$ its cone. We know that $\dim T^1_C(0) = 1$, $\dim T^1_C(-1) = d$ and $T^1_C(\nu) = 0$ for all other ν (6.10). Hence C_X has negative grading. Also if $d \geq 10$, C_X has no smooth deformations (7.5).

(9.2) In this section we improve the results mentioned in (9.1) by using theorem (5.1) and the classification of surfaces of degree d in $\mathbb{P}^d$ (Nagata [33]). Our results are:

(a) When $d \geq 10$ the formal moduli space R of C_X is just a line (the j-line for elliptic curves) with a large nilpotent neighborhood. In other words all the deformations from $T^1(-1)$ are obstructed. It would be interesting to know if R is the product of $k[[j]]$ with an artin ring.

(b) When $5 \leq d \leq 9$, Spec R (the formal moduli space of C_X) has one component V_1 of dimension 11-d containing the j-line V_j, smooth at $V_1 - V_j$, such that the projectivized fibres over $V_1 - V_j$ are biregular to $\mathbb{P}^2$ embedded by a system of cubics with 9-d base points. The fibres above V_j are of course cones over elliptic curves. When d = 8 Spec R has a second component V_2 of dimension 11-8 = 3 smooth at $V_2 - V_j$ with fibres the Veronese transform of a quadric in $\mathbb{P}^3$. V_1 and V_2 intersect along V_j. In all cases Spec R has no other positive dimensional component, except possibly for an embedded component along V_j. In general we do not know how V_1 (and V_2 when d = 8) behave along V_j, but for d = 5 and 6 we have the following results: when d = 5, R is formally smooth by a dimension argument. When d = 6, (9.6) shows that Spec R is the product of V_j with the cone over the Segre embedding of $\mathbb{P}^1 \times \mathbb{P}^2$.

We conclude with the explicit computation in the case d = 6 (9.6).

<u>Theorem</u> (9.3) (Nagata [33]) If F is a surface of degree d in $\mathbb{P}^d$, then F belongs to one of the following 5 classes.

(i) F is the projection from a point x not on the surface of a surface of degree d in $\mathbb{P}^{d+1}$.

(ii) F is biregular to $\mathbb{P}^2$ embedded by a system of cubics with at most six base points, hence $d \leq 9$.

(iii) $d = 8$ and F is the Veronese transform of a quadric in $\mathbb{P}^3$.

(iv) F is a cone over a nonsingular elliptic curve.

(v) $d = 8$ and F is the Veronese transform of a cone over a nonsingular plane conic.

(9.4) We ask which of these surfaces can be projective deformations of $\overline{C}_X$. Certainly not (v), since its hyperplane section is not elliptic, nor (i) since it is not projectively normal (7.3). A result of Schlessinger ([45], §4.3) shows that (iv) occurs, since projective deformations of X lifts to deformations of C_X and it is easily seen that (ii) and (iii) can be realized as deformations of C_X by sweeping out a hyperplane through C_F ((7.6),(iii)).

(9.5) To compute the dimension of the versal deformation Spec R of C_X at points with fibre of type (ii) and (iii) and to show Spec R is smooth there, we proceed as in (8.4).

To find the relative dimension N of $\mathrm{Hilb}_{\overline{C}}$ over D, we compute the dimension of the kernel on tangent spaces:

$$\begin{array}{ccc} \mathrm{Hilb}(k[\epsilon]) & \longrightarrow & D(k[\epsilon]) = T^1_C \\ \| & & \\ \sum\limits_{\nu\leq 0} H^0(X,N_X(\nu)) & & \text{dimension } d+1 \text{ by } (6.10) \end{array}$$

Now (6.10) also shows $h^0(X,N_X(-1)) = 2d$ and a computation using the standard exact sequences shows $h^0(X,N_X) = d^2$. Finally since $T^1(\nu) = 0$ for $\nu \leq -2$ also by (6.10), sequences from [31] (see (6.9)) show that $h^0(X,N_X(\nu)) = 0 \quad \nu \leq -2$. Therefore

$$N = d^2 + d - 1.$$

Next we compute $h^i(F,N_F)$, $i = 0,1$, for surfaces of type (ii) and (iii). As in the rational curve case since $h^2(F,\Theta_F) = 0$ (lemma (8.5)) we obtain $h^1(F,N_F) = 0$ and

$$h^0(F,N_F) = \underbrace{h^0(F,\Theta_{\mathbb{P}^d}|_F)}_{\|\atop (d+1)^2-1} - \chi(F,\Theta_F)$$

By Riemann-Roch (c.f. proof of (b) in (8.4))

$$\chi(\Theta_F) = 2(K^2) - 10$$

where K is a canonical divisor. We must compute K^2.

In case (iii) F is biregular to a (geometrically) ruled rational surface, hence $K^2 = 8$. In case (ii) F is biregular to $\mathbb{P}^2$ with 9-d points blown up. Since K^2 decreases by 1 for each blow up, and $K^2 = 9$ for $\mathbb{P}^2$, $K^2 = d$ for F. So $h^0(F,N_F) = d^2 + 10$ in both cases.

Therefore the components corresponding to both types have dimension $d^2 + 10 - N = 11 - d$, as claimed, and since $h^1(F,N_F) = 0$ they are smooth (since the Hilbert scheme is) at points corresponding to both types.

(9.6) To conclude we carry out the computation for d = 6. Take the elliptic curve $Z_0^3 + Z_1^3 + Z_2^3 = 0$ in $\mathbb{P}^2$. Reembed by $\mathbb{P}^2 \xrightarrow{\mathcal{O}(2)} \mathbb{P}^5$. In this embedding $\mathbb{P}^2$ is defined by

$$f_0 = X_0X_1 - Y_2^2 \qquad g_0 = X_0Y_0 - Y_1Y_2$$

$$f_1 = X_1X_2 - Y_0^2 \qquad g_1 = X_1Y_1 - Y_0Y_2$$

$$f_2 = X_0X_2 - Y_1^2 \qquad g_2 = X_2Y_2 - Y_0Y_1$$

i.e., $\mathbb{P}^2$ is given parametrically by

$$X_0 = Z_0^2 \qquad Y_0 = Z_1Z_2$$

$$X_1 = Z_1^2 \qquad Y_1 = Z_0Z_2$$

$$X_2 = Z_2^2 \qquad Y_2 = Z_0Z_1 .$$

To obtain the elliptic curve we must add three equations

$$h_0 = X_0^2 + X_1Y_2 + X_2Y_1$$

$$h_1 = X_1^2 + X_2Y_0 + X_0Y_2$$

$$h_2 = X_2^2 + X_0Y_1 + X_1Y_0 \; .$$

Note that both the Veronese surface and the elliptic curve are invariant under the projective transformation α such that $\alpha(X_i) = X_{i+1}$, $\alpha(Y_i) = Y_{i+1}$. (All indices are taken mod 3). $\alpha(f_i) = f_{i+1}$, $\alpha(g_i) = g_{i+1}$, $\alpha(h_i) = h_{i+1}$.

We now write down the matrix of relations R^0 between the generators of the ideal of C_X: each row represents a relations. (See Fig. 1.)

Notice the relations are grouped in triples (and one pair) that are permuted under α. Hence we will only have to lift one relation from each group. It is easy to check using projective normality that we have all the generators and relations for the cone over the elliptic curve.

We now exhibit a basis for T^1. (See Fig. 2.) The first six vectors belong to $T^1(-1)$, the last to $T^1(0)$. (Note we write them as row vectors.)

Fig. 1: R^0

	f_0	f_1	f_2	g_0	g_1	g_2	h_0	h_1	h_2
1	X_2	$-X_0$		$-Y_0$		Y_2			
2		X_0	$-X_1$	Y_0	$-Y_1$				
3	Y_0			$-X_1$	$-Y_2$				
4		Y_1			$-X_2$	$-Y_0$			
5			Y_2	$-Y_1$		$-X_0$			
6	Y_1			$-Y_2$	$-X_0$				
7		Y_2			$-Y_0$	$-X_1$			
8			Y_0	$-X_2$		$-Y_1$			
9	$-X_0$				$-X_2$		X_1	$-Y_2$	
10		$-X_1$				$-X_0$		X_2	$-Y_0$
11			$-X_2$	$-X_1$			$-Y_1$		X_0
12			$-X_0$			$-X_1$	X_2		$-Y_1$
13	$-X_1$			$-X_2$			$-Y_2$	X_0	
14		$-X_2$			$-X_0$			$-Y_0$	X_1

Fig. 2: F^1

	f_0	f_1	f_2	g_0	g_1	g_2	h_0	h_1	h_2
v_0	X_2	$-Y_1$	$-Y_0$	$-X_1$	$-X_0$				Y_2
v_1	$-Y_1$	X_0	$-Y_2$		$-X_2$	$-X_1$	Y_0		
v_2	$-Y_0$	$-Y_2$	X_1	$-X_2$		$-X_0$		Y_1	
w_0							X_0	Y_2	Y_1
w_1							Y_2	X_1	Y_0
w_2							Y_1	Y_0	X_2
u							Y_1Y_2	Y_0Y_2	Y_0Y_1

Note here again the symmetry under α. Let

$$F^1 = \sum_{i=0}^{2} t_i v_i + \sum_{i=0}^{2} s_i w_i + ru$$

where t_i, s_i and r are parameters. It is easy to see we have a basis for T^1. We now give the first order lifting R^1 of R^0: see Fig. 3.

Next we lift to second order, i.e., we want to find R^2 and F^2 solving $R^2 \cdot F^0 + R^1 \cdot F^1 + R^0 \cdot F^2 = 0$, introducing a minimum number of relations among s_i, t_i, r. This is done by taking for F^2 the column vector

$$\begin{array}{c} f_0 \\ f_1 \\ f_2 \\ g_0 \\ g_1 \\ g_2 \\ h_0 \\ h_1 \\ h_2 \end{array} \left| \begin{array}{r} s_2t_0 + t_0rY_2 \\ s_0t_1 + t_1rY_0 \\ s_1t_2 + t_2rY_1 \\ t_1^2 - s_2t_2 \\ t_2^2 - s_0t_0 \\ t_0^2 - s_1t_1 \\ - t_0t_2 \\ - t_0t_1 \\ - t_1t_2 \end{array} \right|$$

for R^2 the matrix (rows 3 and 6 are zero) of Fig. 4, and

Fig. 3: R^1

	f_0	f_1	f_2	g_0	g_1	g_2	h_0	h_1	h_2
1							t_1		$-t_0$
3		$-t_2$				$-t_1$		$-t_0$	
6			$-t_1$			$-t_2$	$-t_0$		
9	$-s_0$			$-t_2$	$-s_2-Y_2r$				$-t_1$
12			$-s_0$	$-t_0$		$-s_1-Y_1r$		$-t_1$	

Fig. 4: R^2

	f_0	f_1	f_2	g_0	g_1	g_2	h_0	h_1	h_2
1				t_1r		$-t_0r$			
9						$-t_1r$			
12					$-t_1r$				

introducing the relations
$$s_0t_0 - s_2t_1 = 0$$
$$s_1t_1 - s_0t_2 = 0$$
$$s_2t_2 - s_1t_0 = 0 \quad .$$

For the next order lifting we need introduce no extra relations among the parameters, taking $R^3 = 0$ and F^3 the transpose of

$(t_1t_2r,\ t_0t_2r,\ t_0t_1r,\ 0,\ 0,\ 0,\ -t_1^2r,\ -t_2^2r,\ -t_0^2r)$.

Finally we see that $R^2 \cdot F^2 + R^1 \cdot F^0 \equiv 0$ and $R^2 \cdot F^3 \equiv 0$ mod $s_it_i - s_{i-1}t_{i+1}$ so we are done.

Therefore the formal moduli space is given in $k[[r,s_i,t_i]]$ by the 2×2 minors of the matrix

$$\begin{vmatrix} s_0 & s_1 & s_2 \\ t_1 & t_2 & t_0 \end{vmatrix}$$

so it is the product of the line $k[[r]]$ by the ring of the cone over the Segre embedding of $\mathbb{P}^1 \times \mathbb{P}^2$ in $\mathbb{P}^5$.

CHAPTER III. One-dimensional cones.

10. Computation of T^1 for Gorenstein curves.

In this section we study the dimension of T^1 of any Gorenstein curve singularity, i.e., we do not assume it has a $\mathbb{G}_m$ action. This section can be skipped without loss of continuity. Our main result, which is actually a corollary of the work of Berger, Kunz and Herzog, will be used only once to give an example of an obstructed Gorenstein curve singularity (12.8).

(10.1) A denotes throughout a one-dimensional reduced local ring over k. Since the deformation theory of A only depends on its completion, we will assume A complete. $\bar{A}$ is its normalization, C its conductor ideal. Set

$$\delta = \dim \bar{A}/A \qquad\qquad c = \dim \bar{A}/C$$

$$d = \dim[\mathrm{Coker}(\mathrm{Hom}_A(\Omega_{\bar{A}},C) \longrightarrow \mathrm{Hom}_A(\Omega_A,A)].$$

We write Ω_A for $\Omega_{A/k}$, etc. $\quad i\colon \Omega_A \longrightarrow \Omega_{\bar{A}}$

(10.2) In his thesis Schlessinger showed that the global formal moduli space of a proper reduced curve X, locally a complete intersection, has dimension

$$3g-3 + h^0(X,\Theta_X)$$

where g is the arithmetic genus of X. Rim [38] later showed how to derive from this result the dimension of T^1 for Spec A, A a complete intersection and a domain:

$$\dim T^1 = \delta + \dim(\Omega_{\bar{A}}/i\Omega_A)$$

Of course since A is a complete intersection this is also the dimension of the formal moduli space. A result of Deligne generalizes Rim's formula in this direction:

<u>Theorem</u> (10.3) ([10],2.27) Let E be a component of the formal moduli space of Spec A, A as in (10.1), such that the fibre above the generic point of E is <u>smooth</u>. Then

$$\dim E = 3\delta + d - c.$$

When A is Gorenstein this reduces to the number in Rim's formula: $c = 2\delta$ and $d = \dim(\Omega_{\bar{A}}/i\Omega_A)$. (Note $d = \dim \mathrm{Ext}^1_A(\Omega_{\bar{A}}/i\Omega_A, A)$ and use Gorenstein duality.)

Despite the fact (10.2) and (10.3) are purely local, the proofs are global. It is therefore of interest to obtain local proofs. We obtain a partial result for the dimension of T^1 in case A is a Gorenstein domain.

Recall that since A is reduced, $T^1 = \mathrm{Ext}^1_A(\Omega_A, A)$.

<u>Lemma</u> (10.4) If A is Gorenstein, $\dim T^1 = \dim(\Omega_{A/k})_{\text{torsion}}$.

<u>Proof</u>. By Auslander [5], corollary 8, since Ω_A is finitely generated and $T = (\Omega_A)_{\text{torsion}}$ is the maximum submodule of finite length in Ω_A,

$$\mathrm{Ext}^1_A(\Omega_A, A) = \mathrm{Ext}^1_A(T, A).$$

By Gorenstein duality (Grothendieck [16], 6.3)

$$T = H^0_m(T) \simeq \mathrm{Hom}_A(\mathrm{Ext}^1_A(T,A),\ I)$$

where m = maximal ideal of A, and I is the injective hull of the residue field k of A. Now

$$\dim(\mathrm{Hom}_A(\mathrm{Ext}^1_A(T,A), I) = \dim(\mathrm{Ext}^1_A(T,A))$$

by [24a], 1.36, so that

$$\dim(T) = \dim(\mathrm{Ext}^1_A(T,B)) = \dim(\mathrm{Ext}^1_A(\Omega_A, A)).$$

<u>Remark</u> (10.5). Let X = Spec A be any local curve. It has been conjectured (see Berger [7]) that if $\Omega_{A/k}$ is torsion free then X is regular. If this conjecture is verified, (10.4) implies there are no rigid Gorenstein curve singularities.

(10.6) We establish some notation. Let x be a parameter of A, and let $v = k[[x]]$. $\mathfrak{D}_N, \mathfrak{D}_K, \mathfrak{D}_D$ will denote the Noether, Kähler and Dedekind different, respectively. We recall their definition: let $R \longrightarrow S$ be any ring homomorphism. We follow the presentation of [26].

(1) <u>Noether different</u>. Let $m: S \otimes_R S \longrightarrow S$ be multiplication $m(s_1 \otimes s_2) = s_1 s_2$. Then $\mathfrak{D}_N(S/R)$ is the annihilator of the kernel of m.

(2) <u>Kähler different</u>. Assume $\Omega_{S/R}$ can be finitely presented:

$$F_1 \longrightarrow F_0 \longrightarrow \Omega_{S/R} \longrightarrow 0$$

F_i free and $\dim F_0 = n =$ number of generators of $\Omega_{S/R}$. Then $\mathfrak{D}_K(S/R)$ is the ideal generated by the $n \times n$ minors of the map $F_1 \longrightarrow F_0$ (the 0-th Fitting ideal of $\Omega_{S/R}$).

(3) <u>Dedekind different</u>. This is the classical different. To define it, we must make some restrictions. Let R be a subring of S which is a domain with quotient field K. Assume $S \otimes_R K = L$ is a finite-dimensional separable K algebra, and let $\mathrm{Tr}: L \longrightarrow K$ be the trace. Then

$$S^* = \{x \in L \mid Tr(xs) \in R, \forall s \in S\}$$

$$(S^*)^{-1} = \{y \in L \mid yS^* \subseteq S\}$$

and $$\mathcal{D}_D(S/R) = (S^*)^{-1}.$$

We now return to the specific situation of (10.1). Berger ([7], Satz 6) shows the torsion module T of Ω_A has dimension:

$$\dim \Omega_{\overline{A}}/i\Omega_A + \dim \Omega_{A/v} - \dim \Omega_{\overline{A}/v} - \delta \; .$$

<u>Lemma</u> (10.7) $\dim(A/\mathcal{D}_N(A/v)) = \dim(\overline{A}/\mathcal{D}_N(\overline{A}/v)) + c$ when A is a Gorenstein domain.

<u>Proof</u>. Since A is a complete Gorenstein domain, by Herzog-Kunz [24], 4.7, $\mathcal{D}_N(A/v)\cdot\overline{A} = c\,\mathcal{D}_D(\overline{A}/v)$ and by [24], 4.9 $\mathcal{D}_N(A/v)$ is principal. Therefore by Berger [7],

$$\begin{aligned}\dim(A/\mathcal{D}_N(A/v)) &= \dim(\overline{A}/\overline{A}\cdot\mathcal{D}_N(A/v)) \\ &= c + \dim(\overline{A}/\mathcal{D}_N(\overline{A}/v))\end{aligned}$$

by lemma 3 of [7].

<u>Proposition</u> (10.8) Let A be a Gorenstein domain. Then $\dim T^1 = \dim T = \dim \Omega_{\overline{A}}/i\Omega_A + \delta + \dim \Omega_{A/v} - \dim(A/\mathcal{D}_N(A/v))$.

Proof. Since A is Gorenstein, $c = 2\delta$. Since $\bar{A}$ is regular, $\dim \bar{A}/\mathcal{D}_N(\bar{A}/v) = \dim(\Omega_{\bar{A}/v})$ so combining with the formulas above we are done.

Corollary (10.9) Assume Spec A, A=Gorenstein domain, has smooth deformations. Then

$$\dim \Omega_{A/v} \geq \dim A/\mathcal{D}_N(A/v)$$

with equality holding iff the deformations of Spec A are unobstructed. If A is a complete intersection a result of Kunz [26] shows that equality holds, hence reproving Rim's formula locally.

Proof. By (10.3), since Spec A has smooth deformations a component of the moduli space has dimension $\dim \Omega_{\bar{A}}/i\Omega_A + \delta$. Conclude by (10.8).

11. Deformations of cones of lines

We study in this section the deformations of a cone C_X where X consists of d reduced points in linear general position in $\mathbb{P}^n$, by which we mean no hyperplane $H \subset \mathbb{P}^n$

contains more than n points of X. We prove a criterion for C_X to have negative grading and apply it to show that certain "cones of lines" have no smooth deformations.

<u>Theorem</u> (11.1). If X consists of d points in linear general position in $\mathbb{P}^n$, with $n+1 \leq d \leq 2n+1$, then C_X has negative grading (4.1).

The lower bound on d is mentioned only to insure that C_X is minimally embedded. We can state this result more geometrically as follows: let C_X be as in (11.1), and suppose Y is a curve with singular point y such that the tangent cone of Y at y is formally isomorphic to C_X. Then Y is formally isomorphic to C_X at y. More generally

<u>Proposition</u> (11.2) Let C be <u>any</u> cone with negative grading. Then any variety Y with tangent cone at $y \in Y$ formally isomorphic to C is itself formally isomorphic to C.

Of course by M. Artin's general results the formal isomorphism can be descended to an ètale one.

<u>Proof of (11.2)</u>. Any variety Y can be considered as a deformation of its tangent cone in the following well known way:

Let J be the ideal of Y in $k[X_1,\cdots,X_m]$. Pick a set of generators f_j of J also generating the ideal of the tangent cone. We have placed y at the origin. Now replace in the generators f_j the X_i by tX_i, and factor out in each generator the largest possible power of t. Call the elements thus obtained $g_j(X_i,t)$. It is clear $g_j(X_i,0)$ generate the tangent cone. The family $g_j(X_i,t)$ over $k[[t]]$ is flat with closed fibre isomorphic to the tangent cone and general fibre isomorphic to the variety we started out with.

We now use the negative grading hypothesis to show this deformation is trivial: indeed the monomials of $g_j(X_i,t)$ involving t have strictly higher degree (in the X_i) than those not involving t. Therefore this deformation belongs to the positive part of the grading and is trivial, finishing the proof.

<u>Proof of 11.1</u>.

Let B be the homogeneous coordinate ring of X, hence the affine ring of C_X, localized at the vertex. For simplicity of notation we usually suppress localizations. The normalization $\overline{B}$ of B is $\bigoplus_{\alpha=1}^{d} k[Y_\alpha]$ and the map $B \xhookrightarrow{\varphi} \overline{B}$ is graded in the obvious manner. We write the graded pieces

B_ν and $\overline{B}_\nu$ respectively.

Lemma (11.3). $\varphi_\nu\colon B_\nu \longrightarrow \overline{B}_\nu$ is surjective for all $\nu \geq 2$.

Proof. For each point P_α of X we can find a quadric R_α in $\mathbb{P}^n$ not passing through P_α, but passing through all the other points of X. Indeed since $d \leq 2n+1$ we can do this with two hyperplanes. Clearly $R_\alpha \in B_2$ maps to $(0,\cdots,cY_\alpha^2,\cdots,)$ so we get surjectivity.

Remark (11.4). Saint-Donat [39] uses (11.3) to prove that if $d \leq 2n$ the ideal of X in $\mathbb{P}^n$, hence that of C in $\mathbb{A}^{n+1}$ is generated by quadrics. We will not need this result.

We return to the proof of (11.1). Assume $T^1(\mu) \neq 0$ for some $\mu > 0$. Pick an element $G \in T^1(\mu)$. By the exact sequence

$$\mathrm{Hom}_B(\Omega_P \otimes B, B) \longrightarrow \mathrm{Hom}(I/I^2, B) \longrightarrow T^1 \longrightarrow 0$$

we can view G as an element of $\mathrm{Hom}(I/I^2,B)$. (We are using the notation of (2.6)). We will show that G lies in the image of $\mathrm{Hom}_B(\Omega_P \otimes B, B)$.

Since $T^1(\nu) = 0$ for $\nu >> 0$, for all α the element $Y_\alpha^\nu \cdot G \in \mathrm{Hom}(I/I^2, B)$ maps to zero in T^1. Hence

$$Y_\alpha^\nu \cdot G = \sum_{k=1}^{n} L_{k,\alpha} \frac{\partial F^0}{\partial X_k} \qquad \text{with} \quad L_{k,\alpha} \in B_{\mu+\nu+1}$$

since $\frac{\partial F^0}{\partial X_k}$ has degree -1 for all k (see 2.7). Here G and $\frac{\partial F}{\partial X_k} = \left(\frac{\partial f_i}{\partial X_k}\right)$ are column vectors and $L_{k,\alpha}$ and of course Y_α^ν are scalars. By taking ν sufficiently large we may assume $L_{k,\alpha} = c_{k,\alpha} Y_\alpha^{\mu+\nu+1}$, $c_{k,\alpha} \in k$. Write $c_{k,\alpha} Y_\alpha^{\mu+\nu+1} = (c_{k,\alpha} Y_\alpha^{\mu+1}) Y_\alpha^\nu$. The crucial fact is that since $\mu \geq 1$, $Y_\alpha^{\mu+1} \in B$ and not only $\overline{B}$, by (11.3). Then it is easy to see that

$$G = \sum_\alpha \sum_k c_{k,\alpha} Y_\alpha^{\mu+1} \frac{\partial F^0}{\partial X_k} = \sum_k \sum_\alpha c_{k,\alpha} Y_\alpha^{\mu+1} \frac{\partial F^0}{\partial X_k}$$

and hence is trivial in T^1. (11.1) is proved.

(11.5) Combining (9.1) with (11.1) we see that to study the deformations of C it suffices to study the projective deformations of $\overline{C}$. Since all such deformations of $\overline{C}$ will have the same arithmetic genus g as $\overline{C}$, let us compute it. V, the normalization of $\overline{C}$, consists of d disjoint rational curves. The exact sequence $0 \longrightarrow \mathcal{O}_{\overline{C}} \longrightarrow \mathcal{O}_V \longrightarrow \mathcal{O}_V/\mathcal{O}_{\overline{C}} \longrightarrow 0$ shows that $g = \delta-d+1$, where $\delta = h^0(\mathcal{O}_V/\mathcal{O}_{\overline{C}}) = \dim_k \overline{B}/B$. Since the cokernel of $B \longrightarrow \overline{B}$ on the zeroth (resp. first) graded piece has dimension d-1 (resp. d-n+1) and since for the higher graded pieces φ is surjective (11.3), we see $\delta = 2(d-1)-n$. Therefore $g = d-n-1$.

Note. (11.6) We see that at least in the range $d \leq 2n+1$, g depends linearly on d. For general d this is not true: each time a line is added the cokernel on each graded piece increases by one, roughly speaking.

(11.7) Next we study the d-tuples of points of $\mathbb{P}^n$ which can be a hyperplane section of a smooth curve of genus g in $\mathbb{P}^{n+1}$. Since we have dn independent choices for the coordinates of the points, and since PGL(n+1) has dimension n^2+2n, it is clear that the "moduli space" of d-tuples of points of $\mathbb{P}^n$ form a variety M of dimension $dn-n^2-2n = n(g-1)$. On the other hand, it is easy to find an upper bound for the dimension of the subvariety M_1 of d-tuples appearing as hyperplane sections: there are ∞^{3g-3} curves of genus g, with ∞^g line bundles of degree d and ∞^{n+1} hyperplane sections; therefore we get at most a $3g-3 + g + n+1 = 4g + n-2$ dimensional family. Therefore as soon as

$$(11.8) \qquad n(g-1) > n + 4g - 2$$

M_1 is a subspace of dimension strictly less than that of M.

<u>Proposition</u> (11.9). Take any d-tuple of points $X \in M \setminus$ (closure M_1). Then C_X has no smooth deformations.

<u>Proof</u>. Suppose it did have a smooth deformation. We know then that $\overline{C}_X$ has a smooth projective deformation. Restrict the deformation to the hyperplane at infinity, i.e., in the notation of the proof of (4.1) set $X_{n+1} = 0$. Then we have a family of d-tuples of points in $\mathbb{P}^n$, hyperplane sections of smooth curves in $\mathbb{P}^{n+1}$, degenerating to X, a contradiction to the hypothesis.

Let us now analyze the information contained in (11.8). For $g = 0$, 1 or 2 we get nothing. For $g \geq 3$ we obtain

<u>Theorem</u> (11.10). The general cone consisting of $g+n+1$ lines in $\mathbb{P}^{n+1}$, $3 \leq g \leq n$, does not have smooth deformations for $n > \frac{4g-2}{g-2} = 4 + \frac{6}{g-2}$.

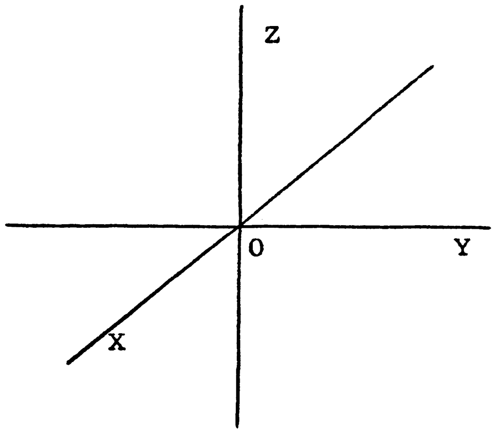
For instance 13 lines in $\mathbb{P}^7$, or 15 lines in $\mathbb{P}^{12}$ cannot in general be smoothed by deformation. This complements the result of Mumford [30].

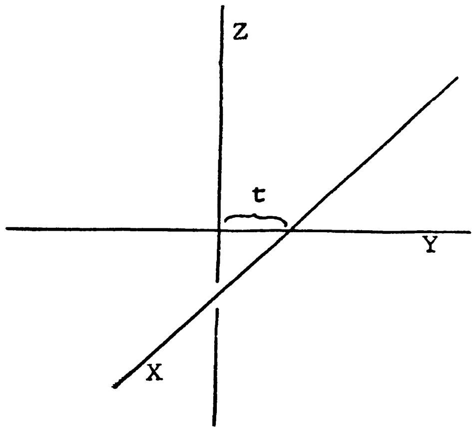
<u>Remarks</u> (11.11). It should be possible to give another proof of (11.10) using a result of Deligne (10.3). With the same notation as before we get an $n(g-1)$-dimensional family of deformations of C_X by taking the cone C_M over a neighborhood of X in M. It is clear that if all the cones in C_M have smooth deformations there will be a component in the formal moduli space of C_X containing C_M with smooth generic fibre. It would have dimension $\geq n(g-1) + 1$. Deligne's result (10.3) tells us exactly what this dimension is; comparing the two numbers we get a contradiction for d and n sufficiently large. Unfortunately the computation in the case of lines of the invariants appearing in (10.3) is awkward, and the result obtained seems weaker than that of (11.10).

(11.12) The bound obtained in (11.10) is probably not the best possible. A more detailed analysis, including a study of the automorphisms of C_X, should improve it; in particular show that when $g = 2$ we also obtain a bound on n.

(11.13) For $g = 0$ and $g = 1$ it is easy to see that the corresponding cone C of $g+n+1$ lines in $\mathbb{P}^{n+1}$ is smoothable. Indeed in this case for each n there is only one cone (in other words $T^1(0) = 0$), so one gets a smooth deformation as follows: let Y be a smooth curve of genus 0 (resp. 1) embedded by a line bundle of degree $n+1$ (resp. $n+2$) in $\mathbb{P}^{n+1}$. Y is projectively normal by (3.5). So just sweep out C_Y by hyperplane sections (cf. (5.6),iii).

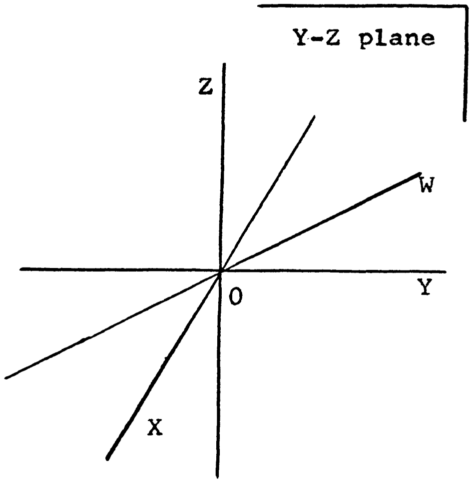
On the other hand it is possible to see this "combinatorially". We treat the genus 0 case first, following Rim. We can deform the system of lines (which in this case has normal crossings, i.e., independent tangent directions) by pulling one line out of the vertex along any other line.

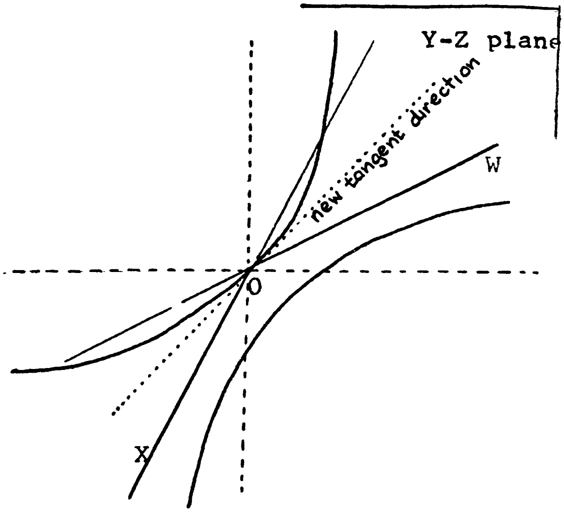




It is easy to see this does not change the arithmetic genus of the configuration. We may now continue by induction on the number of lines with normal crossings.

The case $g = 1$ is more subtle, since we cannot just pull out a line as above: that would lower the genus. Instead we take two lines and deform them, in their plane, to a smooth component still passing through the vertex, with tangent direction depending on those of the remaining lines. We then have the same configuration for n-1, so we continue by induction as before.





the Z and Y axes have deformed to a hyperbola through the origin with tangent direction depending on that of X and W

(11.14) The main result of M. Schaps' thesis [42] states that singular curves in $\mathbb{P}^3$ can be smoothed by deformation. If we examine her proof we note it implies, in the case of a cone of d lines that the smoothing deformation takes place in $\sum_{\nu \leq 0} T^1(\nu)$. Therefore as in (4.1) we may lift it to a deformation of the projective cone. We do not need to make any restriction on d, the number of lines. Let g(d) be the arithmetic genus of the cone: as d becomes large, g(d) increases rapidly. The existence of a smooth deformation implies the existence of a smooth projectively normal curve in $\mathbb{P}^3$ of degree d and genus g(d), an interesting fact since the embedding line bundle will be very special for large g. The dimension of the versal moduli space of the cone should give some information on the dimension of the family of curves with such a line bundle, but the presence of a large group of automorphisms complicates the problem.

Finally, since for a large d, the number of generators of the ideal of C is quite large, we see this implies that we can find smooth projective curves in $\mathbb{P}^3$ for which an arbitrarily large number of generators is required to generate the homogeneous prime ideal.

(11.15) The following question is of some interest: given any $d \geq n+1$, consider any cone of d lines in $\mathbb{P}^{n+1}$

in as general position as possible. Let $g(d)$ be the arithmetic genus of such a cone. Then we ask: do <u>any</u> of these cones have a smooth projective deformation. Note that in the range we considered earlier, it was clear that some such cones did. As d increases, such a smooth projective deformation corresponds to a more and more special embedding. We have just seen that for $n = 2$ such embeddings always exist. What happens for $n > 2$?

CHAPTER IV. Monomial Curves.

12. Monomial curves: definitions and study of T^1

By monomial curve we mean an irreducible affine curve with $\mathbb{G}_m$ action (2.1). It is easiest to study them in parametric form, and to do so we develop some terminology, following Herzog and Kunz [24].

(12.1) H will denote throughout a sub-semigroup of the additive semigroup $\mathbb{N}$ of non-negative integers. Hence $0 \in H$. We always assume the greatest common divisor of the elements of H is 1, so that there are only a finite number of elements of $\mathbb{N}$ not in H. Such elements are called the gaps of H. The number of gaps is called the genus of H, noted $g(H)$. The smallest integer c such that $c + \mathbb{N} \subset H$ is the conductor of H.

Definition (12.2). Let B_H be the subring of the polynomial ring $k[t]$ generated by the monomials t^h, $h \in H$. B_H is called the semigroup ring of H.

Since we deal with only one semigroup H at a time, we write B for B_H. If $\overline{B} = k[t]$, then $g(H) = \dim \overline{B}/B$, and if C is the conductor ideal of B in $\overline{B}$, $\dim \overline{B}/C = c$.

Let $a_1 < a_2 < \cdots < a_n$ be the minimal set of generators of H. We write B as the quotient of the polynomial ring $P = k[X_1,\cdots,X_n]$ by sending X_i to t^{a_i}. If I is the kernel of this map, it is obvious I is invariant under the $\mathbb{G}_m$ action on P with weight a_i on X_i (2.1).

Note (12.3) If n = 2, so H is generated by 2 elements, then I is of course principal. If n = 3, Herzog [23] showed that I is generated by either 2 (in which case it is a complete intersection) or 3 elements. However for $n \geq 4$ H. Brezinsky [8] has shown that I may require an arbitrarily large number of generators.

Lemma (12.4) Let Ω_B be as usual the module of differentials of B over k. Then Ω_B has torsion unless $H = \mathbb{N}$, i.e., unless B is regular.

Proof. Ω_B is the quotient of the free module $dX_1,\cdots,dX_n$ by relations $\sum_{i=1}^{n} \frac{\partial f}{\partial X_i} dX_i$, $f \in I$. We grade Ω_B by assigning to dX_i the weight a_i. Note that in this grading all the relations have degree strictly larger than a_1+a_2.

To test if an element of Ω_B is torsion we have the map $\Omega_B \xrightarrow{\alpha} \Omega_K$, K = quotient field of B, defined as follows: $dX_i \longrightarrow a_i t^{a_i - 1} dt$. It is clear that $\omega \in \Omega_B$ is torsion iff $\alpha(\omega) = 0$. Let

$$\omega = a_1 X_1 dX_2 - a_2 X_2 dX_1.$$

Since ω has degree $a_1 + a_2$ it is clearly not 0 by the above remark unless both a_1 and a_2 are divisible by the characteristic of k, in which case the lemma is trivial.

Now $\alpha(\omega) = a_1 t^{a_1} a_2 t^{a_2 - 1} dt - a_2 t^{a_2} a_1 t^{a_1 - 1} dt$

$= 0.$

Therefore ω is a nonzero torsion element of Ω_B.

By (10.4), $(\Omega_B)_{torsion} \neq 0$ is equivalent in the case of Gorenstein curves to $T^1 \neq 0$. For a general semi-group ring we do not know if $T^1 \neq 0$, though this should not be too difficult to prove using the grading as in the following:

Lemma (12.5) For all monomial curves C, $T^1_C(0) = 0$.

Proof. We use the terminology of (8.6). The proof is very similar to that of (10.1). Pick a nonzero element $G \in T^1(0)$ and consider it as an element of $\mathrm{Hom}(I/I^2, B)$.

The singularity of C being isolated the element $t^m \cdot G$ is trivial in T^1 for m sufficiently large; therefore can be written as

$$\sum_{i=1}^{n} c_i t^{m+a_i} \frac{\partial}{\partial X_i}$$

since $\frac{\partial}{\partial X_i}$ has degree $-a_i$. Therefore, since $t^{a_i} \in B$, we see that

$$G = \sum_{i=1}^{n} c_i t^{a_i} \frac{\partial}{\partial X_i} = \sum_{i=1}^{n} c_i X_i \frac{\partial}{\partial X_i} .$$

Similarly we can prove the following criterion for negative grading.

Lemma (12.6) Suppose there is only one gap of H greater than a_1. Then C, the monomial curve associated to H, has negative grading.

Proof. Using the relation $\sum a_i X_i \frac{\partial}{\partial X_i}$ in $\mathrm{Hom}(I/I^2, B)$ we can push through the proof of (12.5) with only n-1 on the $\frac{\partial}{\partial X_i}$.

Remark (12.7) Consider the semigroup generated by 3 and 5. There are two gaps, 4 and 7, after $a_1 = 3$. Yet one sees easily that $X_1^5-X_2^3$ has negative grading: the element of T^1 with greatest weight is $X_1^3X_2 \in T^1(-1)$. Therefore the condition is not a necessary one.

On the other hand it is easy to construct semigroups without negative grading. Take for example the semigroup generated by 4 and 5. It has 3 gaps: 6, 7 and 11, after $a_1 = 4$. $X_1^5-X_2^4$ does not have negative grading: the element $X_1^3X_2^2 \in T^1(2)$ is not trivial.

(12.8) To conclude we give an example of an obstructed Gorenstein monomial curve in codimension 4 (we were not able to find any in codimension 3).

Let H be the semigroup generated by 6,7,8,9,10. The conductor c of H is 12. H is "symmetric" in the following sense: if a is a gap of H, then $c-1-a = 11-a$ is not. It is known this implies that B_H is Gorenstein [24]. In section 14 we will show that B_H has smooth deformations, hence by (10.3) its formal moduli space has a component of dimension $\dim \Omega_{\overline{B}}/i\Omega_B + \dim \overline{B}/B$. Then by (10.9) to show the deformation space is obstructed we need only show $\dim \Omega_{B/v} > \dim B/\mathscr{D}_N(B/v)$, where $v = k[X_1]$. By Herzog-Kunz

[24], 4.4 the right hand side is $c + a_1 - 1 = 17$. We will now show by a brute force computation that $\dim \Omega_{B/v} = 20$, concluding the proof.

(12.9) It is easily seen that the ideal I can be generated by the 2×2 minors of the following 2 matrices.

$$\begin{vmatrix} x_1 & x_2 & x_3 & x_4 \\ x_2 & x_3 & x_4 & x_5 \end{vmatrix} \qquad \begin{vmatrix} x_1 & x_2 & x_3 & x_5 \\ x_3 & x_4 & x_5 & x_1^2 \end{vmatrix}$$

From this it is easy to compute $\Omega_{B/v}$: it is the quotient of the free module over B generated by four elements $w_i = dX_i$, $i = 2,\cdots,5$ by the relations given by the following matrix:

$$\begin{array}{c} \begin{array}{cccc} w_2 & w_3 & w_4 & w_5 \end{array} \\ \left|\begin{array}{cccc} -2t^7 & t^6 & & \\ -t^8 & -t^7 & t^6 & \\ -t^9 & & -t^7 & t^6 \\ & -2t^8 & & t^6 \\ t^{10} & -t^9 & -t^8 & t^7 \\ & t^{10} & -2t^9 & t^8 \\ -t^{12} & & t^{10} & t^9 \\ & -t^{12} & & 2t^{10} \\ -2t^{13} & & & 2t^{10} \end{array}\right| \end{array}$$

An elementary computation using the grading then shows that $\Omega_{B/V}$ has as a basis over k the starred elements $t^i w_j$

	w_2	w_3	w_4	w_5
t^0	*	*	*	*
t^6	*			
t^7	*	*		
t^8	*	*	*	
t^9	*	*		
t^{10}	*	*	*	
t^{12}	*			
t^{13}	*			
t^{14}	*			
t^{15}	*			
t^{16}	*			

There are 20 such elements.

13. Monomial curves and Weierstrass points

In this section we will show how to construct from the versal deformation space of the semigroup ring B_H a compactification of a "moduli space" for smooth algebraic curves with a Weierstrass point of semigroup H. Definitions and precise statements are made below.[1]

(13.1) Let C = Spec B, where B is the semigroup ring of H (12.2). Then by (2.3) a formal versal deformation of C:

$$\begin{array}{ccc} Y & \longleftarrow & C \\ \downarrow & & \downarrow \\ V & \longleftarrow & \operatorname{Spec} k \end{array}$$

can be constructed so that $\mathbb{G}_m$ acts compatibly on Y, V and C. In the notation of (2.9) we have

$$V = \operatorname{Spec} R, \qquad R = k[[t_1,\cdots,t_r]]/J$$

$$Y = \operatorname{Spec} A, \qquad A = R[[X_1,\cdots,X_n]]/(F^{\infty})$$

and $\mathbb{G}_m$ acts with weight a_i on X_i and weight $-e_i$ on t_i.

Unfortunately it is easy to give examples of semigroup rings for which some of the e_i are positive: for example

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[1] See (1.20)

take the semigroup generated by 4 and 5 and the deformation: $X_1^5 - X_2^4 + tX_1^3X_2^2$. t has weight -2, hence the corresponding e is 2. In other words the action of $\mathbb{G}_m$ on R may be partly attractive, partly repulsive. We will take the quotients R' and A' of R and A respectively obtained by setting all t_k with $e_k > 0$ equal to zero. Since by (12.5) $T^1(0) = 0$ all the remaining variables t_i have weight $-e_i$ positive. More geometrically we intersect Spec R with the linear subspace of Spec $k[[t_1,\ldots,t_r]]$ on which $\mathbb{G}_m$ acts attractively. This puts us in a situation of negative grading as in our preceding work. In the terminology of Orlik and Wagreich [34] R' and A' have "good $\mathbb{G}_m$ action". Let Y' = Spec A', V' = Spec R'. Note that the construction of (2.9) shows that the generators of the ideal of R' and A' in $k[[t_i]]$ and $k[[t_i,X_j]]$ are polynomials; therefore we replace R' and A' by the corresponding quotient rings of $k[t_i]$ and $k[t_i,X_j]$, which we continue to call R' and A'.

(13.2) We have a flat graded morphism φ: $R' \longrightarrow A'$, where R' and A' are graded in positive degrees. Following Grothendieck [15], 3.5, this gives a rational map

Proj A' ---→ Proj R' which is a morphism except at the unique point of Proj A' above the irrelevant ideal R_+ of R. It is natural to blow up the ideal R_+A in A to obtain a morphism from the blown up variety to Proj R. Unfortunately in general it is not true that this morphism is flat. This arises from the $\mathbb{G}_m$ quotient we are taking in the base variety Spec R'.

(13.3) Instead we will projectivize the fibres without projectivizing the base. This is done as follows: in the equations f_k^∞ of A' substitute $t_iX_{n+1}^{-e_i}$ for t_i. Recall $-e_i > 0$ for all i by (13.1). Call the polynomials so obtained $\overline{f}_k^\infty$. Then let $\overline{A}$ be the ring $R'[X_1,\cdots,X_{n+1}]/(\overline{f}_k^\infty)$. (Since R' is graded with weights $-e_i$, $\overline{A}$ is an R' algebra). As in the proof of (4.1) we see that $\overline{A}$ is flat over R'. We consider $\overline{A}$ as a graded algebra in $X_1,\cdots,X_{n+1}$ alone with $X_1,\cdots,X_n$ having the same weights a_i as before and X_{n+1} having weight 1. Thus

<u>Proposition</u> (13.4). The morphism

$$\pi:\ X = \mathrm{Proj}\ \overline{A} \longrightarrow \mathrm{Spec}\ R' = V'$$

is flat and proper with fibres reduced projective curves.

Let σ be the section of π obtained by $X_{n+1} = 0$.

<u>Definition</u> (13.5) Let U be the open set of points $u \in V'$ such that the fibre of Spec A' $\longrightarrow$ Spec R' above u is smooth. U is obviously $\mathbb{G}_m$ invariant.

U may of course be void.

<u>Remark</u> (13.6) All the fibres of Proj $\bar{A} \longrightarrow$ Spec R' (and for that matter Spec A' $\longrightarrow$ Spec R') above a given $\mathbb{G}_m$ orbit of Spec R' are isomorphic.

Before we state our main theorem we make some definitions for smooth projective curves.

<u>Definition</u> (13.7) Let P be a point of a smooth projective curve X. Let H_P be the following semigroup of non negative integers: $h \in \mathbb{N}$ is in H_P iff there exists a meromorphic function on X, holomorphic on X-P, of degree exactly h. H_P is called the semigroup of P.

The genus g of the semigroup H_P (12.1) is equal to the genus of X. If the first nonzero element of H_P is g+1, P is called an ordinary point of X. If not, P is called a Weierstrass point. In characteristic 0 all curves have only a finite number of Weierstrass points; rational and elliptic curves have none. For an introduction to Weierstrass points, see Gunning [19], p. 120.

(13.8) $\mathcal{M}_{g,1}$ will denote the coarse moduli space of smooth projective curves of genus g with a section ("pointed" curves of genus g). Consult [11] and [32a] for definitions and details.

Theorem (13.9) Given a semigroup H, construct $\pi: X \longrightarrow V'$ with its section σ as in (13.4). Restrict π and σ to $U \hookrightarrow V'$. The fibres of $\pi: X \times_{V'} U \longrightarrow U$ are smooth projective curves of genus $g = g(H)$, and σ picks out a point of semigroup H. Hence we get a morphism $U \longrightarrow \mathcal{M}_{g,1}$, which by (13.6) factors through the quotient $\overline{U}$ of U by $\mathbb{G}_m$. Then $\overline{U} \longrightarrow \mathcal{M}_{g,1}$ is a bijection of $\overline{U}$ with the subscheme of $\mathcal{M}_{g,1}$ parametrizing pairs $\{(X_0,P) \mid X_0$ smooth projective curve of genus g, $P \in X_0$ a point of semigroup $H\}$.

The next few pages will be devoted to a proof of (13.9).

<u>Theorem</u> (13.10) With the same notation as before consider the family $W = \operatorname{Spec} \bar{A} \longrightarrow V'$. The fibre W_u above any point $u \in U$ is an affine surface with a normal singularity and a good $\mathbb{C}_m$ action. W_u is obtained from X_u, the fibre of $X \longrightarrow V'$ as follows: let $P = \sigma(u)$. Embed X_u in the affine line bundle Z with sheaf of sections $\mathcal{O}_{X_u}(-P)$, i.e., $Z = V(\mathcal{O}_{X_u}(P))$. X_u has self intersection -1 in Z. Then W_u is isomorphic to Z with X_u blown down to a point. $P = \sigma(u)$ has semigroup H, hence X_u has genus $g(H)$.

<u>Proof of (13.10)</u>. We first show W_u is normal. Let T be the closure of the orbit of u in V', and let $\bar{T}$ be its normalization.[1] Consider the cartesian diagram

$$\begin{array}{ccc} W_0 & \longrightarrow & \operatorname{Spec} A' \\ \downarrow & & \downarrow \\ \bar{T} & \longrightarrow & V' \end{array}$$

Then by construction W_u is isomorphic to W_0. It is easy to see that W_0 is normal by Serre's criterion ([1],VII.2.13) since the flatness of $W_0 \longrightarrow \bar{T}$ implies S_2 and the smoothness of the general fibre of the deformation on $\bar{T}$ implies R_1.

[1] A ramified equivariant cover of $\bar{T}$ may be needed.

We now construct the canonical equivariant resolution of W_u, à la Orlik-Wagreich [34]. Let Z be the graph of the rational map $W_u \dashrightarrow X_u$. This replaces the singular point of W_u by a copy of X_u. In general the blown up surface Z obtained in this way may have rational singularities along the exceptional locus, corresponding to the exceptional orbits of $\mathbb{G}_m$ on W_u, i.e., those orbits with a non trivial stabilizer. Such orbits can only occur when at least one of the coordinates X_i is equal to zero. If $X_{n+1} = 0$, we get back the original curve C. Since the greatest common divisor of the a_i is 1, $\mathbb{G}_m$ acts without stabilizer on C. If X_{n+1} is not zero, then no stabilizer is possible since X_{n+1} has weight 1. Therefore there are no exceptional orbits, so Z is smooth.

Now Z is fibred over the exceptional divisor X_u by the orbits of $\mathbb{G}_m$; Noether's theorem (see [46], p. 24) shows Z is a line bundle over X_u: $Z = V(L^{-1})$, L an invertible sheaf on X_u. The ring of holomorphic functions on Z and hence on W_u is

$$\bigoplus_{i=0}^{\infty} \Gamma(X_u, L^i)$$

and therefore has a natural grading which by the very definition of the line bundle must coincide with the original $\mathbb{G}_m$ action on $X_1, \cdots, X_{n+1}$. Therefore $X_{n+1} \in \Gamma(X_u, L)$ since it has weight 1. Since in X_u the point obtained by setting $X_{n+1} = 0$ is P, we see that $L = \mathcal{O}(P)$. It is now clear that H is the semigroup of P; in particular X_u has genus $g(H)$. The proof of (13.10) is complete.

To prove (13.9) it suffices to prove the following proposition:

Proposition (13.11) Let X_0 be a smooth projective curve, $P \in X_0$ a point with semigroup H. Then there exists one and only one orbit O of $\mathbb{G}_m$ in V' such that for all $u \in O$, X_u is isomorphic to X_0 and $\sigma(u) = P$. Of course $O \subset U$.

Proof of (13.11). We will just reverse the construction made above. Let $L = \mathcal{O}(P)$ and let W_0 be the normal surface with coordinate ring

$$A_0 = \bigoplus_{i=0}^{\infty} \Gamma(X_0, L^i).$$

If $a_1,\cdots,a_n$ are generators of H, then A_0 can be generated as an algebra over $\Gamma(X_0,\mathcal{O}_{X_0}) = k$ by suitably chosen $X_i \in \Gamma(X_0,L^{a_i})$ and $X_{n+1} \in \Gamma(X_0,L)$. Adjusting constants if necessary we see that $W_0 \cap (X_{n+1}=0) = C$, the original monomial curve. Therefore we have constructed a deformation of C over Spec $k[X_{n+1}]$ which clearly maps to V' (and not only V). Requiring that Spec $k[X_{n+1}]$ map to an orbit of V' makes the map Spec $k[X_{n+1}] \longrightarrow V'$ unique.

This gives existence of an orbit O. To get uniqueness note that the only choices made in the construction were the X_i, $i = 1,\cdots,n$. A different set of X_i will just induce a coordinate change in $k[X_1,\cdots,X_{n+1}]$, and will not affect the orbit obtained.[1]

The proof of (13.9) is now complete.

Remark (13.12) The idea of constructing a moduli space for algebraic curves with Weierstrass points goes back to that of putting curves with a given type of Weierstrass point in normal form. See for example Baker [6], p. 59 or Hensel-Landsberg [22], p. 547. Instead of using all the generators of the semigroup they used only two

[1] This argument breaks down in characteristic $p > 0$.

relatively prime ones in order to obtain a plane curve, and then counted constants.

14. The existence of smooth deformations of monomial curves in certain special cases

We will study here the following two questions:

(1) When is U nonvoid?

(2) What is the dimension of Proj R'?

We conclude with some examples and some questions. (See section 13 for notation.)

As we have already noted (13.9) does not give us any information on whether U is void or not.

Definition (14.1) If U is non void we said C can be smoothed negatively.

In view of (13.1) and (13.5) this is reasonable terminology. Thus the existence of a projective curve with a point of semigroup H is equivalent to negative smoothing for $C = \mathrm{Spec}(B_H)$.

Theorem (14.2) If H is a complete intersection, i.e., if the ideal of $C = \mathrm{Spec}(B_H)$ in $P = k[X_1,\cdots,X_n]$ can be generated by n-1 elements, then there exists a smooth projective curve of genus g(H) and a point $P \in X$ of semigroup H.

Using M. Schaps' main theorem [42] one can probably prove that semigroups generated by 3 elements can be smoothed negatively, but we have not worked out the details.

(14.3) On the other hand since all curves have ordinary points, and since almost all curves have normal Weierstrass points, we see the semigroups H_g and $H_{g,1}$ generated by

$g+1, g+2, \cdots, 2g+1$ and $g, g+2, g+3, \cdots, 2g-1$

respectively have rings that can be smoothed negatively.

The question of existence of Weierstrass points with given semigroup seems not to have been seriously studied. Rauch [36], Farkas [13] and Arborello [2] have studied the case where only the first non-gap is specified, and have proved dimension formulae for the moduli space in this case. M. Haure [21] studied the general case, applying the methods of (13.12). Unfortunately his claim that certain semigroups must be excluded is based on a faulty analysis in the case the functions considered do not have relatively prime orders at the Weierstrass point: see [21], p. 151, last paragraph. That the plane curve obtained is not birational to the original curve was already noticed by Baker [6] and Hensel-Landsberg [22]. The only example he gives is clearly invalid: it is the semigroup generated by 4, 6 and 9. It is immediately seen to be symmetric, hence its ring is Gorenstein. Since it has embedding codimension 2 it is a complete intersection, so we can apply (14.2). On the other hand the tables at the end of his article seem to show that all possible semigroups can occur for genus ≤ 7.

We have been mainly interested in semigroups with only one gap after the first non gap, in part because their rings have negative grading (12.6) which will make the computation of Proj R' easy using (10.3).

<u>Definition</u> (14.4) Let $H_{g,k}$ be the semigroup of integers generated by $g, g+1, \cdots, g+k-1, g+k+1, \cdots, 2g-1$, $1 \leq k \leq g-1$, that is, $g+k$ is the only gap of $H_{g,k}$ after the first non gap g.

<u>Theorem</u> (14.5) The semigroup ring of $H_{g,k}$, $1 \leq k \leq g-1$, can be negatively smoothed; in other words there are smooth projective curves with a point of semigroup $H_{g,k}$.

By (14.3) we already know this is true for $k = 1$. Our method of proof will allow us to also treat that case.

<u>Proof</u>. Let H be <u>any</u> semigroup of integers with minimal set of generators $a_1 < a_2 < \cdots < a_n$. Hence $B_H = k[t^{a_1}, \cdots, t^{a_n}]$. We consider the family of curves over $k[s]$ generated by

$$t^{a_1}-s,\ t^{a_2-a_1}(t^{a_1}-s),\ \cdots,\ t^{a_n-a_1}(t^{a_1}-s).$$

In the case $n = 2$, $a_1 = 2$, $a_2 = 3$, these are parametric equations for a node degenerating to a cusp. In the general case we want to determine when this is a deformation of $k[t^{a_1}, t^{a_2}, \cdots, t^{a_n}]$, i.e., when the closed fibre of this family is B_H. To do this it is sufficient to check the arithmetic genus of a fibre $s \neq 0$ and compare it with the arithmetic genus of B_H: if they agree we have a deformation. There is no problem in what we mean by arithmetic genus since we can projectivize our affine curves. So fix $s \neq 0$ and consider the fibre B_s above s. Since it is parametrized by t it is rational, so that the arithmetic genus g_s of $B_s = \sum_{x \in \operatorname{Spec} B_s} \delta_x$, where $\delta_x = \dim \overline{B_{s,x}}/B_{s,x}$. $B_{s,x}$ = local ring of Spec B_s at x, and $\overline{B}_{s,x}$ its normalization. Therefore we must find the singular points x of Spec B_s, and compute δ_x. Now the singular points of Spec B_s fall into 2 categories:

(a) a single branch where the "velocity" $\frac{d}{dt}$ is 0,

(b) several branches through one point.

It is clear from the parametrization that the only point of type (a) is obtained when $t = 0$; the singular point Q_1 is <u>monomial</u> with semigroup H_{a_1} generated by $a_1, a_2-a_1, a_3-a_1, \cdots, a_n-a_1$. $\delta_{Q_1} = g(H_{a_1})$. Therefore when $a_2 = a_1+1$ the point obtained is actually not singular. What about singular points of type (b)? When can 2 values

t_1 and t_2 give the same point on the curve. We must have

$$t_1^{a_1}-s = t_2^{a_1}-s$$

so $t_1 = \omega t_2$, where ω is an a_1-th root of unity. But

$$t_1^{a_i-a_1}(t_1^{a_1}-s) = t_2^{a_i-a_1}(t_2^{a_1}-s)$$

for a_i relatively prime to a_1, so $t_1^{a_1}-s = 0$. Hence we get only one point Q_2 of type (b): a_1 distinct branches through the origin in linear general position.

We return to the hypotheses of the theorem. Suppose first that $k \geq 2$. Then $a_1 = g$, $n = g-1$, so that by (11.5) $\delta_{Q_2} = g$. By the remark above this is the only singular point, so the arithmetic genus of B_s coincides with that of B_H. Therefore we do have a deformation.

When $k = 1$, $a_1 = g$, $n = g$, so $\delta_{Q_2} = g-1$. But now $\delta_{Q_1} = 1$, so the arithmetic genus is again g. So in this case too we obtain a deformation.

To complete the proof of (14.5) it suffices to show $H_{g,k}$ can be smoothed by deformation, since by (12.6) they are negatively graded . Let

$$\begin{array}{ccc} Y & \longleftarrow & \operatorname{Spec} B_H \\ \downarrow & & \downarrow \\ V & \longleftarrow & \operatorname{Spec} k \end{array}$$

be the versal deformation space of Spec B_H. As noted in (13.1) V is algebraic. B_H is a specialization of B_s. Therefore if $0 \in V$ corresponds to $\operatorname{Spec}(B_H)$ and $s \in V$ is a point near 0 such that the fibre of $Y \longrightarrow V$ at s is isomorphic to B_s, then $Y \longrightarrow V$ localized at s is a complete (but not necessarily minimal) family of deformations of B_s. In both cases (when $k = 1$ and $k \geq 2$) it is clear by (11.13) that B_s has smooth deformations. Therefore B_H has smooth deformations, proving the theorem.

<u>Remark</u> 14.6 The construction of B_s also works for H_g. It would be interesting to understand in exactly what generality it holds.

<u>Theorem</u> 14.7 The open set U (13.5) associated to the semigroup $H_{g,k}$ is nonvoid of dimension $3g-k-1$, hence the number of moduli of smooth curves with a Weierstrass point of semigroup $H_{g,k}$ (the dimension of $\overline{U}$) is

$$3g - k - 2.$$

Proof. By (10.3) and (13.9) the dimension of U is $3\delta-c+d$, since all the semigroup rings in question have negative grading. Easy computations show:

$$\delta = g$$

$$c = g + k + 1$$

$$d = g$$

giving the dimension. The fact that U is nonvoid follows from (14.5).

Remark (14.8) In general it is difficult to compute the dimension of $U \subset \operatorname{Spec} R'$, since (10.3) only gives us the dimension of certain components of Spec R.

(14.9) When the semigroup H is Gorenstein, then Rim's formula ((10.2) and (10.3)) shows that the versal deformation space of B_H has dimension $2g(H)$[1]. Therefore $\overline{U}$ has at most dimension $2g(H) - 1$. This maximum is attained when B_H has negative grading: this is the case when H is the hyperelliptic semigroup (generated by 2 and $2g+1$). The dimension of $\overline{U}$ can drop much lower however.

[1]in characteristic 0.

If H is the semigroup generated by 4, 6 and 2g-3, then $\bar{U}$ has dimension 2g-3 (elementary computation). The ideal of B_H in $k[X_1,X_2,X_3]$ is generated by

$$X_1^3 - X_2^2$$

$$X_3^2 - X_1^{2g-6}X_2$$

so it is actually a complete intersection.

(14.10) We have just seen that the number of moduli of curves with a hyperelliptic point (i.e., a point with a semigroup generated by 2 and 2g+1) is 2g-1. This was of course well known, since they are the hyperelliptic curves. The next case to consider is that of a semigroup with first nonzero element 3. Rauch [36] has shown that the number of moduli of curves of genus g with a point of semigroup starting with N is N+2g-3. When N = 3 the number of moduli is 2g. In each genus g we will exhibit a semigroup with that number of moduli.

<u>Case I</u>. g even.

Set g = 2n, and consider the semigroup generated by 3, 3n+1, 3n+2. It is easy to see the number of gaps is g.

The semigroup ring of this semigroup is $B = k[X_1,X_2,X_3]/I$ where I is generated by the 2×2 minors of

$$\begin{vmatrix} X_1^n & X_2 & X_3 \\ X_2 & X_3 & X_1^{n+1} \end{vmatrix} .$$

Elementary computations show that B has negative grading; it would not be difficult to write down explicitly all the deformations, since they arise from deforming the matrix. We compute the dimension of Spec R using (10.3). $\delta = g$, $c = 3n$. It is slightly more difficult to compute d, but using the grading we find $d = n+1$. Hence the dimension of Spec R is $6n-3n+1 = 4n+1 = 2g+1$, so Proj R has dimension $2g$, proving the result.

Case II. g odd.

Set $g = 2n+1$ and let H be the semigroup generated by 3, $3n+2$, $3(n+1) + 1$. The ideal of B_H in $k[X_1,X_2,X_3]$ is generated by the 2×2 minors of

$$\begin{vmatrix} X_1^n & X_2 & X_3 \\ X_2 & X_3 & X_1^{n+2} \end{vmatrix} .$$

B_H has negative grading, $c = 3n+2$, $d = n+2$, so Spec R has dimension $3\delta + d - c = 3(2n+1) + n + 2 - 3n = 2 = 4n + 3 = 2g + 1$, as claimed.

(14.11) To conclude here are some questions to which we hope to devote attention at a later date.

(1) We have shown here in some special cases that U is non void, and we have computed its dimension. Is this true in general, and what is the dimension?

(2) $X \longrightarrow V'$ is a flat family of projective curves, with smooth fibre above $U \subset V'$. What are the fibres above $V'-U$? What is the significance of the section σ? Will it give us information concerning the degeneration of Weierstrass points?

(3) By forgetting the section σ, we get a map $U \longrightarrow \mathcal{M}_g$, the coarse moduli space for curves of genus g. As before this map factors through $\overline{U}$. What is the generic degree of $\overline{U} \longrightarrow \mathcal{M}_g$ above its image? This degree is known to be $2(g+1)$ in the case of the hyperelliptic semigroup, and $(g-1)g(g+1)$ in the case of the normal semigroup $H_{g,1}$ [19]. In general one expects it to be 1: if it is $d > 1$ then the

general curve with a point of semigroup H must have d-1 other points (up to isomorphism) with the same semigroup.

(4) Can one use deformation theory to obtain information about R or R'? For example when is R irreducible? reduced? When is Proj R' unirational?

(5) Probably the most interesting semigroup is H_g, since U is then an open subspace of $\mathcal{M}_{g,1}$. Recall H_g is generated by $g+1, g+2, \cdots, 2g+1$. The ideal of B_{H_g} in $k[X_1, \cdots, X_{g+1}]$ is generated by the 2×2 minors of

$$\begin{vmatrix} X_1 & X_2 & X_3 & \cdots\cdots\cdots & X_{g+1} \\ X_2 & X_3 & \cdots\cdots\cdots & X_{g+1} & X_1^2 \end{vmatrix}$$

B_{H_g} has negative grading by (12.6), and Spec R has dimension $3g-1$ by (10.3). We obtain a $2g+1$ dimensional subspace of Spec R by deforming the matrix, and it is not hard to see we obtain only the hyperelliptic curves this way. Even when $g = 3$ it is difficult to compute what R is: T^1 already has dimension 11.

(6) H_g has one other special feature: it is the only semigroup with semigroup ring having a 2-dimensional family of automorphisms, at least formally. Indeed besides $\mathbb{G}_m$ we have $\mathbb{G}_a$ acting by $t \longrightarrow t/1-at$ (t is the parameter of the normalization $k[t]$ of B_{H_g}). This heuristically explains why the moduli space Spec R of B_{H_g} has dimension 2 greater than that of smooth curves of genus g. Does this $\mathbb{G}_a$ action lift to the versal deformation? This is probably not the case.

15. An example

We now compute the simplest case of degeneration mentioned in (14.11), question (2), that is, the case of an ordinary point on a genus 2 curve degenerating to a hyperelliptic point. To simplify notation we write H for the semigroup generated by 3, 4, 5 (previously called H_2) and G for the one generated by 2, 5 (previously called $H_{2,1}$). These are the only semigroups in genus 2.

The versal deformation space of B_H is given, over $R = k[t_1, \ldots, t_5]$ by the vanishing of the maximal minors of the matrix

$$\begin{vmatrix} X_1 & X_2 & X_3 \\ X_2+t_3 & X_3+t_1X_1+t_4 & X_1^2+t_2X_1+t_5 \end{vmatrix} \tag{1}$$

Since B_H has negative grading, by (13.9) Proj R, where t_i has weight i+1, is a compactification of $\bar{U}$ (13.9) which is isomorphic to the open subspace $W_H \hookrightarrow \mathcal{M}_{2,1}$ consisting of pairs of genus 2 curves with an ordinary point. Now the complement of W_H in $\mathcal{M}_{2,1}$ is the analogously defined subspace W_G. Fix a point $w \in W_G$. Hence w represents a curve C and a hyperelliptic point $P \in C$. Let S be a discrete valuation ring and a: Spec S $\longrightarrow \mathcal{M}_{2,1}$ a curve in $\mathcal{M}_{2,1}$ with closed point w and generic point in W_H. Via the isomorphism of $\bar{U}$ with W_H, and using the valuative criterion we deduce a curve a': Spec S $\longrightarrow$ Proj R. We want to determine the image w' of the closed point of Spec S, and also the fibre of the versal deformation space of B_H over any inverse image of w' in Spec R. If we do this for all w W_G and all possible a, we will obtain a great deal of information concerning the structure of W_G in $\mathcal{M}_{2,1}$.

More generally we can ask the same question for any pair of semigroups H and G, both of genus g, such that W_G is contained in the closure of W_H in $\mathcal{M}_{g,1}$.

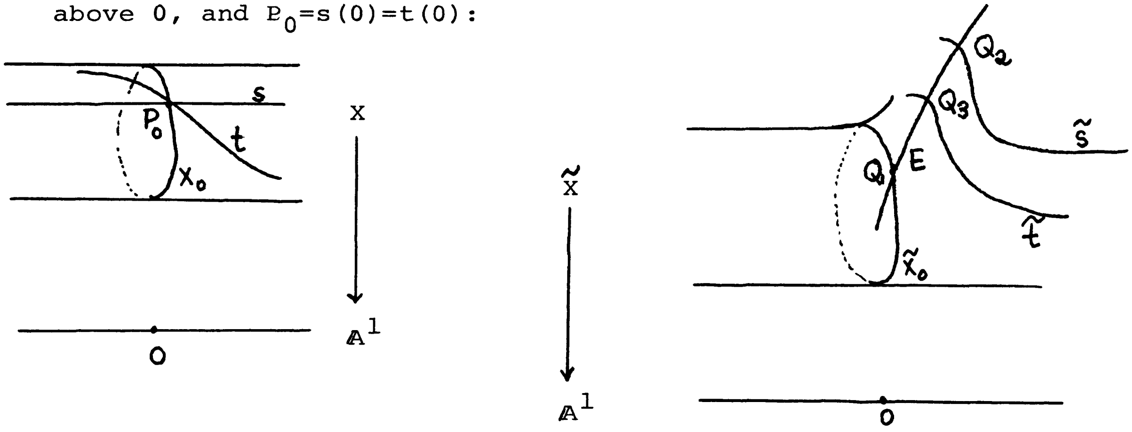
We return to the special case. Note that a should be thought of as a one parameter family of curves $X \longrightarrow \operatorname{Spec} S$ of genus 2, with closed fibre C and a section picking out generically an ordinary point and, at the closed point of Spec S, the hyperelliptic point P. For simplicity we work out the degeneration only when the curve a has the following special form: we have the obvious map $\mathcal{M}_{2,1} \xrightarrow{f} \mathcal{M}_2$ by forgetting the section. We let a be the localization of $f^{-1}(f(w))$ at w. This means we are approaching w = (C, P) along C.

<u>Claim 1</u>: the point w' obtained for this special a does not depend on the choice of $w \in W_G$.

We will give explicit coordinates for w' later on. Note that the claim is false if we do not assume a has the special form mentioned above; in other words w' does depend on the <u>direction</u> in which we are approaching w. Thus, roughly speaking, we get Proj R from $\mathcal{M}_{2,1}$ (in a neighborhood of W_G) by first blowing up along W_G and then contracting the proper transform of W_G.

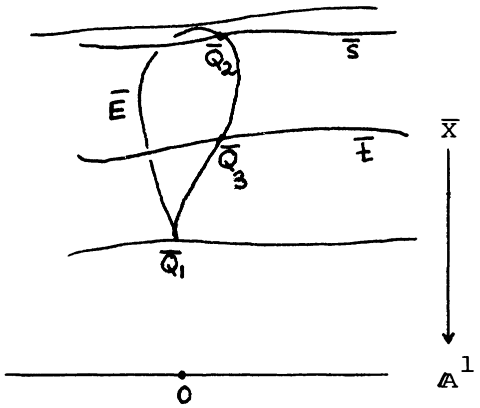
To prove the claim we make a construction which gives the orbit of (C, P) in the versal deformation space of B_G by a method slightly different from the one described in (13.11).

Let X be the product of C with the affine line $\mathbb{A}^1$. Let s be the section of $X \xrightarrow{p_1} \mathbb{A}^1$ picking out in each fibre the point P, and let t be any other section with t(0)=P and not tangent to s at that point. Denote by X_0 the fibre of p_1 above 0, and P_0=s(0)=t(0):



We now blow up P_0 on the surface X. Call the transformed surface $\tilde{X}$, the exceptional divisor E and the proper transforms of X_0, s and t, $\tilde{X}_0$, $\tilde{s}$ and $\tilde{t}$ respectively. Let Q_1, Q_2 and Q_3 be the points of intersection of E with $\tilde{X}_0$, $\tilde{s}$ and $\tilde{t}$ respectively. By hypothesis they are distinct.

$\tilde{X}_0$ has self-intersection -1 in $\tilde{X}$, hence by Grauert's theorem we can contract it analytically. We obtain a (singular) surface $\bar{X}$ and a flat map $\bar{X} \longrightarrow \mathbb{A}^1$ with sections $\bar{s}$ and $\bar{t}$, such thatthe fibre $\bar{E}$ above the origin is a rational curve, singular at $\overline{Q}_1$ with two other distinguished points $\overline{Q}_2$ and $\overline{Q}_3$. As t varies $\overline{Q}_3$ varies over all the points of $\overline{E}$ except $\overline{Q}_1$ and $\overline{Q}_2$, which are uniquely determined. All the other fibres of $\overline{X} \longrightarrow \mathbb{A}^1$ are isomorphic to C. It is not hard to see that



$\overline{E} \smallsetminus \overline{Q}_2$ is isomorphic to Spec B_G and hence that the family $\overline{X} \longrightarrow \mathbb{A}^1$ with the section $\overline{s}$ represents (the normalization of) the orbit corresponding to (C,P) in the versal deformation space of B_G.

<u>Claim 2</u>: the pairs $(\overline{E}, \overline{Q}_3)$, as $\overline{Q}_3$ varies, parametrize the fibres of the versal deformation space of Spec B_H over the points of Spec R lying above $w' \in$ Proj R.

This is proved by examining the graded ring

$$\bigoplus_{i \geq 0} H^0(\overline{X}, \mathcal{O}_{\overline{X}}(i\overline{t}))$$

and showing that its Spec is the total space of a 2 dimensional family of deformations of Spec B_H. This is essentially equivalent to showing that $\overline{Q}_3$ is an ordinary point on $\overline{E}$, i.e., that the meromorphic functions on $\overline{E}$ with poles only at $\overline{Q}_3$ have orders in H.

Claim 2 implies claim 1, since $(\overline{E}, \overline{Q}_3)$ does not depend on the curve C we started out with.

We now show how to find w' explicitly. The point is to notice that the representation (1) can be used to find the hyperelliptic covering of the curves in the family: we get a rational map to $\mathbb{P}^1$ by taking the common ratio of the columns.

$$s = \frac{X_2+t_3}{X_1} = \frac{X_3+t_1X_1+t_4}{X_2} = \frac{X_1^2+t_2X_1+t_5}{X_3}$$

Note that we have a natural section $X_1=X_2=Z=0$ (where Z is the homogenizing coordinate, the X_i having weight 1). s is infinite at that point. The other point with s infinite is $X_1=X_2=X_3=0$, hence there never is a ramification point at infinity. Indeed an easy computation shows that the ramification points of this map are given by the roots of the equation

$$(2) \qquad s^6- 2t_1s^4-2t_2s^3+(t_1^2-4t_3)s^2+(2t_1t_2-4t_4)s+(t_2^2-4t_5) = 0$$

Two fibres of the family are isomorphic if and only if their ramification points are equivalent under the natural action of the projective linear group on $\mathbb{P}^1$.

Consider the fibre C above $t_1=t_2=t_3=t_4=0$. Cis easily seen to be nonsingular by the jacobian criterion; another way of seeing this is to notice that s^6-4t_5 has six distinct roots.

Next we determine the curve in $\overline{U} \hookrightarrow \text{Proj } R$ above which the fibre of the versal deformation is isomorphic to C, and then find its limit points in Proj R. There is only one such limit point which is the w' we are after.

To find this curve we let SL_2 act on $\mathbb{P}^1$, with inhomogeneous coordinate s, in the usual way:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2, \qquad s \longmapsto \frac{as+b}{cs+d}$$

We ask which elements of SL_2 send s^6+T^6, $T^6=-4t_5$, to an equation of the form (2). Necessary and sufficient conditions are that

$$\left.\begin{aligned} a^6+T^6c^6 &= 1 \\ a^5b+c^5dT^6 &= 0 \end{aligned}\right\} \qquad \text{and of course } \; ad-bc=1$$

so that the matrices are of the form $\begin{pmatrix} a & -T^6c^5 \\ c & a^5 \end{pmatrix}$ with $a^6+Tc^6=1$.

We let a tend to infinity. A computation shows that the coefficients of the transform of s^6+T^6 under this matrix tend to $(1, 0, -15T^2, -40T^3, -45T^4, -24T^5, -5T^6)$ in the sense of invariant theory, so that we obtain the point in Proj R with projective coordinates

$$t_1=3\cdot 5/2;\; t_2=2^2\cdot 5;\; t_3=5\cdot 3^4/2^4;\; t_4=3^4;\; t_5=5\cdot 3^4/2^2.$$

The T factor disappears by homogeneity. The equation for the discriminant is $(s-5)(s+1)^5$. It is easy to see that the fibre of the versal deformation above this point is a rational curve with a singular point above $s=-1$ and that the map to $\mathbb{P}^1_s$ is its normalization. Hence in the notation of the earlier part of this section, the singular point above $s=-1$ is $\overline{Q}_1$, the section $X_1=X_2=Z=0$ is $\overline{Q}_3$ and the point above $s=5$ is $\overline{Q}_2$. Since we have a 5-fold root of the discriminant at $s=-1$ it is easy to see directly that the singularity at $\overline{Q}_1$ is isomorphic to Spec B_G.

This completes the study of this simple type of degeneration.

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