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FAMILIES OF CURVES OF GENUS TWO

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§ 0 . - INTRODUCTION .

Let $\pi : X \to D$ be a proper surjective holomorphic map of a 2-dimensional complex manifold to a disk $D = \{t \mid |t| < \epsilon\}$. Assume that :

(1) π is smooth at every point on $\pi^{-1}(D - \{0\})$,

(2) for every point $t \in D - \{0\}$, $X_t = \pi^{-1}(t)$ is a non-singular curve of genus genus 2 .

In view of the theory of an exceptional curve of the first kind we can also assume that

(3) the surface X does not contain exceptional curves of the first kind.

By a (singular) fibre X_0 of $\pi : X \to D$ over the origin, we shall mean a divisor on X defined by $\pi = 0$. X_0 is written in the form

$$\sum_{i=1}^{N} n_i C_i , \; n_i > 0$$

where C_i is an irreducible curve over X .

These (singular) fibres are classified complex analytically by three invariants "Picard-Lefschetz transformation", "modulus point", "degree" associated to each family $\pi : X \to D$. ([6], [7], [8]).

In my talk I pointed out that the notion of a "stable curve" ([2]) played an important role in our theory.

Here I will discuss the relationship between singular and stable curves.

§ 1 . - PICARD-LEFSCHETZ TRANSFORMATIONS .

To any family $\pi : X \to D$ we can associate a Picard-Lefschetz transformation. The following proposition is a special case of a theorem due to Clemens ([1]).

PROPOSITION 1 . -

Assume that the family $\pi : X \to D$ satisfies (1) (2) and

(3') the divisor $X_o = \sum n_i C_i$ has normal crossings.

Let M be a Picard-Lefschetz transformation of $\pi : X \to D$ and let n be the least common multiple of the integers $n_1, n_2, \ldots, n_N$. Then M^n is a unipotent matrix.

Remark. – Applying a finite numbers of successive blowing-ups at points over the origin, the condition (3') will be satisfied. This process does not change the Picard-Lefschetz transformation.

This proposition implies that if a singular fibre over the origin has the form $\sum_{i=1}^{N} C_i$ with normal crossings (i.e. is a reduced curve with ordinary double points), the Picard-Lefschetz transformation is unipotent.

PROPOSITION 2. –

Let $\pi : X \to D$ be a family of curves of genus two which satisfies the conditions (1), (2), (3). Let n be the natural number appearing in the above proposition.

Suppose $f : E = \{s \mid |s| < \varepsilon^{\frac{1}{n}}\} \to D$ is a ramified covering over D defined by

$$s \longrightarrow t = s^n ,$$

and $\tilde{X}$ is a minimal non-singular model of the fibre product $X \times_D E$. Then, we have

(i) the fibre of X over the origin of E is a reduced curve with ordinary double points (i.e. the fibre has the form $\sum C_i$ with normal crossings.)

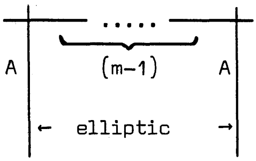
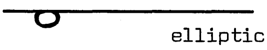
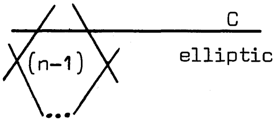
(ii) the cyclic group G of order n of automorphisms of E generated

$$g : s \longmapsto \exp\left(\frac{2\pi\sqrt{-1}}{n}\right). s$$

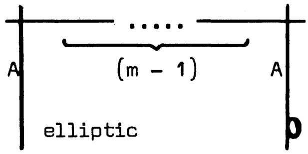
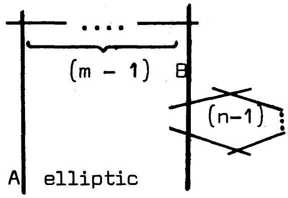
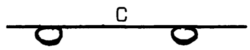
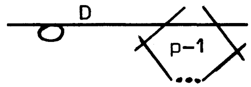
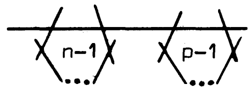
can be lifted to a group G of analytic automorphisms of $\tilde{X}$,

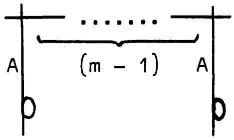
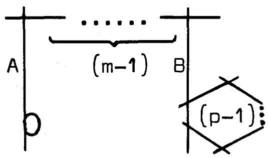
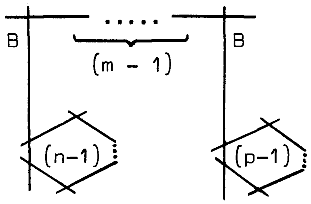
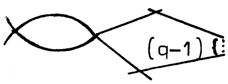
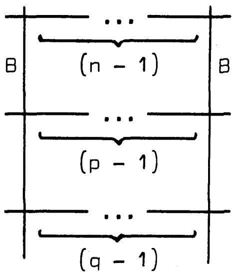
(iii) the minimal non-singular model $\hat{X} \to D$ of the quotient space $\tilde{X}/G \to D$ is complex analytically isomorphic to $\pi : X \to D$.

In view of Proposition 1 the Picard-Lefschetz transformation of the above family $\pi : \tilde{X} \to E$ is unipotent. This proposition implies that the study of families reduces to the study of families whose fibre over the origin is a reduced curve with ordinary double points. By a numerical calculation we find that all possible types of singular fibres, which are reduced curves with ordinary double points, are as follows. ([4], [8]) (*). In the following configurations almost all curves are non-singular rational curves.

[Type] Modulus point	Monodromy	Numerical type (Ogg)	Configuration
$[I_{0-0-0}]$ S_2	$\begin{pmatrix} 1&0&0&0 \\ 0&1&0&0 \\ 0&0&1&0 \\ 0&0&0&1 \end{pmatrix}$	[0]	regular curve of genus 2
$[I_0-I_0-m]$ $(m>0)$ $\begin{pmatrix} z_1 & 0 \\ 0 & z_2 \end{pmatrix}$	$\begin{pmatrix} 1&0&0&0 \\ 0&1&0&0 \\ 0&0&1&0 \\ 0&0&0&1 \end{pmatrix}$	[13]	
$[I_{n-0-0}]$ $(n>0)$ $\begin{pmatrix} z & * \\ * & \infty \end{pmatrix}$	$\begin{pmatrix} 1&0&0&0 \\ 0&1&0&n \\ 0&0&1&0 \\ 0&0&0&1 \end{pmatrix}$	$n=1$ [0] $n>1$ [1]	 

(*) [Note du rapporteur] : The notations of [Type] as $[I_{0-0-0}]$, $[I_0-I_0-m]$ etc.. used in the following table are the classifying notations for fibres in pencils of curves of genus two, which are already used in previous works of the author ([5], [6], [7]) .

[Type] Modulus point	Monodromy	Numerical type (Ogg)	Configuration
$[I_n - I_0 - m]$ $(n > 0)$ $(m > 0)$ $\begin{pmatrix} z & 0 \\ 0 & \infty \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & n \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$	$n = 1$ [13]	
		$n > 1$ [14]	
$[I_{n-p-0}]$ $(n > 0)$ $(p > 0)$ $\begin{pmatrix} \infty & * \\ * & \infty \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & p & 0 \\ 0 & 1 & 0 & n \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$	$n = p = 1$ [0]	
		$n=1,\ p > 1$ and vice versa [1]	
		$n > 1,\ p > 1$ [2]	

[Type] Modulus point	Monodromy	Numerical type (Ogg)	Configuration
$[I_n - I_p - m]$ $(n > 0)$ $(p > 0)$ $(m > 0)$ $\begin{pmatrix} \infty & 0 \\ 0 & \infty \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & p & 0 \\ 0 & 1 & 0 & n \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$	$n = p = 1$ [13]	
		$n=1$, $p>1$ and vice versa [14]	
		$n>1$, $p>1$ [39]	
$[I_{n-p-q}]$ $(n > 0)$ $(p > 0)$ $(q > 0)$ $\begin{pmatrix} \infty & \infty \\ \infty & \infty \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & p+q & -q \\ 0 & 1 & -q & n+q \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$	$n=p=q=1$ [34]	
		$n=p=1$ [35]	
		otherwise [40]	

We adopt, as in Ogg [8] , the following symbol for a component Γ of singular fibres.

Symbol	Genus	Γ^2	$\Gamma \cdot K$
A	1	-1	1
B	0	-3	1
C	1	-2	2
D	0	-4	2
none	0	-2	0

§ 2 . - <u>STABLE CURVES</u> .

First we recall the definition of a family of stable curves in our situation.

<u>DEFINITION</u> . - The fibre space $\pi : X \to D$ is called a family of stable curves over D if the following conditions are satisfied :

(a) X is a 2-dimensional "normal complex space" .
(b) π is proper, surjective and flat and every fibre is a reduced connected curve.
(c) $X_t = \pi^{-1}(t)$ has only ordinary double points.
(d) If C is a non-singular rational component of X_t , then C meets $\overline{X_t - C}$ in at least three points.
(e) $\dim_{\mathbb{C}} H^1(X_t, \mathcal{O}_{X_t}) = 2$.

For our study it is enough to consider the case when $\pi : X \to D$ satisfies the conditions (a) $\sim$ (e) and π <u>is smooth at every point of</u> $\pi^{-1}(D - \{0\})$. In this case the fibre over the origin (we shall call it a stable curve of genus two) is one of the curves of type I_{0-0-0}, $I_0 - I_0 - 1$, I_{1-0-0}, $I_1 - I_0 - 1$, I_{1-1-0} , $I_1 - I_1 - 1$, I_{1-1-1} .

On the other hand, in the above definition, X is only assumed to be a normal complex space. What is the relationship between minimal non-singular models of families of stable curves and families whose fibres over the origin are reduced curves with ordinary double points ?

Let $\pi : X \to D$ be a family of stable curves such that π is smooth at every point over $\pi^{-1}(D - \{0\})$.

Then, all singularities of X are isolated and are contained in the fibre over the origin. One can prove that these singularities are rational double points of type A_m for some natural number m ([2], [7]). An isolated singularity is called <u>a rational double point of type</u> A_m if the singular point is analytically isomorphic to the singular point of the surface in $\mathbb{C}^3$ defined by the equation

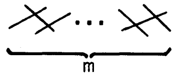
$$x \, . \, y - t^{m+1} = 0 \quad .$$

The minimal resolution of this singularity is well known and it is easy to show that the fibre over the origin of the minimal non-singular model $\pi : \tilde{X} \to D$ is one of the configurations that has appeared in the table. Hence the Picard-Lefschetz transformation of $\pi : \tilde{X} \to D$ (= the Picard-Lefschetz transformation of $\pi : X \to D$) is unipotent.

Conversely, let $\pi : X \to D$ be a family of curves of genus 2 which satisfies the conditions (1), (2), (3). Assume that the fibre over the origin is one of the configurations that has appeared in the table. (i.e. the fibre over the origin is a reduced curve with ordinary double points.) Then, the surface X contains a certain chain of non-singular rational curves  . The intersection matrix of these curves has the form

$$\begin{pmatrix} -2 & 1 & & & & \\ 1 & -2 & 1 & & \bigcirc & \\ & 1 & -2 & & & \\ & & & \ddots & & \\ & \bigcirc & & & -2 & 1 \\ & & & & 1 & -2 \end{pmatrix}$$

and, hence, this matrix is negative definite.

Therefore, by the theorem of Grauert, these curves can be contracted to one point and at that point the new surface is normal ([3]). Moreover, the singular point is a rational double point of type A_m . In this way, from the family $\pi : X \to D$, we can construct the family $\pi : \hat{X} \to D$ which does <u>not</u> contain chains of non-singular rational curves  . Moreover it is not difficult to show that the family $\pi : \hat{X} \to D$ satisfies all the conditions (a) ~ (e) . Hence $\pi : \hat{X} \to D$ is a family of stable curves. We remark that a chain  is contracted to a rational double point of type A_m. In this way there is a one to one correspondance between families $\pi : X \to D$ whose fibre over the origin is a reduced curve with ordinary double points and families of stable curves $\pi : X \to D$ plus their types of singularities. Therefore, by proposition 2, all families can be constructed from families of stable curves. Moreover, we can prove the converse of proposition 1 ([6], [7]).

PROPOSITION 3. –

Let $\pi : X \to D$ be a family which satisfies conditions (1), (2), (3). Then, the following are equivalent.

(i) The fibre over the origin is a reduced curve with ordinary double points.

(ii) The Picard–Lefschetz transformation is unipotent.

§ 3. – EXAMPLES OF FAMILIES OF STABLE CURVES.

We set

$$\tau(t) = \begin{pmatrix} \tau_1 & t^m \\ t^m & \tau_2 \end{pmatrix}, \quad \mathrm{Im}(\tau_1) > 0, \ \mathrm{Im}(\tau_2) > 0, \ m > 0 .$$

We choose a positive number ϵ such that $\mathrm{Im}\,\tau(t)$ is positive definite for all t such such that $|t| < \epsilon$. We set $D = \{t \mid |t| < \epsilon\}$.
For each element $\nu = (\nu_1, \nu_2, \nu_3, \nu_4) \in \mathbb{Z}^4$, let g_ν denote an analytic automorphism of $D \times \mathbb{C}^2$ defined by

$$g_\nu : (t, (u_1, u_2)) \longmapsto \left(t, (u_1, u_2) + \nu \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ & \tau(t) \end{pmatrix}\right) .$$

Then, $G = \{g_\nu\}_{\nu \in \mathbb{Z}^4}$ is an abelian group isomorphic to $\mathbb{Z}^4$ and a properly discontinuous group of analytic automorphisms of $D \times \mathbb{C}^2$.
The quotient space B is a complex manifold and the natural projection $D \times \mathbb{C}^2 \to D$ induces a holomorphic surjective map $\rho : B \to D$. Each fibre of $\rho : B \to D$ is an abelian variety and ρ is smooth at every point. For any point $(t, (u_1, u_2)) \in D \times \mathbb{C}^2$, we denote the corresponding point of B by (t, u_1, u_2).

Let X be a subvariety of B defined by the equation

$$\theta(\tau(t),\ u_1,\ u_2)$$

$$= \sum_{n_1, n_2 \in \mathbb{Z}^2} e\left(\tfrac{1}{2}(n_1+\tfrac{1}{2}, n_2+\tfrac{1}{2})\tau(t)\begin{pmatrix} n_1+\frac{1}{2} \\ n_2+\frac{1}{2} \end{pmatrix} + (n_1+\tfrac{1}{2},\ n_2+\tfrac{1}{2})\begin{pmatrix} u_1+\frac{1}{2} \\ u_2+\frac{1}{2} \end{pmatrix}\right)$$

$$= 0$$

where $e(\ ) = e^{2\pi i(\ )}$. The function (t, u) is usually called the theta function of the first order of characteristic $(1, 1, 1, 1)$. The mapping $\rho : B \to D$ is a non-singular curve of genus two for $0 \neq t \in D$. The fibre over the origin is two elliptic curves which intersect transversally at one point. (i.e. $I_0 - I_0 - 1$). Hence, $\pi : X \to D$ is a family of stable curves. If $m = 1$, the surface X is non-singular and the fibre space $\pi : X \to D$ is imbedded into a topologically trivial fibre space. $\rho : B \to D$. It is not known if the fibre space $\pi' : X' \to D'$ $(D' = D - \{0\},\ X' = \pi^{-1}(D'))$ is topologically trivial or not.

If $m \geq 2$ the surface X has only one singular point $(0, [0, 0])$. This singular point is a rational double point of type A_{m-1} ([7]).

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