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PSEUDO-DIFFERENTIAL EQUATIONS AND THETA FUNCTIONS

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1.- It has been known since a year ago that any system of pseudo-differential equations, i.e. any admissible coherent left module $\mathcal{M}$ of the sheaf of rings $\mathcal{P}$ of pseudo-differential operators, given on the conormal sphere bundle $\sqrt{-1}S^*M$ of an analytical manifold M , is isomorphic to a combined system of de RHAM equations, CAUCHY-RIEMANN equations and LEWY -MIZOHATA equations, when considered micro-locally in the neighborhood of a generic point on the real characteristic variety of $\mathcal{M}$, provided that the complex characteristic variety of $\mathcal{M}$, i.e. the support of the sheaf $\mathcal{M}$ in a complex neighborhood of $\sqrt{-1}S^*M$, meets with its complex conjugate non-tangentially (SATO-KAWAI-KASHIWARA [1] , [2]).

In the simplest case where the characteristic variety is real, the cited structure theorem for pseudo-differential equations says in particular that $\mathcal{M}$ is micro-locally isomorphic to a de RHAM system.

Theoretically this process of transforming $\mathcal{M}$ to the de RHAM system consists of two steps. In the first steps, the celebrated classical theory of JACOBI on the involutory system of first order (non-linear) partial differential equations assures that characteristic variety V of our system $\mathfrak{M}$, which is proved to be a real involutory submanifold in the contact manifold $\sqrt{-1}S^*M$, is brought to the form $V_o = \{(x,i\eta) \in \sqrt{-1}S^*M \mid \eta_1 = \ldots\ldots = \eta_m = 0\}$, $m < \dim M$, by application of a contact tranformation, and consequently our system $\mathfrak{M}$ is, by application of a corresponding quantized contact tranformation, brought to a system of the form

$$\mathfrak{M}_o : \qquad \frac{\partial}{\partial x_j} u = P_j(x,D')u \quad , \qquad j=1,\ldots,m.$$

Here D' means $\frac{\partial}{\partial x_j}$ for $j>m$, u denotes a column vector of unknown functions (i.e. generators of the $\mathcal{P}$-module $\mathfrak{M}$), and $P_j(x,D')$ denote matrices of pseudo-differential operators of finite order satisfying the following two conditions : first, they should satisfy the compatibility condition

$$\frac{\partial P_j}{\partial x_k} + P_j P_k = \frac{\partial P_k}{\partial x_j} + P_k P_j \quad , \quad i,j = 1,\ldots\ldots,m \quad ;$$

second, they should be matrix operators "of orders smaller than 1 ", so that $\mathfrak{M}_o$ would have V_o as its characteristic variety.

The second step in the process of transforming $\mathfrak{M}$ is to bring $\mathfrak{M}_o$ further to the de RHAM type : $\frac{\partial}{\partial x_j} u_o = 0$, by eliminating the " lower orders terms", i.e. the terms $P_j(x,D')u$ in $\mathfrak{M}_o$. And this elimination is achieved as follows by using pseudo-differential operators of infinite order (which of course ar micro-local operators). Namely, we construct inversible pseudo-differential operators $U(x,D')$ satisfying

$$\frac{\partial}{\partial x_j} . U(x,D') - U(x,D') . \frac{\partial}{\partial x_j} = P_j(x,D') . U(x,D') \quad , \quad j=1,\ldots\ldots,m \quad ,$$

$$U(x,D') \big|_{x_1=\ldots=x_m=0} = I \quad (=\text{the unit matrix}) \quad ,$$

and then by putting $u_o = U^{-1} u$, $u = Uu_o$, we see that the system $\mathfrak{M}_o$ is readily tranformed to the de RHAM system for u_o. The matrix $U(x,D')$ will be called the wave operator for $\mathfrak{M}_o$, because this operator describes solutions of $\mathfrak{M}_o$ in terms of its initial data : $u(x) = U(x,D')[u(x)]_{x_1=\ldots=x_m=0}$. The characteristic variety V_0 has a natural foliation structure where the leaves are (m-dimensional) bicharacteristic strips defined by $x_j = \text{const.}$, $\eta_j = \text{const.}$ for $j > m$. The wave operator describes the programtion of initial data along each leaf.

Now let us suppose further that the characteristic variety V has a fiber structure $V \xrightarrow{f} W$ (smooth) rather than a foliation structure. The fibers of f are bicharacteristic strips, which we assume to be all isomorphic to a typical one, an m-dimensional manifold F , and V is isomorphic to $F \times W$. The base space W has the structure of a contact manifold, and is identified with a conormal bundle $\sqrt{-1}S^*N$ whose points we describe by $(t, i\tau)$. Denoting by x the coordinates of a point on the universal covering manifold $\tilde{F}$ of F , our equations will now assume the form :

$$\frac{\partial}{\partial x_j} u(x,t) = P_j(x,t,\frac{\partial}{\partial t})u(x,t).$$

On taking into account the fact that $\mathfrak{M}$ is a system on $F \times W = (\tilde{F} \times W)/\pi_1(F)$, $\pi_1(F)$ denoting the fundamental group of F , we observe that finding solutions of $\mathfrak{M}$ on $F \times W$ amounts to finding a solution of the above equations on $\tilde{F} \times W$ which possesses a quasi-periodicity condition of the form

$$u(\sigma(x),t) = T_\sigma(x,t,\frac{\partial}{\partial t})u(x,t). \qquad \forall\ \sigma \in \pi_1(F) ,$$

where $T_\sigma(x,t,\frac{\partial}{\partial t})$ are family of invertible pseudo-differential operators in t subject to the conditions

$$T_{\sigma'\sigma}(x,t,\frac{\partial}{\partial t}) = T_{\sigma'}(\sigma(x),t,\frac{\partial}{\partial t}).T_\sigma(x,t,\frac{\partial}{\partial t}),$$

$$\frac{\partial}{\partial x_j} T_\sigma(x,t,\frac{\partial}{\partial t}) = - T_\sigma(x,t,\frac{\partial}{\partial t}).P_j(x,t,\frac{\partial}{\partial t}) + \sum_k \frac{\partial \sigma(x)_k}{\partial x_j} P_k(\sigma(x),t,\frac{\partial}{\partial t}).T_\sigma(x,t,\frac{\partial}{\partial t}).$$

Defining S-matrices by $S_\sigma(t,\frac{\partial}{\partial t}) = T_\sigma^{-1}(0,t,\frac{\partial}{\partial t}).U(\sigma(0),t,\frac{\partial}{\partial t})$ by means of T_σ and wave operator $U(x,t,\frac{\partial}{\partial t})$, we see that $S_\sigma(t,\frac{\partial}{\partial t})$ are invertible pseudo-differential operators in t and satisfy the relation $S_{\sigma'\sigma}(t,\frac{\partial}{\partial t}) = S_{\sigma'}(t,\frac{\partial}{\partial t}).S_\sigma(t,\frac{\partial}{\partial t})$, and that an initial data $u(0,x)$ admits the corresponding global solution of our system (which clearly is uniquely determined) if and only if $S_\sigma(t,\frac{\partial}{\partial t})u(0,t) = u(0,t)$ hold for all $\sigma \in \pi_1(F)$, i.e. if and only if $u(0,t)$ is a simultaneous eigenfunction of $S_\sigma(t,\frac{\partial}{\partial t})$ of eigenvalues 1.

2.- We now apply the preceding observations to the situation where the fiber F is a $2n$-dimensional torus $\mathbb{R}^{2n}/\mathbb{Z}^{2n}$. $\tilde{F}$ and $\pi_1(F)$ are $\mathbb{R}^{2n}$ and $\mathbb{Z}^{2n}$ respectively, and σ is given by $x \to x+\nu$, $\nu \in \mathbb{Z}^{2n}$.

First we give the following definition :

Definition.- A set of $2n$ pseudo-differential operators on $W = \sqrt{-1}\, S^*N$, $P_j(t,\frac{\partial}{\partial t})$, $j = 1,\ldots, 2n$, (or rather, the linear set spanned by them $\{\nu_1 P_1 + \ldots + \nu_{2n}P_{2n} \mid \nu \in \mathbb{Z}^{2n}\}$) is called a Jacobi structure on W if the following conditions are satisfied :

(1) P_j satisfy the commutation relation $P_kP_j - P_jP_k = 2\pi i e_{jk}$, with $e_{jk} = - e_{kj} \in \mathbb{Z}$, $\det(e_{jk}) \neq 0$.

(2) P_j are pseudo-differential operators of orders smaller than 1.

(From (1) and (2) follows that $c_1P_1 + \ldots + c_nP_{2n}$ also has an order smaller than 1, for any $c \in \mathbb{C}^{2n}$.)

Suppose that a Jacobi structure $(P_j(t,\frac{\partial}{\partial t}))_{j=1,\ldots,2n}$ is given on W. Then, defining the operators $P_j(x,t,\frac{\partial}{\partial t})$ by

$$P_j(x,t,\frac{\partial}{\partial t}) = \pi i(Ex)_j + P_j(t,\frac{\partial}{\partial t})$$

with $E = (e_{jk})$ and $(Ex)_j = \Sigma_k e_{jk}x_k$, and choosing $T_\sigma(x,t,\frac{\partial}{\partial t})$ to be a multiplication operator by a factor $c(\nu)e^{\pi i\langle E\nu,x\rangle}$ (where $\langle E\nu,x\rangle = \Sigma_{j,k}e_{jk}\nu_kx_j$ while $c(\nu)$ is a non-zero constant satisfying the relation $c(\nu'+\nu)/c(\nu')c(\nu) = (-1)^{\langle E\nu',\nu\rangle}$ and is given e.g. by $c(\nu) = (-1)^{\Sigma_{j<k}e_{jk}\nu_k\nu_j}$), we see that all requirements imposed in the preceding paragraph are satisfied. The S-matrices are given by $S_\nu(t,\frac{\partial}{\partial t}) = c(\nu)^{-1}e^{\pi i(\nu_1P_1+\ldots+\nu_{2n}P_{2n})}$. They are mutually commutative although P_j are not.

Definition.- A column vector of microfunctions on W is called a Jacobi function if it is a simultaneous eigenfunction of $e^{\pi i P_1}, \ldots, e^{\pi i P_{2n}}$ of eigen-value 1 (and hence a simultaneous eigenfunction of $e^{\pi i(\nu_1 P_1 + \cdots + \nu_{2n} P_{2n})}$ of eigen-value $c(\nu)$ for all $\nu \in \mathbb{Z}^{2n}$).

Definition.- A column vector of microfunctions $\Theta(x|t)$ on $\tilde{F} \times W = \mathbb{R}^{2n} \times W$ is called a theta function, associated to the Jacobi structure, if the followings hold.

$$(1) \quad \left(\frac{\partial}{\partial x_j} - (Ex)_j\right) \Theta(x|t) = P_j\left(t, \frac{\partial}{\partial t}\right) \Theta(x|t)$$

$$(2) \quad (x+\nu|t) = c(\nu)\, e^{\pi i \langle E\nu, x\rangle}\, \Theta(x|t), \qquad \nu \in \mathbb{Z}^{2n} .$$

From the observations of preceding paragraph we obtain

THEOREM.- If $\Theta(x|t)$ is a theta function associated to the Jacobi structure then the 'zero-value' $\Theta(0|t)$ is a Jacobi function. Conversely, any Jacobi function $f(t)$ on W determines uniquely a theta function $\Theta(x|t)$ with the property $\Theta(0|t) = f(t)$ uniquely.

It is known that, from micro-local stand point, it is not very restricting to assume that the underlying contact manifold $W = \sqrt{-1}\, S^*N$ has the dimension $2n-1$ (i.e. $\dim N = n$). If this is the case, we can show that the number of linearly independent theta functions or Jacobi functions is finite. Still more important is the case where operators P_j are of the orders $\frac{1}{2}$, because we can then introduce a natural representation of the symplectic group $Sp(n)$ by infinitesimal operators $\frac{1}{2}(P_j P_k + P_k P_j)$, can prolong the germ of the manifold W to a $(2n-1)$-dimensional projective space of $2n$-dimensional symplective vector space in a natural way, and can deduce the automorphy property of Jacobi function $\Theta(0|t)$ under the action of $Sp(n,\mathbb{Z})$. (The 'factor of automorphy' appears to be a pseudo-differential operator of infinite order in general).

It is known that our concept of theta function includes a wide class of functions, of which the well-known class of theta functions of Siegel-Hilbert type is a very special example.

Some detailed account for what is stated here is found in [3]. Complete details will appear elsewhere.

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