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SOME APPLICATIONS OF MICRO-LOCAL ANALYSIS
TO THE GLOBAL STUDY OF LINEAR DIFFERENTIAL EQUATIONS (*)

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0.- INTRODUCTION.-

The purpose of this report is to give a survey on the application of micro-local analysis to the global study of linear differential equations. The micro-local analysis means the local analysis on the cotangential sphere bundle, which has been recently fully developed in Sato, Kawai and Kashiwara [1], [2]. See also Kawai [2], [3], Maslov [1], Egorov [1], Hörmander [1], [2] and Duistermaat and Hörmander [1].

§1 treats the problem of the global existence of real analytic solutions of single linear differential equations with constant coefficients. At first sight the micro-local analysis seems not to give any informations about the real analytic solutions, because the object of micro-local analysis is the sheaf of microfunctions

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which Sato [1] has introduced in order to analyze the structure of hyperfunctions modulo real analytic functions. However, the introduction of microfunctions allows us to investigate explicitly the role of bicharacteristic strips themselves played on the propagation of singularities of solutions of (pseudo-)differential equations (Kawai [2], see also Sato, Kawai and Kashiwara [1], [2], Hörmander [1], Duistermaat and Hörmander [1] and Andersson [1]), and this fact combined with one of the fundamental properties of the sheaf of microfunctions, i.e. its flabbiness, is an essential step in the proof of the global existence of real analytic solutions as is shown below.

§2 is devoted to present some theorems on the finite dimensionality of the cohomology groups of some differential complexes. The study of such cohomology groups essentially relies on the micro-local analysis developed in Sato, Kawai and Kashiwara [2] Chapter III and the local study of boundary value problems of linear differential equations developed in Komatsu and Kawai [1] and Kashiwara [1].

The details of §1 will appear in Kawai [6] and those of §2 will appear in Kawai [7]. See also Kawai [4], [5].

1.- GLOBAL EXISTENCE OF REAL ANALYTIC SOLUTIONS.-

In this section we always assume the following condition (1.1).

(1.1) The linear differential operator $P(D)$ under consideration has only constant multiple real characteristics, that is, its principal symbol $P_r(\eta)$ has the form $(Q_m(\eta))^s$ where $Q_m(\eta)$ has real coefficients and $\text{grad}_{\eta} Q_m(\eta)$ never vanishes whenever $Q_m(\eta) = 0$ with $\eta \in \mathbb{R}^n - \{0\}$.

In this report we do not discuss the case where condition (1.1) does not hold. See Kawai [4], [6] where some generalizations are given.

Condition (1.1) assures the existence of the following two "good" elementary solutions $E_+(x)$ and $E_-(x)$ of $P(D)$ satisfying condition (1.2) $_+$ and (1.2) $_-$ respectively. (See Kawai [2], [6].)

$$\begin{aligned}
 (1.2)_+ \quad & S.S.E_+(x) \subset \{(x, \sqrt{-1}\eta) \in \sqrt{-1} S^*R^n | \\
 & x = 0\} \cup \{(x, \sqrt{-1}\eta) \in \sqrt{-1} S^*R^n | \\
 & x = t \operatorname{grad}_{\eta} Q_m(\eta) \quad (t \geq 0), Q_m(\eta) = 0\} \\
 (1.2)_- \quad & S.S.E_-(x) \subset \{(x, \sqrt{-1}\eta) \in \sqrt{-1} S^*R^n | \\
 & x = 0\} \cup \{(x, \sqrt{-1}\eta) \in \sqrt{-1} S^*R^n | \\
 & x = t \operatorname{grad}_{\eta} Q_m(\eta) \quad (t \leq 0), Q_m(\eta) = 0\}.
 \end{aligned}$$

Here and in the sequel we use freely the notions and notations of the theory of microfunctions. The reader who is not familiar with them is referred to Sato, Kawai and Kashiwara [2] Chapter I.

Now we show how the existence of good elementary solutions proves the existence of real analytic solutions of the differential equation $P(D)u = f$. We first investigate the case where the given function $f(x)$ belongs to $\mathcal{A}(K)$ for a suitable compact set $K \subset \mathbb{R}^n$ (Theorem 1.1). Here $\mathcal{A}(K)$ denotes the space of real analytic functions on K , i.e. $\mathcal{A}(K) = \lim_{\overleftarrow{V \supset K}} \mathcal{O}(V)$, where V is a complex neighborhood of K and $\mathcal{O}(V)$ denotes the space of holomorphic functions on V . This case may be treated by the method of functional analysis combined with the result of Kawai [2] and the abstract form of the uniqueness theorem of Holmgren

concerning hyperfunctions with a real analytic parameter (Kawai [1]), since $\mathcal{A}(K)$ becomes a DFS-space. (See Kawai [6] §6.). However, we prefer here more direct proof partly because the following proof is the prototype of the proof in the case where $f(x)$ is only assumed to be analytic on an open set Ω (Theorem 1.2). Note that the linear topological structure of $\mathcal{A}(\Omega)$, the space of real analytic functions on an open set Ω , is so complicated that it is very difficult to use the method of functional analysis in that case. (See Ehrenpreis [2].)

In the sequel the image under the natural projection π from $\sqrt{-1} S^*R^n$ to R^n of the bicharacteristic strip of $Q_m(D)$ passing through $(x^0, \sqrt{-1}\eta^0)$ will be called by an abuse of language the bicharacteristic curve of $P(D)$ passing through $(x^0, \sqrt{-1}\eta^0)$. We denote the curve in R^n by $\ell_{(x^0, \sqrt{-1}\eta^0)}$.

THEOREM 1.1.

Assume that the compact set K is the closure of an open set $\Omega = \{x \in R^n \mid \varphi(x) < 0\}$ where $\varphi(x)$ is a real valued real analytic function defined in a neighborhood of K satisfying $\text{grad}_x \varphi \neq 0$ on $\partial\Omega$, the boundary of Ω . Assume further that Ω satisfies the following condition (1.3). Then for any $f(x) \in \mathcal{A}(K)$ we can find $u(x) \in \mathcal{A}(\Omega)$ such that $P(D)u(x) = f(x)$ holds in Ω . For any characteristic boundary point $x^0 \in \partial\Omega$, i.e. the point where $P_r(\text{grad}_x \varphi(x)|_{x=x^0}) = 0$ holds, the bicharacteristic curve $\ell_{(x^0, \sqrt{-1} \text{grad}_x \varphi(x)|_{x=x^0})}$ of $P(D)$ passing through $(x^0, \sqrt{-1} \text{grad}_x \varphi(x)|_{x=x^0})$ never intersects Ω .

PROOF.- We first define a hyperfunction $\tilde{f}(x)$ with compact support K by $f(x)Y(\varphi(x))$. Here $Y(t)$ denotes the 1-dimensional step function of Heaviside and $Y(-\varphi(x))$ is a well-defined hyperfunction in n -variables since $\text{grad}_x \varphi(x)$ never

vanishes whenever $\phi(x) = 0$. We next define a hyperfunction $\tilde{u}(x)$ by $\int_+ (x-y)\tilde{f}(y)dy$.

The theory of integral along fibers of hyperfunctions and microfuctions (Sato, Kawai and Kashiwara [2] Chapter 1) assures that the above integral is well-defined and that

$$(1.4) \quad \begin{aligned} S.S.\tilde{u}(x) \subset \{ (x, \sqrt{-1}\eta) \in \sqrt{-1} S^*\mathbb{R}^n | \\ x \in \partial\Omega \} \cup \{ (x, \sqrt{-1}\eta) \in \sqrt{-1} S^*\mathbb{R}^n | \end{aligned}$$

there exists y in $\partial\Omega$ such that

$$x = y + t \operatorname{grad}_{\eta_m} Q(\eta) \quad (t \geq 0), \quad P_r(\operatorname{grad}_y \phi(y)) = 0$$

and that

$$\eta = \frac{t}{2} \operatorname{grad}_y \phi(y)$$

holds. Condition (1.3) combined with (1.4) clearly proves that the hyperfunction

$\tilde{u}(x)$ is real analytic in Ω by the definition of $S.S.\tilde{u}(x)$. On the other hand

$P(D)\tilde{u}(x) = \tilde{f}(x)$ holds as an equality for hyperfunctions, since $P(D)E_+(x) = \delta(x)$

and since $\tilde{f}(x)$ has compact support. Therefore the restriction $u(x)$ of $\tilde{u}(x)$ to Ω

is evidently the required analytic solution of the equation $P(D)u(x) = f(x)$ in Ω .

Q.E.D.

Remark.— Condition (1.3) may be weakened a little. It is also possible to modify conditions in the theorem so that it gives the existence of real analytic solutions on K . For these topics we only refer to Kawai [6] §2 and do not discuss them here.

We next prove the existence of real analytic solutions on a relatively compact open set $\Omega \subset \mathbb{R}^n$ with smooth boundary satisfying the following condition (1.5).

THEOREM 1.2.

$$(1.5) \quad \left| \begin{array}{l} P(D)a(\Omega) = a(\Omega) \text{ holds under the following assumption :} \\ \text{Any bicharacteristic curve of } P(D) \text{ intersects } \Omega \text{ in an} \\ \text{open interval.} \end{array} \right.$$

PROOF.- Since the sheaf $\mathcal{B}$ of hyperfunctions is flabby, we can find a hyperfunction $\tilde{f}(x)$ for any $f(x) \in \mathcal{A}(\Omega)$ so that $\tilde{f}(x) = f(x)$ in Ω and that $\text{supp } \tilde{f}(x) = \Omega$. Then

$$(1.6) \quad S.S.\tilde{f}(x) \subset M = \{(x, \sqrt{-1}\eta) \in \sqrt{-1} S^*\mathbb{R}^n | x \in \partial\Omega\}$$

holds by the definition. Moreover the smoothness of the boundary assures that the set $N = \{(x, \sqrt{-1}\eta) \in \sqrt{-1} S^*\mathbb{R}^n | x \in \partial\Omega, P_r(\eta) = 0\}$ can be decomposed into the union of two closed set $N_+, S.S.\tilde{f}_+(x) \subset M$ and $N_-, S.S.\tilde{f}_-(x) \subset M$, where

$N_+ = \{(x, \sqrt{-1}\eta) \in \sqrt{-1} S^*\mathbb{R}^n | x \in \partial\Omega, P_r(\eta) = 0 \text{ and the positive half of bicharacteristic curve through } (x, \sqrt{-1}\eta), \text{ i.e., } x+t \text{ grad}_{\eta} Q_m(\eta) \text{ } (t \geq 0), \text{ does not intersect } \Omega\}$ and $N_- = \{(x, \sqrt{-1}\eta) \in \sqrt{-1} S^*\mathbb{R}^n | x \in \partial\Omega, P_r(\eta) = 0 \text{ and the negative half of the bicharacteristic curve through } (x, \sqrt{-1}\eta), \text{ i.e., } x+t \text{ grad}_{\eta} Q_m(\eta) \text{ } (t \leq 0), \text{ does not intersect } \Omega\}$. Since sheaf $\mathcal{C}$ of microfunctions is flabby (Sato, Kawai and Kashiwara [2] Chapter III Corollary 2.1.5) and since $H^1(\mathbb{R}^n, \mathcal{A}) = 0$, we can find two hyperfunctions $\tilde{f}_+(x)$ and $\tilde{f}_-(x)$ so that

$$(1.7) \quad S.S.(\tilde{f}(x) - \tilde{f}_+(x) - \tilde{f}_-(x)) = \emptyset,$$

$$(1.8)_+ \quad S.S.\tilde{f}_+(x) \cap N \subset N_+$$

and

$$(1.8)_- \quad S.S.\tilde{f}_-(x) \cap N \subset N_-$$

hold. Then conditions (1.2) $_{\pm}$, (1.6) and (1.8) $_{\pm}$ prove by the aid of the theory of integration of microfunctions along fibers that the following integral along fibers $\tilde{u}(x)$ is well-defined as a microfunction :

$$(1.9) \quad \tilde{u}(x) = \int E_+(x-y) \tilde{f}_+(y) dy + \int E_-(x-y) \tilde{f}_-(y) dy.$$

Moreover condition (1.7) implies that

$$(1.10) \quad P(D)\tilde{u}(x) = \tilde{f}(x)$$

holds as an equality for microfunctions. It immediately means that we can find a hyperfunction $V(x)$ so that

$$(1.11) \quad P(D)V(x) = \tilde{f}(x) + g(x)$$

holds on $\mathbb{R}^n$ as an equality for hyperfunctions where $g(x) \in \mathcal{A}(\mathbb{R}^n)$, since $H^1(\mathbb{R}^n, \mathcal{A}) = 0$.

Now, taking a sufficiently large ball as K , we apply Theorem 1.1. and find a

real analytic function $h(x)$ on Ω satisfying $P(D)h(x) = g(x)$ in Ω . Then,

subtracting $h(x)$ from $V(x)$, we obtain the required real analytic solution $u(x)$

on Ω of the equation $P(D)u(x) = f(x)$. Q.E.D.

Remark.- If Ω has the form $\{x \in \mathbb{R}^n \mid \varphi(x) < 0\}$ with the same regularity condition

on $\varphi(x)$ and $\partial\Omega$ as in Theorem 1.1, we can localize condition (1.5) at least

concerning the bicharacteristic curve of $P(D)$ which does not touch Ω . As for

this point we refer the reader to Kawai [6] Theorem 3.2.

At the end of this section we note that de Giorgi and Cattabriga [1]

have obtained an affirmative answer to the problem of the global existence of real

analytic solutions of the equation $P(D)u(x) = f(x)$ on $\mathbb{R}^2$.

2.- FINITE-DIMENSIONALITY OF COHOMOLOGY GROUPS ATTACHED TO SYSTEMS OF LINEAR DIFFERENTIAL EQUATIONS.-

Throughout this section we always assume that the system $\mathcal{M}$ of linear differential equations, i.e. the coherent left $\mathcal{D}$ -Module, is defined on a real analytic manifold M , that it is admissible and that it has the following free resolution by $\mathcal{D}$:

$$(2.1) \quad 0 \leftarrow \mathcal{M} \leftarrow \mathcal{D}^{r_0} \xleftarrow{Q_0} \mathcal{D}^{r_1} \xleftarrow{Q_1} \mathcal{D}^{r_2} \xleftarrow{\quad} \dots\dots\dots,$$

where $\mathcal{D}^{r_j}$ denotes r_j -tuple of $\mathcal{D}$, the sheaf of linear differential operators.

In the sequel we freely use the notions and fundamental properties concerning systems of pseudo-differential equations (in hyperfunction theory) and we refer to Sato, Kawai and Kashiwara [2] for them. Here we only emphasize the fact that (pseudo-) differential operators under consideration may be of infinite order.

This fact plays its essential role in investigating the structure of general systems of pseudo-differential equations. (See Sato, Kawai and Kashiwara [2], especially Chapter II §5.). Here we only recall the following instructive example : the structure of distribution solution sheaf of the equation $(\partial^2 / \partial_1^2 - \partial / \partial x_2)u = 0$ is completely different from that of $\partial^2 \partial x_1^2 u = 0$, while the employment of (pseudo-)differential operators of infinite order, which is inherent in hyperfunction theory, reveals that the structures of hyperfunction solution sheaf of the above two equations are the same as left $\mathcal{D}$ -Modules. We also note that Guillemin [1] has announced results close to ours under the assumption of simple characteristic condition on $\mathcal{M}$.

At first we consider the case where the underlying manifold M is compact and oriented.

THEOREM 2.1.

Assume that the generalized Levi form $L(V)$ attached to the characteristic variety $V = S.S. \mathcal{M}$ of the system $\mathcal{M}$ has always at least q negative eigenvalues on $V \cap \sqrt{-1} S^*M$. Then

$$(2.2) \quad \dim_{\mathbb{C}} \text{Ext}_{\mathcal{D}}^j(M, \mathcal{M}, \mathcal{A}) = \dim_{\mathbb{C}} \text{Ext}_{\mathcal{D}}^j(M, \mathcal{M}, \mathcal{B}) < \infty \quad \text{holds for } j < q.$$

THEOREM 2.2.

Assume that there exists an integer q which is equal to or larger than

$$\sup_{(x, \sqrt{-1}\eta) \in U_{x_0^*}} (\dim_{(x, \sqrt{-1}\eta)} \text{proj}_{\Pi^{-1}\mathcal{D}} (\mathcal{P}^{\otimes} \Pi^{-1}\mathcal{M})) - p \quad \text{for any}$$

$x_0^* = (x_0, \sqrt{-1}\eta^0) \in V \cap \sqrt{-1} S^*M$, where $U_{x_0^*}$ is a suitable neighborhood of x_0^* , Π denotes the projection from $\sqrt{-1} S^*M$ to M , $\dim_{(x, \sqrt{-1}\eta)} \text{proj}_{\Pi^{-1}\mathcal{D}} (\mathcal{P}^{\otimes} \Pi^{-1}\mathcal{M})$ denotes the projective dimension of left $\mathcal{P}$ -Module $\mathcal{P}^{\otimes} \Pi^{-1}\mathcal{M}$ and p is equal to or smaller than the number of positive eigenvalues in U_{x^*} of the generalized Levi form $L(V)$ attached to the characteristic variety V of $\mathcal{M}$. Then

$$(2.3) \quad \dim_{\mathbb{C}} \text{Ext}_{\mathcal{D}}^j(M, \mathcal{M}, \mathcal{A}) = \dim_{\mathbb{C}} \text{Ext}_{\mathcal{D}}^j(M, \mathcal{M}, \mathcal{B}) < \infty \quad \text{holds for } j > q+1$$

and

$$(2.4) \quad \dim_{\mathbb{C}} \text{Ext}_{\mathcal{D}}^{q+1}(M, \mathcal{M}, \mathcal{B}) < \infty$$

holds.

For simplicity we say that the system M is q -convex (resp. q -concave) at x_0^* when the assumption in Theorem 2.1. (resp. Theorem 2.2) holds in a neighborhood $U_{x_0^*}$ of $x_0^* \in V \cap \sqrt{-1} S^*M$. Using these terminologies we have the following

generalization of the above theorems.

THEOREM 2.3.

Assume that the system $\mathcal{M}$ is either $(q+1)$ -convex or $(q-1)$ -concave for any point x_0^* in $\bigcap_{i=1}^n S^*M$. Then

$$(2.5) \quad \dim_{\mathbb{C}} \text{Ext}_{\mathbb{C}}^q(M, M_{\mathbb{C}}(\mathcal{A})) < \infty \quad \text{holds.}$$

The proof of these theorems are given in the following way. As for the detailed arguments we refer the reader to Kawai [7].

Sketch of the proof of Theorem 2.1 ~ Theorem 2.3. By the aid of Theorem 2.3.10 in Chapter III of Sato, Kawai and Kashiwara [2] we easily conclude in the case of Theorem 2.1 that

$$(2.6) \quad \text{Ext}_{\mathcal{A}}^j(M, \mathcal{B}/\mathcal{A}) = 0$$

holds for $j < q$. Then, taking into account the following long exact sequence (2.7) of cohomology groups, we find that $\text{Ext}_{\mathcal{D}}^j(M, \mathcal{M}_A)$ is (at least algebraically) isomorphic to $\text{Ext}_{\mathcal{D}}^j(M, \mathcal{M}_B)$ for $j < q$.

$$(2.7) \quad \begin{aligned} 0 \longrightarrow \text{Ext}_{\mathcal{P}}^0(M, \mathcal{N}, a) &\longrightarrow \text{Ext}_{\mathcal{P}}^0(M, \mathcal{N}, b) \longrightarrow \text{Ext}_{\mathcal{P}}^0(M, \mathcal{N}, b/a) \\ &\longrightarrow \text{Ext}_{\mathcal{P}}^1(M, \mathcal{N}, a) \longrightarrow \dots \dots \dots \\ &\dots \dots \dots \longrightarrow \text{Ext}_{\mathcal{P}}^{j-1}(M, \mathcal{N}, b/a) \\ &\longrightarrow \text{Ext}_{\mathcal{P}}^j(M, \mathcal{N}, a) \longrightarrow \text{Ext}_{\mathcal{P}}^j(M, \mathcal{N}, b) \longrightarrow \text{Ext}_{\mathcal{P}}^j(M, \mathcal{N}, b/a) \\ &\longrightarrow \text{Ext}_{\mathcal{P}}^{j+1}(N, \mathcal{N}, a) \longrightarrow \dots \dots \dots \end{aligned}$$

On the other hand we can endow $\text{Ext}_{\mathcal{A}}^j(M, \mathcal{N}_b, \mathcal{A})$ and $\text{Ext}_{\mathcal{B}}^j(M, \mathcal{N}_b, \mathcal{B})$ with the natural quotient topologies (not necessarily Hausdorff, a priori) by the aid of free resolution (2.1) of $\mathcal{N}_b$, since $\mathcal{A}(M)$ (resp. $\mathcal{B}(M)$) is a DFS- (resp. FS-) space.

Using the arguments usually employed in the theory of analytic functions of several complex variables, we can show that $\text{Ext}_{\mathcal{D}}^j(M, \mathcal{M}, \mathcal{B})$ ($j < q$) is a quotient space of an FS-space by its closed subspace. Hence $\text{Ext}_{\mathcal{D}}^j(M, \mathcal{M}, \mathcal{B})$ ($j < q$) is an FS-space.

The essential step in obtaining this statement is the employment of the celebrated theorem of Schwartz [1] concerning compact perturbations of a linear operator with closed range. (See for example Hitotumatu [1] Chapter 12. and Bony and Schapira [1]).

Note that what is necessary at this point is the surjectivity of the natural map i_j from $\text{Ext}_{\mathcal{D}}^j(M, \mathcal{M}, \mathcal{A})$ to $\text{Ext}_{\mathcal{D}}^j(M, \mathcal{M}, \mathcal{B})$. Since i_j is continuous with respect to the natural quotient topologies on $\text{Ext}_{\mathcal{D}}^j(M, \mathcal{M}, \mathcal{A})$ and $\text{Ext}_{\mathcal{D}}^j(M, \mathcal{M}, \mathcal{B})$, the injectivity of i_j ($j < q$) proves that $\text{Ext}_{\mathcal{D}}^j(M, \mathcal{M}, \mathcal{A})$ is Hausdorff. This immediately implies that $\text{Ext}_{\mathcal{D}}^j(M, \mathcal{M}, \mathcal{A})$ is a DFS-space and that $\text{Ext}_{\mathcal{D}}^j(M, \mathcal{M}, \mathcal{B})$ is topologically isomorphic to $\text{Ext}_{\mathcal{D}}^j(M, \mathcal{M}, \mathcal{A})$ for $j < q$.

Hence we may easily conclude that (2.2) holds under the assumption of Theorem 2.1.

In the same way as above we may show that $\text{Ext}_{\mathcal{D}}^{q+1}(M, \mathcal{M}, \mathcal{B})$ (resp. $\text{Ext}_{\mathcal{D}}^q(M, \mathcal{M}, \mathcal{B})$) is topologically isomorphic to a suitable DFS-space under the assumption of Theorem 2.2 (resp. Theorem 2.3). It proves (2.4) (resp. (2.5)). The proof of (2.3) is just the same as that of Theorem 2.1 Q.E.D.

Now we go on to the investigation of the case where the underlying manifold is not necessarily compact. In the rest of this section we assume that the system $\mathcal{K}$ of linear differential equations under consideration is defined on a compact, oriented and real analytic manifold L and that N is an open set with C^ω -boundary M contained in L . Then our main result in this case is the following.

THEOREM 2.4.

Assume that $\mathcal{K}$ is elliptic, that is, $S.S.\mathcal{K}(\overline{N-1}) S^*L = \emptyset$
 Assume further that the tangential system $\mathcal{M} = \mathcal{I} * \mathcal{M}$
 induced from $\mathcal{K}$ by $\iota : M \hookrightarrow L$ satisfies the condition of
 Theorem 2.3, i.e., $\mathcal{M}$ is either $(q+1)$ -convex or
 $(q-1)$ -concave at any point in $S.S.M(\overline{N-1}) S^*M$. Then

$$(2.8) \quad \dim_{\mathbb{C}} \text{Ext}_{\mathcal{D}}^q(N, \mathcal{K}, \mathcal{A}) = \dim_{\mathbb{C}} \text{Ext}_{\mathcal{D}}^q(N, \mathcal{K}, \mathcal{B}) < \infty \quad \text{holds.}$$

Sketch of the proof. We first recall the following long exact sequence

of cohomology groups :

$$(2.9) \quad \begin{aligned} 0 \longrightarrow \text{Ext}_{\mathcal{D}, M}^0(L, \mathcal{K}, \mathcal{A}) &\longrightarrow \text{Ext}_{\mathcal{D}}^0(L, \mathcal{K}, \mathcal{A}) \longrightarrow \text{Ext}_{\mathcal{D}}^0(L-M, \mathcal{K}, \mathcal{B}) \\ &\longrightarrow \text{Ext}_{\mathcal{D}, M}^1(L, \mathcal{K}, \mathcal{A}) \longrightarrow \dots\dots\dots \\ &\dots\dots\dots \longrightarrow \text{Ext}_{\mathcal{D}}^{j-1}(L-M, \mathcal{K}, \mathcal{B}) \\ &\longrightarrow \text{Ext}_{\mathcal{D}, M}^j(L, \mathcal{K}, \mathcal{A}) \longrightarrow \text{Ext}_{\mathcal{D}}^j(L, \mathcal{K}, \mathcal{A}) \longrightarrow \text{Ext}_{\mathcal{D}}^j(L-M, \mathcal{K}, \mathcal{B}) \\ &\longrightarrow \text{Ext}_{\mathcal{D}, M}^{j+1}(L, \mathcal{K}, \mathcal{A}) \longrightarrow \dots\dots\dots \end{aligned}$$

On the other hand we have the following canonical isomorphism (2.10) in the derived category. (Kashiwara [1], Sato, Kawai and Kashiwara [2] Corollary 3.5.8 in Chapter II. See also Komatsu and Kawai [1].

$$(2.10) \quad R \text{Hom}_{\mathcal{D}}(\mathcal{K}, \mathcal{A})[-1]^{\otimes \omega_M \otimes \omega_L} \xrightarrow{\sim} R\Gamma_M R \text{Hom}_{\mathcal{D}}(\mathcal{K}, \mathcal{A}), \quad \text{where } \omega_M \text{ and } \omega_L \text{ denote the orientation bundle of } M \text{ and } L \text{ respectively, } \mathcal{A} \text{ and } \mathcal{A}' \text{ denote the sheaf of linear differential operators and hyperfunctions on } M \text{ respectively. Note that isomorphism (2.10) holds only under the non-characteristic assumption of } M \text{ with respect to } \mathcal{K}, \text{ which is secured in our case by the assumption of ellipticity of } \mathcal{K}.$$

Using (2.10) we immediately find that

$$(2.11) \quad \text{Ext}_{\mathcal{D}}^{j-1}(M, \mathcal{M}, \mathcal{K}) \cong \text{Ext}_{\mathcal{D}, M}^j(L, \mathcal{M}, \mathcal{K})$$

holds for any j . On the other hand Theorem 2.3 implies that $\text{Ext}_{\mathcal{D}}^q(M, \mathcal{M}, \mathcal{K})$ is finite-dimensional. Moreover $\text{Ext}_{\mathcal{D}}^j(L, \mathcal{M}, \mathcal{K})$ is finite-dimensional for any j since L is compact and $\mathcal{M}$ is elliptic by the assumption. Therefore, taking into account (2.9), we easily conclude that (2.8) holds under the assumptions of the theorem.

Q.E.D.

Remark.— As is clear from the above arguments, the assumption of compactness of L may be replaced by relative compactness of N if $\text{Ext}_{\mathcal{D}}^j(L, \mathcal{M}, \mathcal{K}) = 0$ ($j \geq 1$) and if $q \geq 1$. One of the most important cases where this remark is applicable is the case where $L = \mathbb{R}^n$ and $\mathcal{M}$ is with constant coefficients. (See Komatsu [1].)

In this case our result may be regarded as a geometrical interpretation of the results of Ehrenpreis [1] and Malgrange [1]. Note that the tangential system $\mathcal{M}$ is not with constant coefficients even when $\mathcal{K}$ is with constant coefficients.

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