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ALGEBRAIC THEORY OF FUNDAMENTAL GROUPS

INTRODUCTION TO THE ALGEBRAIC THEORY OF FUNDAMENTAL GROUPS

HERBERT POPP

Recently topologists became interested, in connection with monodromy and deformation problems, in calculating explicitly the topological fundamental group of the projective space \mathbb{P}^N minus a hypersurface C . Compare [1].

This problem is difficult to handle, if the hypersurface C has singularities. We indicate here an attack by algebraic methods and describe the main results. A more complete treatment of the matter can be found in [6].

Let k an algebraically closed field of characteristic $p \geq 0$. V shall be an irreducible, quasi projective k -scheme which is reduced and normal.

DEFINITION 1. An irreducible, reduced k -scheme V' together with a surjective morphism $f: V' \rightarrow V$, which is finite, is called a covering of V .

DEFINITION 2.

- a) - For a covering $f: V' \rightarrow V$ of V the degree of the field extension $k(V')/k(V)$ is called the degree of the covering. ($k(V)$ = function field of V .)
- b) - A covering $f: V' \rightarrow V$ is called galois if the field extension $k(V')/k(V)$ is galois and abelian if $k(V')/k(V)$ is abelian.

DEFINITION 3. A covering $f: V' \rightarrow V$ of V is called unramified at $P' \in V'$ if

POPP

a) - the maximal ideal $\mathfrak{m}_{P',V'}$ of the local ring $(\mathcal{O}_{P',V'}, \mathfrak{m}_{P',V'})$ of P' on V' is generated by the maximal ideal $\mathfrak{m}_{P,V}$ of $\mathcal{O}_{P,V}$ where $P = f(P')$ is the image of P' and $(\mathcal{O}_{P,V}, \mathfrak{m}_{P,V})$ the corresponding local ring of P on V .

b) - The residue field $\mathcal{O}_{P',V'} / \mathfrak{m}_{P',V'}$ is a separable extension of the residue field $\mathcal{O}_{P,V} / \mathfrak{m}_{P,V}$.

If every point $P' \in V'$ of a covering $f: V' \longrightarrow V$ is unramified, we say that the covering $V' \xrightarrow{f} V$ is an unramified covering of V .

Using local algebra, see [6], lecture 1, one can prove the following propositions.

PROPOSITION 1. Let $f: V' \longrightarrow V$ be a covering of V of degree n . Then f is unramified if and only if for every closed point $P \in V$ the inverse image $f^{-1}(P)$ contains n distinct points.

PROPOSITION 2. Let $f: V' \longrightarrow V$ be an unramified covering of V . Then V' is the normalisation of V in the field $k(V')$ and the map f is the canonical map from the normalisation onto V .

We are interested in describing the finite, unramified coverings of V by a group $\pi_1(V)$ via galois theory, i.e., we want a group $\pi_1(V)$ such that the subgroups of $\pi_1(V)$ of finite index correspond in a natural way to the (finite) unramified coverings of V . Proposition 2 can be used to obtain such a group.

ALGEBRAIC THEORY OF FUNDAMENTAL GROUPS

Let $K = k(V)$ be the field of rational functions on V and \bar{K} a separable algebraic closure of K .

If $V' \xrightarrow{f} V$ is an unramified covering of V , the function field $k(V')$ of V' can be embedded into \bar{K} . By proposition 2, V' is the normalisation of V in $k(V')$ and is therefore determined by $k(V')$.

Consider now the set Σ of subfields of \bar{K} , defined as follows

$\Sigma = \{ K' ; K' \text{ subfield of } \bar{K} \text{ containing } K, \\ K'/K \text{ finite ; the normalisation of } V \\ \text{ in } K' \text{ is unramified over } V. \}$

One shows, see [6], lecture 1, that the composit of two fields K_1, K_2 in Σ is again in Σ . Therefore with any field $K' \in \Sigma$ one has also the galois closure of K' over K in Σ . Let $\hat{K} = \bigcup_{K' \in \Sigma} UK'$. Then \hat{K} is a galois fields extension of K . Denote by $\pi_1(V)$ the galois group of the extension \hat{K}/K . By general galois theory the subgroups of $\pi_1(V)$ of finite index are in 1-1 correspondence in a natural way with the fields $K' \in \Sigma$. Using proposition 2 we obtain that the group $\pi_1(V)$ has the desired properties.

DEFINITION 4. $\pi_1(V)$ is called the fundamental group of V .

Over the complex number field \mathbb{C} what is the relation between the topological fundamental group $\pi_1^{\text{top}}(V)$ of an irreducible, reduced, normal and quasi projective \mathbb{C} -scheme V and the group $\pi_1(V)$?

By the general RIEMANN existence theorem due to GRAUERT and REMMERT [4], one knows that the finite connected topological coverings of the \mathbb{C} -scheme V are in 1-1 correspondence with the unramified coverings of V .

in the sense of definition 1.1. This shows that $\pi_1(V)$, which is by definition a profinite group, is the profinite completion of $\pi_1^{\text{top}}(V)$.

REMARK 1. : It is well known (compare [6] , lecture 1) that for an irreducible, reduced and normal k -scheme V the connected etale coverings in the sense of [5] coincide with the unramified covering of V in the sense of definition 1. We conclude from this that $\pi_1(V)$ coincides with the $\pi_1(V)$ introduced in [5] by Grothendieck.

REMARK 2. : Serre has shown in [7] by an example that the natural map from $\pi_1^{\text{top}}(V)$ into its profinite completion is not always injective. Hence, $\pi_1(V)$ contains less information than $\pi_1^{\text{top}}(V)$.

We have already stated at the beginning that we want to determine by algebraic methods the structure of $\pi_1(\mathbb{P}^N - C)$, where C is a hypersurface of \mathbb{P}^N .

For simplicity we restrict from now on to the case where the ground field k has characteristic 0. In the case when characteristic $k = p > 0$, the method described can be used to obtain results on the p -prime part of the fundamental group. Details are in [6] .

THEOREM 1. Let C be an irreducible, smooth hypersurface of degree d in \mathbb{P}^N . Then $\pi_1(\mathbb{P}^N - C)$ is the cyclic group of order d .

PROOF : Let X_0, \dots, X_N be projective coordinates in \mathbb{P}^N and H_0 be the hyperplane defined by $X_0 = 0$. Let $A_0^N = \mathbb{P}^N - H_0$ and $x_1 = \frac{X_1}{X_0}, \dots, x_N = \frac{X_N}{X_0}$ affine coordinates in A_0^N . Assume

that $C \neq H_0$. Let $f(x_1, \dots, x_N) = f(x)$ be a defining polynomial for $C \cap \mathbb{A}_0^N$ of degree d . We describe first the abelian, unramified coverings of $\mathbb{P}^N - C$. Let $V \longrightarrow \mathbb{P}^N - C$ be such an abelian covering of $\mathbb{P}^N - C$ and $k(V) \supset k(x)$ ($k(x) =$ function field of \mathbb{P}^N) the corresponding field extension. Then $k(V)/k(x)$ is a Galois extension with an abelian Galois group and as such the composite of the cyclic subfields of $k(V)$ which contain $k(x)$. We can restrict our attention therefore to cyclic unramified coverings of $\mathbb{P}^N - C$.

Let $V \longrightarrow \mathbb{P}^N - C$ be such a cyclic unramified covering and $k(V)/k(x)$ the corresponding cyclic field extension. By general Kummer theory $k(V)$ can be generated over $k(x)$ by an element z which satisfies an irreducible equation $z^n = h(x)$ with $h(x) \in k(x)$. We can change z to $z' = \alpha \cdot z$ with $\alpha \in k(x)$. Then z' satisfies then the equation $z'^n = \alpha^n \cdot h$. Write $h(x) = \frac{h_1(x)}{h_2(x)}$ with $h_1, h_2 \in k[x]$. By taking $z' = h_2(x)z$ as a generator of $k(V)$ we find that z' satisfies an equation of the form

$$z'^n = h(x), \text{ with } h(x) \in k[x].$$

Let $h(x) = h_1^{a_1}(x) \dots h_r^{a_r}(x)$ be the prime factorisation of h . By changing the generator, if necessary, we may assume that the equation for the generator z is of the form

$$(1) \quad z^n = h_1^{a_1} \dots h_r^{a_r}$$

with $0 < a_i < n$.

We return to the cyclic field extension $k(V) \supset k(x)$ and the covering $V \longrightarrow \mathbb{P}^N - C$ where V is the normalisation of $\mathbb{P}^N - C$ in $k(V)$.

Let V^* be the normalisation of \mathbb{P}^N in $k(V)$ and $f: V^* \longrightarrow \mathbb{P}^N$ the natural covering map. Then $f: V^* \longrightarrow \mathbb{P}^N$ is ramified at most along C and the restriction of f to $V^* - \hat{f}^{-1}(C)$ is the covering $f': V \longrightarrow \mathbb{P}^N - C$. (Notice that $V^* - \hat{f}^{-1}(C)$ is isomorphic to V .)

The irreducible polynomials h_1, \dots, h_r which are divisors of h in the equation (1) define hypersurfaces C_1, \dots, C_r in \mathbb{P}^N which are easily shown to be ramified in the covering $V^* \longrightarrow \mathbb{P}^N$. This implies that (1) is of the form

$$(2) \quad z^n = f^a, \quad 0 < a < n,$$

where f is a defining polynomial of C of degree d .

The degree of the covering $V \longrightarrow \mathbb{P}^N - C$ is n . Therefore, the integers a and n in equation (2) are relatively prime.

Let u, v be integers such that $un + v a = 1$. Then

$$(z^v)^n = f^{va} = f^{1-un}$$

or

$$(z^v \cdot f^u)^n = f.$$

This shows that $z' = z^v \cdot f^u$ generates $k(V)$ over $k(x)$ and z' satisfies the equation

$$(3) \quad (z')^n = f.$$

We have not considered so far the hyperplane H_0 , which is by assumption unramified in the covering $V^* \longrightarrow \mathbb{P}^N$. To consider H_0 we pass to the affine space $\mathbb{A}_1^N = \mathbb{P}^N - H_1$, where H_1 is the hyperplane defined by

$x_1 = 0, \frac{x_0}{x_1}, \frac{x_2}{x_1}, \dots, \frac{x_N}{x_1}$ are affine coordinates in A_1^N . If we write (3)

in terms of this coordinates, we get the equation

$$(z')^n = (x_1)^d f\left(\frac{x_0}{x_1}, \frac{x_2}{x_1}, \dots, \frac{x_N}{x_1}\right).$$

The equation for $H_0 \cap A_1^N$ is $\frac{x_0}{x_1} = 0$. By assumption H_0 is unramified in the covering $V^* \rightarrow \mathbb{P}^N$. This implies that d is a multiple of n .

Combining we have the following proposition :

PROPOSITION 3. Let $V \rightarrow \mathbb{P}^N - C$ be an unramified abelian covering and let $f(x_1, \dots, x_N) = 0$ be a defining equation of $C \cap A_0^N$. (The notation is as above). Then, the function field $k(V)$ is of the form $k(x_1, \dots, x_N) (\sqrt[n]{f})$

where n is an integer which divides the degree of C and V is the normalisation of $\mathbb{P}^N - C$ in $k(x) (\sqrt[n]{f})$.

Next we have to show that every unramified covering $V \rightarrow \mathbb{P}^N - C$ is abelian. For this purpose it is necessary to recall some well known facts about the galois theory of local rings. Let $V' \xrightarrow{f} V$ be a normal galois covering of V , i.e., V' is normal, and let G be the galois group of the covering $f: V' \rightarrow V$. Suppose W is an irreducible subvariety of V and W' an irreducible component of $f^{-1}(W)$. By $(\mathcal{O}_{W', V'}, \mathfrak{m}_{W', V'})$ we denote the local ring of W' on V' .

DEFINITION 4. The group $I = I(W') = \{ \sigma \in G; \sigma(x) \equiv x \text{ modulo } \mathfrak{m}_{W', V'} \}$ is called the inertia group of W' .

The significance of this group is described by the following proposition (compare [6], lecture 1) :

PROPOSITION 4. Let $k(V')$ be the function field of V' and $k(V')^I$ be the fixed field of I . V'^I shall be the normalisation of V in $k(V')^I$. Consider the diagram $V' \xrightarrow{f_1} V'^I \xrightarrow{f_2} V$ and let $f_1(W') = W'^I$ be the image of W' in V'^I . Then W'^I is unramified over W in the covering $V'^I \xrightarrow{f_2} V$ and V'^I is the largest normal covering of V between V' and V with this property.

We need the following 4 general theorems from algebraic geometry (compare [6], lecture 2, for comments).

THEOREM 2. (Purity of the branch locus). Let V be a smooth, irreducible k -scheme and $V' \xrightarrow{f} V$ be a normal covering. Let Δ be the branch locus of $f: V' \rightarrow V$, i.e. $\Delta = \{P \in V; \text{at least one point } P' \in V' \text{ lying over } P \text{ is ramified in } f: V' \rightarrow V\}$. Then Δ is a closed subvariety of V which is of pure codimension 1.

Let V be a normal irreducible k -scheme and L a linear system of effective divisors of V . Suppose dimension $L \geq 1$. Let $\phi_L: V \rightarrow \mathbb{P}^N$ be the rational map which is determined by L . We say that L is composite with a pencil if the dimension of the image variety $\phi_L(V)$ is 1. Zariski has proved the following theorem :

THEOREM 3. (Theorem of Bertini). Let L be a linear system of divisors of an irreducible normal k -scheme V . Assume that L has no fixed components

and dimension ≥ 2 . Then L is composite with a pencil if and only if every divisor of L is reducible.

THEOREM 4. (Connectedness Theorem). The specialisation of a connected, projective variety is connected.

THEOREM 5. (Local non splitting). Let $V' \xrightarrow{f} V$ be a (connected) normal covering of a smooth variety V , i.e. V' is connected and normal. Let $P \in \Delta$, the branch locus, and assume that Δ has at P only strong normal crossings as singularities. *) If Δ_1 is an irreducible component of Δ with $P \in \Delta_1$, and $P' \in V'$ a point lying over P , only one component of $f^{-1}(\Delta_1)$ passes through P' .

Consider now the coverings of \mathbb{P}^N which are ramified at most along C .

Let $V^* \rightarrow \mathbb{P}^N$ be a galois covering of this type, where V^* is normal and G is the galois group.

Let C_1^*, \dots, C_r^* be the irreducible components of the inverse image of C in V^* and $I_i = I_i(C_i^*)$ the inertia groups of the C_i^* .

By definition $I_i(C_i^*)$ is the inertia group of the local ring of C_i^* which is a discrete valuation ring of rank 1. This implies that $I_i(C_i^*)$ is cyclic.

CLAIM : G is equal to the subgroup $I = \langle I_1, \dots, I_r \rangle$ generated by the inertia groups I_i .

*) This means that there exist elements x_1, \dots, x_r in the local ring $(\mathcal{O}_{P,V}, \mathfrak{m}_{P,V})$ of P on V which are a part of a regular parameter system of $\mathcal{O}_{P,V}$ such that $x_1 x_2 \dots x_r = 0$ is a local equation for Δ at P .

PROOF : Consider $k(V^*)^I$, the fixed ^{field} of $K(V^*)$ by I . We have to show that $k(V^*)^I = k(\mathbb{P}^N)$. Let V^{*I} be the normalisation of \mathbb{P}^N in $k(V^*)^I$ and consider the diagram :

$$\begin{array}{ccccc} V^* & \xrightarrow{f_1} & V^{*I} & \xrightarrow{f_2} & \mathbb{P}^N \\ & & \xrightarrow{f} & & \end{array}$$

Applying proposition 4 we conclude that the general point of C is unramified in $V^{*I} \xrightarrow{f_2} \mathbb{P}^N$. Hence, $V^{*I} \xrightarrow{f_2} \mathbb{P}^N$ is unramified in codimension 1 and by theorem 2 unramified everywhere. We show in the lemma below that the \mathbb{P}^N has no unramified (connected) coverings and obtain from these considerations that $V^{*I} = \mathbb{P}^N$ and $G = \langle I_1, \dots, I_r \rangle$.

Lemma : \mathbb{P}^N has no non trivial unramified coverings.

PROOF : Let $V \xrightarrow{f} \mathbb{P}^N$ be an unramified covering and L the linear system of hyperplanes of the \mathbb{P}^N . $f^*(L)$ shall denote the pullback of L to V . By considering the rational map $\emptyset \xrightarrow{f^*(L)}$ determined by $f^*(L)$ we see that $\emptyset \xrightarrow{f^*(L)} = \emptyset \xrightarrow{L} \circ f$ and obtain that the linear system $f^*(L)$ is not composite with the pencil. This implies, because $f^*(L)$ has no fixed components, (use theorem 3), that the general member of $f^*(L)$ is irreducible. In other words, if H is a general hyperplan section of the \mathbb{P}^N , the inverse image $H' = f^{-1}(H)$ is an irreducible variety of V . Furthermore H' is, with the induced map $f : H' \rightarrow H$, an irreducible, unramified covering of H of the same degree as $V \rightarrow \mathbb{P}^N$. We can continue this argument and obtain in this way an irreducible, unramified covering $W \rightarrow \mathbb{P}^1$ which is of the same degree as the covering $V \rightarrow \mathbb{P}^N$.

Using Hurwitz's genus formula we conclude that the degree of $W \longrightarrow \mathbb{P}^1$ is 1.

We have shown so far that, in a normal galois covering $V \xrightarrow{f} \mathbb{P}^N$ with galois group G which is at most ramified along C , the group G is generated by the inertia groups I_i of the irreducible components of $f^{-1}(C)$. If we can show that $f^{-1}(C)$ is irreducible, we will have that G is the inertia group of $f^{-1}(C)$ and hence that G is cyclic.

Consider, for this purpose, the pullback $f^*L(C)$ of the complete linear system $L(C)$ of \mathbb{P}^N which is determined by C . This system has no fixed components and is not composite with a pencil. Therefore, the general member of $f^*L(C)$ is irreducible.

By the connectedness theorem 4 we conclude that the specialisation $f^{-1}(C)$ of the general member of $f^*L(C)$ is connected.

But C was assumed to be smooth. We have, therefore, local non splitting for $f^{-1}(C)$. This together with the connectedness of $f^{-1}(C)$ implies that $f^{-1}(C)$ is irreducible.

Theorem 1 is proved.

The arguments in the proof of theorem 1 will sometimes determine the structure of the fundamental group of the complement of the curve C in \mathbb{P}^2 , even if the curve C has singularities.

THEOREM 6. Let C be an irreducible curve in \mathbb{P}^2 of degree d . If C does not have "big singularities", the fundamental group $\pi_1(\mathbb{P}^2 - C)$ is cyclic of order d .

PROOF. The notion^{of} "big singularities" will be made precise during the proof of the theorem. Let $V^* \xrightarrow{f} \mathbb{P}^2$ be an irreducible, normal galois covering of \mathbb{P}^2 , ramified at most along C . Let G be the galois group of $V^* \xrightarrow{f} \mathbb{P}^2$. We want to show that $f^{-1}(C)$ is irreducible by using the arguments that we used in the proof of theorem 1. But for C singularities are allowed. We therefore do not have theorem 5 available. We prepare the situation by resolving the singularities of C by quadratic transformations. It is known that there exists a sequence of smooth surfaces $W_1 = \mathbb{P}^2, W_2, \dots, W_r = W$, all defined over k , such that :

- 1) W_i is the quadratic transform of W_{i-1} with a k -closed point $P_{i-1} \in W_{i-1}$ as center.
- 2) The reduced total transform of the curve C in W has only strong normal crossings as singularities. (Compare [6] , p.50).

Let C_W be the proper transform of C in W and let W^* be the normalisation of W in the function field $k(V^*)$ of V^* . $g : W^* \longrightarrow W$ shall be the covering map.

CLAIM. If $g^{-1}(C_W)$ is irreducible, $f^{-1}(C)$ is irreducible.

To prove this claim notice that by making quadratic transforms we do not change the general point of the curve C . Hence, the local rings of C and C_W on \mathbb{P}^2 , respectively W , are the same. If $g^{-1}(C_W)$ is irreducible, only one local ring of $k(V^*) = k(W^*)$ lies over the local rings of C and C_W . But this means then that $f^{-1}(C)$ is irreducible. (Use that the irreducible components of $f^{-1}(C)$

are in 1-1 correspondence with the valuation rings of $k(V^*)$ which lie over the local ring of C on P^2).

We can conclude the irreducibility of $g^{-1}(C_W)$, as in the proof of theorem 1, if the dimension of the complete linear system $L(C_W)$ of C_W on W is ≥ 2 .

We know that the dimension of the linear system $L(C)$ of C on P^2 is equal to $\frac{d(d+3)}{2}$, $d = \text{degree of } C$. However, the dimension of $L(C)$ changes in a sequence of quadratic transformations.

In [6], p.52 (Satz 6.3) the following is shown :

PROPOSITION 5 . Let V be a smooth, irreducible surface over k , P a closed point of V and C be a curve on V which has multiplicity e at P . Suppose $V' \xrightarrow{\sigma} V$ is the quadratic transformation of V at P and C' the proper transform of C in V' . Then,

$$\dim L(C') > \dim L(C) - \frac{e(e+1)}{2} .$$

Proposition 5 can be used to define the weight of the singularities of C :

Let

$$\begin{array}{l} P^2 \leftarrow W_2 \leftarrow \dots \leftarrow W_{r-1} \leftarrow W_r = W \\ P_1 \leftarrow P_2 \leftarrow \dots \leftarrow P_{r-1} \\ C = C_1 \leftarrow C_2 \leftarrow \dots \leftarrow C_{r-1} \leftarrow C_r = C_W \end{array}$$

be a minimal desingularisation of the curve C . (Compare [6], p.51.)

C_2, \dots, C_r are the proper transforms of C in the surfaces W_2, \dots, W_r and P_i the centers of the quadratic transformations. Let e_i be the multiplicity of C_i at P_i , $i = 1, \dots, r-1$. Then we define :

DEFINITION 5. The integer $v(C) = \sum_{i=1}^{r-1} \frac{e_i(e_i+1)}{2}$ is called the weight of the singularities of C .

Theorem 6 can now be reformulated as follows.

THEOREM 7. Let C be an irreducible curve in \mathbb{P}^2 such that $\dim L(C) - v(C) \geq 2$. Then $\pi_1(\mathbb{P}^2 - C) = \mathbb{Z}/d$, where d is the degree of C .

EXAMPLES :

1) If C has s nodes as singularities, every node contributes 3 to the weight $v(C)$. Hence $v(C) = 3s$.

2) If C has s nodes and t ordinary cusps, we see easily that $v(C) = 3s + 5t$.

Further examples can be found in [6], lecture 7.

REMARK 3 : It is shown in [6], lecture 5, that one can determine explicitly the structure of the galois groups of the abelian unramified coverings of $\mathbb{P}^N - C$, C any hypersurface in \mathbb{P}^N . This is the result :

THEOREM 8. Let $C = C_1 \cup \dots \cup C_r$ be a hypersurface in \mathbb{P}^N with C_1, \dots, C_r as irreducible components. Assume that the ground field k has characteristic zero. Let d_1, \dots, d_r be the degrees of C_1, \dots, C_r respectively.

Then, the factor commutator group of $\pi_1(\mathbb{P}^N - C)$ is the profinite abelian group with r generators $\alpha_1, \dots, \alpha_r$ and the relation $\alpha_1^{d_1} \dots \alpha_r^{d_r} = 1$.

From theorem 8 and the consideration in the proof of theorem 1 follows (see [6], lecture 5) :

THEOREM 9. Let $C = C_1 \cup \dots \cup C_r$ be a hypersurface in \mathbb{P}^N , defined over a field of characteristic 0. Let C_i , $i = 1, \dots, r$, be the irreducible components of C , and let d_i denote the degree of C_i . Assume further that C has only strong normal crossings as singularities. Then $\pi_1(\mathbb{P}^N - C)$ is the profinite abelian group with r generators $\alpha_1, \dots, \alpha_r$ and the relation $\alpha_1^{d_1} \dots \alpha_r^{d_r} = 1$.

REMARK 4. We have excluded so far the case where the field of definition k has characteristic $p > 0$. There the methods described can also be applied, if one restricts to coverings which are galois of a degree prime to p . Denote by $\pi_1^{(p)}(\mathbb{P}^N - C)$ the factor group of $\pi_1(\mathbb{P}^N - C)$ modulo the p -Sylow subgroups of $\pi_1(\mathbb{P}^N - C)$ ([6], lecture 1). The normal subgroups of $\pi_1^{(p)}(\mathbb{P}^N - C)$ parametrize then the finite unramified galois coverings of $\mathbb{P}^N - C$ with an order prime to p .

THEOREM 10. Let C be an irreducible, smooth hypersurface in \mathbb{P}^N of degree d defined over the field k . Assume that the characteristic of k is $p > 0$. Write $d = p^a d'$, with $(d', p) = 1$. Then, $\pi_1^{(p)}(\mathbb{P}^N - C) = \mathbb{Z}/(d')$. Further results for the group $\pi_1^{(p)}(\mathbb{P}^N - C)$ can be found in [6].

REMARK 5. The situation changes drastically if one considers, in characteristic $p > 0$, arbitrary coverings.

In characteristic 0 the affine line has no unramified covering as one checks easily using Hurwitz's genus formula. In characteristic $p > 0$ one can show the following theorem, ([6], p.119):

THEOREM 11. Let \mathbb{P}^1/k be the projective line over the algebraic closed field k of characteristic $p > 0$ and let P be a k -valued point of \mathbb{P}^1 . Then, every smooth, irreducible, projective curve (of any genus) defined over k is up to isomorphism an irreducible and normal covering of \mathbb{P}^1 ramified at most at P .

REMARK 6. Theorem 7 gives a method which shows that in characteristic 0 the group $\pi_1(\mathbb{P}^2 - C)$ is cyclic of order d if C is irreducible with singularities of not too high weight ($d = \text{degree of } C$). It is of particular interest in algebraic geometry to determine $\pi_1(\mathbb{P}^2 - C)$ if C is an irreducible curve with only cusps and nodes as singularities, because every smooth surface can be considered as a covering of \mathbb{P}^2 which is ramified at most along such a curve. Theorem 7 can handle this situation if one has not too many nodes and cusps. In this connection it is interesting that there is an other method which allows to show that $\pi_1^{(p)}(\mathbb{P}^2 - C)$ is the cyclic group of order d' (d' as in theorem 10), if C is an irreducible curve of degree d with s nodes as singularities, provided s is greater than $2g^2$, where g is the genus of C . Compare [6], lecture 14. This result follows from a close study of the behaviour of $\pi_1^{(p)}(\mathbb{P}^2 - C)$, if C varies in an algebraic family of plane curves and the fact that an irreducible curve C of degree d with s nodes as singularities can be degenerated into d lines in general position if $s > 2g^2$.

It is not known that $\pi_1^{(p)}(\mathbb{P}^2 - C)$ is abelian (and therefore cyclic) if C is an arbitrary irreducible curve with only nodes as singularities.

Edmunds [2] and Geyer [3] have shown the following theorem :

THEOREM 12. Let C be an irreducible curve in \mathbb{P}^2 with only nodes as singularities. Then every unramified galois covering of $\mathbb{P}^2 - C$ with a solvable galois group is abelian.

POPP

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