

Astérisque

ROBERT KAUFMAN

M-sets and distributions

Astérisque, tome 5 (1973), p. 225-230

[<http://www.numdam.org/item?id=AST_1973__5__225_0>](http://www.numdam.org/item?id=AST_1973__5__225_0)

© Société mathématique de France, 1973, tous droits réservés.

L'accès aux archives de la collection « Astérisque » (<http://smf4.emath.fr/Publications/Asterisque/>) implique l'accord avec les conditions générales d'utilisation (<http://www.numdam.org/conditions>). Toute utilisation commerciale ou impression systématique est constitutive d'une infraction pénale. Toute copie ou impression de ce fichier doit contenir la présente mention de copyright.

NUMDAM

Article numérisé dans le cadre du programme
Numérisation de documents anciens mathématiques
<http://www.numdam.org/>

M-SETS AND DISTRIBUTIONS

Robert Kaufman

0. INTRODUCTION.

A closed subset S of the circle group T is called a weak Kronecker set (K_O -set) if each complex measure μ carried by S has the property $\sup |\hat{\mu}(n)| = \|\mu\|$; S is called an M -set if it carries a distribution $\tau \neq 0$ (of L. Schwartz) whose Fourier transform $\hat{\tau}(n)$ vanishes for $n = \pm\infty$; and S is called M_O if the distribution τ is a finite measure. In 1954 Pyatečĭi-Šapiro [3] showed the existence of sets of type M , not of type M_O ; this work is still striking because it exhibits a specific set S . Then Körner [1] showed the existence of sets of type $M \cap K_O$. In this note we modify the method of [3] to prove.

THEOREM. Each closed set S of type M contains a closed set S_1 of type $M \cap K_O$.

1. Let the N -dimensional torus T^N be represented as the product of intervals $[-\pi, \pi]$ and let V_ϵ^N be the set of N -tuples $(x_1, \dots, x_N)$ such that $|x_k| \leq \epsilon$ for at least $(1-\epsilon)N$ indices $k = 1, 2, \dots, N$ ($0 < \epsilon < 1$). We need a chain of lemmas to prove.

LEMMA 1. For $N > N_\epsilon$ we can find a function F , continuous on T^N , vanishing off V_ϵ^N , such that $|\hat{F}(\chi)| < \epsilon |\hat{F}(0)|$ for all characters $\chi \neq 0$ (in additive notation).

To prove Lemma 1 we construct a special kind of product probability measure on T^N . Let $0 \leq t \leq 1$ and let σ_t be the measure $(2\pi)^{-1} t dx + (1-t) \delta_0$ on t , and λ_t the N -fold product of σ_t , a probability measure on T^N . (The index N is suppressed in λ_t).

LEMMA 2. $\lambda_t(T^N \sim V_\epsilon^N) \rightarrow 0$ as $N \rightarrow +\infty$, uniformly on the interval $0 \leq t \leq \frac{\epsilon}{2}$.

This is a simple consequence of Chebyshev's inequality, because $[-\epsilon, \epsilon]$ has σ_t measure $\geq 1 - \frac{\epsilon}{2}$.

LEMMA 3. Let V be an open set in a compact abelian group G , with dual Γ ; suppose that $\sup \{|\hat{F}(\chi)| : \chi \neq 0\} \geq \epsilon |\hat{F}(0)|$ for every F continuous on G and vanishing on $G \sim V$. Then there is an identity

$$1 = \sum \alpha_\chi \chi(y), \quad \sum |\alpha_\chi| \leq \epsilon^{-1}, \quad \alpha_0 = 0,$$

valid for all y in V .

Proof. The Fourier transform associates to each continuous F an element of the space $C_0(\Gamma)$; assuming that $\epsilon |\hat{F}(0)| \leq \sup \{|\hat{F}(\chi)| : \chi \neq 0\}$ for all F in our subspace, we can write $\hat{F}(0) = \sum' b_\chi \hat{F}(\chi)$, with $\sum |b_\chi| \leq \epsilon^{-1}$. Since V is open, this implies $1 = \sum b_\chi \overline{\chi(y)}$ identically in V .

Proof of Lemma 1. We shall prove that an equality of the type mentioned in Lemma 3 can be valid for only finitely many integers $1, 2, \dots, N_\epsilon$. The key to this is the

formula $\lambda_t(\chi) = (1-t)^k$ ($k = 1, 2, 3, \dots$) for each character $\chi \neq 0$ on T^N .

Suppose $\lambda_t(V_\epsilon^N) \geq 1 - \eta_N$ for $0 \leq t \leq \epsilon/2$, and integrate the identity with respect to λ_t . Then $|1 - \sum_1^\infty C_k^N (1-t)^k| \leq \epsilon^{-1} \eta_N$ over $0 \leq t \leq \epsilon/2$, with $\sum |C_k^N| \leq \epsilon^{-1}$. Since the functions $\sum_1^\infty C_k^N s^k$ form a normal family for $|s| \leq 1$ in the plane, and $\eta_N \rightarrow 0$, the identities in question are possible only for $N \leq N_\epsilon$.

Let F_N be the function just constructed, corresponding to an $\epsilon > 0$ and $N > N_\epsilon$. We must replace F_N by a smooth function, since τ , being a distribution rather than a measure, does not admit multiplication by continuous functions. Let $\psi(x)$ be a smooth approximation to δ_0 vanishing outside a small interval $[-\delta, \delta]$ and let G_N be the convolution $F_N * (\psi(x_1) \dots \psi(x_N))$. Then $\hat{G}_N(0) = \hat{F}_N(0) = 1$ (say) and G_N vanishes outside $V_{\epsilon+\delta}^N \subseteq V_{2\delta}^N$ when $0 < \epsilon < \delta$. Also $|\hat{G}_N(\chi)| < \epsilon |\hat{\psi}(k_1) \dots \hat{\psi}(k_N)|$ when $\chi \neq 0$ and χ has components $(K_1, \dots, K_N)$.

2. Proof of Theorem. Let τ be a distribution such that $\hat{\tau}(\infty) = 0$, and $g(x)$ a real function of class $C^\infty(T)$. For integers $p \geq 1$ we are going to use distributions of the form $\tau_1 = G_N(g(x) - px, \dots, g(x) - p^N x)$. $\tau(dx)$, and observe first of all that the multiplier of τ is smooth on T , so the product is defined. Using the expansion of G_N as a Fourier series on T^N , we can write τ_1 as a sum
$$\sum C(k_1, \dots, k_N) \exp i(k_1 + \dots + k_N)g(x) \cdot \exp -i(pk_1 + \dots + p^N k_N)x \cdot \tau(dx).$$

The distributions with bounded Fourier transforms form a Banach space with the norm $\|\sigma\| = \sup |\hat{\sigma}|$. The sum above converges in norm, uniformly with respect to p .

For $\|\exp -ikx \cdot \tau(x)\| = \|\tau\|$; the $C^1(T)$ -norm of $\exp i(k_1 + \dots + k_N)g(x)$ is $O(1) + O(|k_1 + \dots + k_N|)$, and $|C(k_1, \dots, k_N)| \leq |\hat{\psi}(k_1) \dots \hat{\psi}(k_N)|$ with ψ in

$C^\infty(T)$, so $\hat{\psi}$ decreases rapidly.

We assert now that for large p $\|\tau_1 - \tau\|$ exceeds by $o(1)$ the maximum norm of the summands with $|k_1| + \dots + |k_N| > 0$, a number bounded in turn by $\|\tau\| \cdot \sup |C(k_1, \dots, k_N)| \cdot \|\exp i(k_1 + \dots + k_N)g(x)\|_{C^1}$. In view of the uniform convergence mentioned above, it is sufficient to verify this for finite sums, say for $1 \leq |k_1| + \dots + |k_N| \leq A$. Each distribution in the sum has a transform vanishing at infinity; to each B , and $p > p(B)$, the values of $p k_1 + \dots + p^N k_N$, generated by the N -tuples in question, differ by at least B . This in fact suffices for the necessary bound on $\|\tau_1 - \tau\|$.

Recall that p was chosen after N ; we now study the effect of increasing N , and assert that $|\hat{\psi}(k_1) \dots \hat{\psi}(k_N)| \cdot |k_1 + \dots + k_N|$ remains bounded for all N . In the argument we can assume $1 \leq k_1 \leq \dots \leq k_N$, and observe that $|\hat{\psi}(k)| \leq 1 - \eta$ for $k \geq 1$. Cancellation of k_1 effects a multiplication by at least $(1 - \eta)^{-1}(1 - N^{-1})$, and this exceeds 1 provided $N > \eta^{-1}$. Thus the problem is reduced to the special case $N \leq \eta^{-1}$ and here the inequality $|\hat{\psi}(k)| < k^{-1}$ is at hand. Finally, for large N we have the additional factor $\varepsilon > 0$.

Before applying this to the last step, we recapitulate what has been attained. Given g in $C^\infty(T)$ and $\delta > 0$ we found a function H in $C^\infty(T)$ such that $\|H(x)\tau(x) - \tau(x)\| < \delta$. Moreover there exist integers $p \geq 1$ and $N \geq 1$ such that $H(x) = 0$ unless at least $(1 - \delta)N$ of the inequalities $|g(x) - p^r x| \leq 2\delta$ (modulo 2π) are fulfilled. Of course the closed support of $H(x)\tau(x)$ is contained in that of τ , and also in the set just mentioned.

Beginning with a distribution $\tau \neq 0$, we choose a sequence $(g_j)_1^\infty$, uniformly dense in the real Banach space $C(T)$ and perform a sequence of operations of the kind

just completed. We obtain a distribution $\tau_1 \neq 0$, whose closed support S_1 is contained in the support of S . For each $j \geq 1$ there are integers p_j and N_j so that at least $(1-2^{-j})N_j$ of the N_j inequalities (with $p = p_j$, $g = g_j$, $N = N_j$) $|g(x) - p^r x| \leq 2^{-j}$ (modulo 2π), $1 \leq r \leq N$ are fulfilled at each point in S_1 . Thus S_1 is a K_0 -set; S_1 has the property, somewhat stronger, that each finite measure on S_1 is nearly carried by a Kronecker set.

REFERENCES

- [1] KÖRNER, T. W. A pseudofunction on a Helson set, *ce volume*.
- [2] PYATEČKII-ŠAPIRO, I. I. On the problem of uniqueness of the expansion of a function in a trigonometric series (Russian). *Moskov. Gos. Univ. Uč Zap.* 155 (Math V), (1952), 54-72.
- [3] Supplement to the same. *Ibidem* 165 (Math VII), (1954), 79-97.