Ann. Sc. Norm. Super. Pisa Cl. Sci. (5)
Vol. XII (2013), 93-139

The fundamental solution of nonlinear equations with natural growth terms

BENJAMIN J. JAYE AND IGOR E. VERBITSKY

Abstract. We find bilateral global bounds for the fundamental solutions associated with some quasilinear and fully nonlinear operators perturbed by a nonnegative zero order term with natural growth under minimal assumptions. Important model problems involve the equations $-\Delta_p u = \sigma\, |u|^{p-2}\, u + \delta_{x_0}$, for $p > 1$, and $F_k(-u) = \sigma\, |u|^{k-1}\, u + \delta_{x_0}$, for $k \geq 1$. Here Δ_p and F_k are the p-Laplace and k-Hessian operators respectively, and σ is an arbitrary positive measurable function (or measure). We will in addition consider the Sobolev regularity of the fundamental solution away from its pole.

Mathematics Subject Classification (2010): 35J60 (primary); 42B37, 31C45, 35J92, 42B25 (secondary).

1. Introduction

1.1. In this paper we study the fundamental solution associated with certain nonlinear operators perturbed by natural growth terms. Consider, for $1 < p < \infty$, the quasilinear operator

$$\mathcal{L}(u) = \mathcal{L}^{(p)}(u) = -\Delta_p u - \sigma\, |u|^{p-2}\, u, \tag{1.1}$$

where $\Delta_p u = \operatorname{div}(\nabla u\, |\nabla u|^{p-2})$ is the p-Laplacian operator and σ is a nonnegative Borel measure, on $\mathbb{R}^n$.

Our main goal is to investigate the interaction between the differential operator $-\Delta_p u$, and the lower order term $\sigma\, |u|^{p-2}\, u$, under necessary conditions on σ. This interaction between the differential operator and the lower order term turns out to be highly nontrivial. We will also study the corresponding problem when the p-Laplacian is replaced by a more general quasilinear operator, or a fully nonlinear operator of Hessian type.

Our theorems extend to nonlinear operators very recent results [16,17,19] regarding the behavior of the Green function of the time independent Schrödinger

Supported in part by NSF grants DMS-0556309 and DMS-0901550.
Received October 5, 2010; accepted in revised form June 2, 2011.

operator $-\Delta u - \sigma u$. Our approach, which combines some nonhomogeneous harmonic analysis, nonlinear potential theory and PDE methods, is based on a certain discrete "pseudo-probabilistic" model of equation (1.1), which employs a family of nonlinear expectation operators (see Section 4 below).

This method allows us to construct fundamental solutions of the operator $\mathcal{L}$ under assumptions on σ so in general the Harnack inequality fails for positive solutions u of $\mathcal{L}(u) = 0$. The Harnack inequality formed an essential part in classical arguments concerning the construction of fundamental solutions to both linear and nonlinear operators [41,54–56]. For example, our results hold for the Hardy potential $\sigma(x) = c|x|^{-p}$ for $0 < c < ((n-p)/p)^p$.

Now consider the equation:

$$\mathcal{L}(u) = \delta_{x_0} \quad \text{in } \mathbb{R}^n, \qquad \inf_{x \in \mathbb{R}^n} u(x) = 0, \tag{1.2}$$

where δ_{x_0} is the Dirac delta measure concentrated at x_0. A solution $u(x, x_0)$ of (1.2) understood in a suitable weak, or potential theoretic sense (see Definition 2.1), is called a *fundamental solution* of the operator $\mathcal{L}$, with pole at x_0.

It is well known [55,56,64] that, under stringent assumptions on σ, there exists a positive constant c so that

$$\frac{1}{c}\, G(x, x_0) \le u(x, x_0) \le c\, G(x, x_0), \tag{1.3}$$

if $|x - x_0| < R$ for some $R > 0$, where $G(x, x_0)$ is the fundamental solution of Δ_p on $\mathbb{R}^n$:

$$G(x, x_0) = \gamma_{p,n}\, |x - x_0|^{\frac{p-n}{p-1}}, \qquad \text{when } 1 < p < n. \tag{1.4}$$

Here $\gamma_{p,n} = \left(\frac{p-1}{n-p}\, n\omega_{n-1}\right)^{-\frac{1}{p-1}}$ and ω_{n-1} is the surface area of the $n-1$ dimensional sphere in $\mathbb{R}^n$. Moreover, it was shown recently by L. Verón (see [53, Lemma 5.1]) that $\lim_{x \to x_0} u(x, x_0)/G(x, x_0) = c$ if $\sigma \in L^\infty_{\mathrm{loc}}(\mathbb{R}^n)$. However, as we will see below, $u(x, x_0)$ may behave very differently in comparison to $G(x, x_0)$, both locally and globally.

In this paper we will obtain sharp global estimates for the behavior of fundamental solutions: *Suppose* $1 < p < n$. *Then any fundamental solution* $u(x, x_0)$ *with pole at* x_0 *satisfies the following lower bound:*

$$\begin{aligned} u(x, x_0) \ge c\, |x - x_0|^{\frac{p-n}{p-1}} \exp\left(c \int_0^{|x-x_0|} \left(\frac{\sigma(B(x,r))}{r^{n-p}} \right)^{1/(p-1)} \frac{dr}{r} \right) \\ \cdot \exp\left(c \int_0^{|x-x_0|} \frac{\sigma(B(x_0,r))}{r^{n-p}} \frac{dr}{r} \right), \end{aligned} \tag{1.5}$$

for any $x, x_0 \in \mathbb{R}^n$ under necessary conditions on the measure σ. Here c is a positive constant depending on n and p, and $B(x, r)$ is a ball of radius r centered at x.

The sharpness of this lower bound is illustrated explicitly by our primary result: *Under a natural assumption on σ, there exists a fundamental solution $u(x, x_0)$ of $\mathcal{L}$ satisfying the corresponding upper bound, i.e. for another positive constant c, depending on n, p and σ, it holds that:*

$$u(x, x_0) \leq c\,|x - x_0|^{\frac{p-n}{p-1}} \exp\left(c \int_0^{|x-x_0|} \left(\frac{\sigma(B(x, r))}{r^{n-p}} \right)^{1/(p-1)} \frac{dr}{r} \right) \cdot \exp\left(c \int_0^{|x-x_0|} \frac{\sigma(B(x_0, r))}{r^{n-p}} \frac{dr}{r} \right). \tag{1.6}$$

See Theorems 2.2 and 2.5 below for more precise statements. Furthermore, it follows that there is a minimal fundamental solution which obeys (1.5) and (1.6); see Corollary 3.10. These results had previously been announced without proofs in [63].

In addition to the pointwise bounds presented above, the regularity of the constructed fundamental solution $u(x, x_0)$ away from the pole x_0 will be considered. In particular it will be proved that $u(\cdot, x_0) \in W^{1,p}_{\text{loc}}(\mathbb{R}^n \setminus \{x_0\})$, see Theorem 2.8. This is the optimal regularity that one can hope for under our assumption on σ, see Remark 2.9 below.

Remark 1.1. It is somewhat surprising that expressions involving both the linear potential $\mathbf{I}^{\rho}_{p}\sigma(x_0) = \int_0^{\rho} \frac{\sigma(B(x_0, r))}{r^{n-p}} \frac{dr}{r}$ of fractional order p, and the nonlinear Wolff's potential, introduced in [21],

$$\mathbf{W}^{\rho}_{1,p}\sigma(x) = \int_0^{\rho} \left(\frac{\sigma(B(x, r))}{r^{n-p}} \right)^{1/(p-1)} \frac{dr}{r},$$

should appear, in the exponential form, in global bounds of solutions of the equation $-\Delta_p u - \sigma\,|u|^{p-2}\,u = \delta_{x_0}$.

We observe that local Wolff's potential estimates of solutions of the equation $-\Delta_p u = \sigma$ were established by Kilpeläinen and Maly in [33], while the fully nonlinear analogues for Hessian equations are due to Labutin [36].

A simple corollary of our results (Corollary 7.2 below) gives necessary and sufficient conditions on σ which ensure that $u(x, x_0)$ and $G(x, x_0)$ are pointwise comparable globally. This requires the uniform boundedness of the Riesz potential $\mathbf{I}_p\sigma$ when $1 < p \leq 2$ and the Wolff potential $\mathbf{W}_{1,p}\sigma$ when $p > 2$:

Suppose there is a constant $c > 0$ so that (1.3) *holds for all $x, x_0 \in \mathbb{R}^n$. Then necessarily,*

$$\sup_{x \in \mathbb{R}^n} \int_0^{\infty} \frac{\sigma(B(x, r))}{r^{n-p}} \frac{dr}{r} < \infty \quad \text{if} \quad 1 < p \leq 2, \tag{1.7}$$

$$\sup_{x \in \mathbb{R}^n} \int_0^{\infty} \left(\frac{\sigma(B(x, r))}{r^{n-p}} \right)^{1/(p-1)} \frac{dr}{r} < \infty \quad \text{if} \quad p > 2. \tag{1.8}$$

Conversely, (1.7)–(1.8) *are sufficient for* (1.3) *to hold for all* $x, x_0 \in \mathbb{R}^n$*, under a natural smallness assumption on* σ *discussed below.*

In a recent paper of Liskevich and Skrypnik [40], an indication of this behavior involving the linear potential $\mathbf{I}_p(\sigma)$ when $1 < p \leq 2$ appeared for the first time. They studied isolated singularities of operators of the type $\mathcal{L}u = -\Delta_p u - \sigma\, |u|^{p-2}\, u$, under the assumption that σ is in the quasilinear Kato class (see, *e.g.*, [7]):

$$\lim_{\rho \to 0^+} \sup_{x \in \mathbb{R}^n} \int_0^\rho \left(\frac{|\sigma|(B(x,r))}{r^{n-p}} \right)^{1/(p-1)} \frac{dr}{r} = 0. \tag{1.9}$$

In this paper we will assume that σ is a positive Borel measure satisfying the following capacity condition:

$$\sigma(E) \leq C\, \mathrm{cap_p(E)} \;\; \text{for any compact set } \mathrm{E} \subset \mathbb{R}^{\mathrm{n}}, \tag{1.10}$$

where cap_p is the standard p-capacity:

$$\mathrm{cap}_p(E) = \inf\{\, \|\nabla f\|^p_{L^p} \;:\; f \geq 1 \text{ on } E,\; f \in C_0^\infty(\mathbb{R}^n)\,\}. \tag{1.11}$$

This intrinsic condition, which originated in the work of Maz'ya in the context of linear problems (see [44]), is less stringent than the quasilinear Kato condition (1.9). However, when working in this generality, we cannot expect solutions to be continuous or satisfy a Harnack inequality.

It is easy to see that (1.10) with constant $C = 1$ is necessary in order that $u(\cdot, x_0)$ be finite a.e., which is an immediate consequence of the inequality

$$\int_{\mathbb{R}^n} |h|^p\, d\sigma \leq \int_{\mathbb{R}^n} |\nabla h|^p\, dx, \qquad h \in C_0^\infty(\mathbb{R}^n). \tag{1.12}$$

The preceding inequality holds whenever there exists a positive supersolution u so that $-\Delta_p u \geq \sigma u^{p-1}$ (see Section 4). We observe that, in its turn, (1.10) with $C = (p-1)^p/p^p$ yields (1.12) (see [44]).

1.2. Recall that the fundamental solution of the Laplacian operator plays an important role in the theory of harmonic functions not only because of the principle of superposition, but also because of its importance in understanding how solutions near an isolated singularity can behave, see *e.g.* [3, Theorem 1.3.7]. The latter theory carries over to the theory of the quasilinear and fully nonlinear operators considered here, and hence from the bounds for the fundamental solution we deduce a rather complete analysis of the behavior of solutions of $\mathcal{L}(u) = 0$, and the analogue for the k-Hessian operator, in the punctured space. For the quasilinear operator, this has been considered under a variety of assumptions on σ in [40,49,55,56,64]. Isolated singularities of nonlinear operators have been studied recently in [35,38]. We will present this application in a forthcoming note, where we will also consider other applications, for instance to the study of sign changing solutions of the equation:

$$-\Delta_p u = |\nabla u|^p + \sigma, \tag{1.13}$$

see, for instance [2,14,20,27,46] for some of the existing literature regarding (1.13).

1.3. The plan of the paper is as follows. In Section 2 we precisely state our main results regarding the fundamental solution of (1.1) and its fully nonlinear analogue.

In Section 3, we rapidly review some elements of the theory of nonlinear PDE from a potential theoretic perspective. We are essentially interested in two aspects of this theory: potential estimates for solutions, and weak continuity of the elliptic operators. In this section we also collect a few facts about capacities, and discuss minimal fundamental solutions. After this, in Section 4, we discuss how the potential estimates reduce matters to the study of certain nonlinear integral inequalities. In this section we also discuss the necessary capacity conditions on the measure σ in order for positive solutions of the differential inequalities $\mathcal{L}u \geq 0$ or $\mathcal{G}u \geq 0$ to exist.

Section 5 is concerned with finding a lower bound for any positive solution of a certain nonlinear integral inequality. This bound is proved by estimating successive iterations of the inequality by induction. From this bound Theorems 2.2 and 2.11 are deduced, and their proofs conclude Section 5.

In Section 6, we consider the problem of constructing a positive solution to the integral inequality of Section 5. This construction forms the main technical step in the arguments asserting Theorems 2.5 and 2.12, which we prove in Section 7. In this section we also discuss criteria for the fundamental solutions of $\mathcal{L}$ and $\mathcal{G}$ to be pointwise equivalent to the fundamental solutions of the unperturbed differential operators.

Finally, in Section 8, we consider the Sobolev regularity of the fundamental solution away from its pole. This is the content of Theorem 2.8 below.

2. Main results

We need to introduce some notation before we can state our results. The global bounds will involve two local potentials, a nonlinear Wolff potential, and a linear Riesz potential. If $s > 1, \alpha > 0$ with $0 < \alpha s < n$, we define the local Wolff potential of a measure σ, for $\rho > 0$, by:

$$\mathbf{W}^{\rho}_{\alpha,s}\sigma(x) = \int_0^{\rho} \left(\frac{\sigma(B(x,r))}{r^{n-\alpha s}} \right)^{1/(s-1)} \frac{dr}{r}. \tag{2.1}$$

For $0 < \alpha < n$ the local Riesz potential of σ is defined by:

$$\mathbf{I}^{\rho}_{\alpha}\sigma(x) = \int_0^{\rho} \frac{\sigma(B(x,r))}{r^{n-\alpha}} \frac{dr}{r}. \tag{2.2}$$

We make the convention that when $\rho = +\infty$, then we write $\mathbf{W}_{\alpha,s}\sigma$ and $\mathbf{I}_{\alpha}\sigma$ for $\mathbf{W}^{\infty}_{\alpha,s}\sigma$ and $\mathbf{I}^{\infty}_{\alpha}\sigma$ respectively. In particular,

$$\mathbf{I}_{\alpha}\sigma(x) = \int_0^{+\infty} \frac{\sigma(B(x,r))}{r^{n-\alpha}} \frac{dr}{r} = (n-\alpha)^{-1} \int_{\mathbf{R}^n} \frac{d\sigma(y)}{|x-y|^{n-\alpha}}. \tag{2.3}$$

When $d\sigma = f(x)\,dx$ where $f \in L^1_{\rm loc}(dx)$, we will denote the corresponding potentials by $\mathbf{W}_{\alpha,s} f$ and $\mathbf{I}_\alpha f$ respectively.

2.1. Let us first state our main result for the quasilinear operator $\mathcal{L}$ defined by (1.1). We choose to work with solutions in the potential theoretic sense, see Section 3 below. The reader should note that these solutions are by definition lower semicontinuous, and hence defined everywhere, and so it makes sense to talk about pointwise bounds. We could have alternatively worked with solutions in the *renormalized sense*, see [12] for a thorough introduction.

Definition 2.1. A fundamental solution (with pole at x_0) of the operator $\mathcal{L}$ defined by (1.1), is a positive p-superharmonic function $u(\,\cdot\,, x_0)$, such that $u \in L^{p-1}_{\rm loc}(\sigma)$, satisfying equation (1.2). The equality in (1.2) is understood in the p-superharmonic sense, *i.e.* in the sense of Definition 3.1 in Section 3 below.

When we write $u(x, x_0)$ *is a fundamental solution of* $\mathcal{L}$, with no mention of the pole, we tacitly assume that it has pole at x_0.

The first theorem concerns the lower bound for fundamental solutions. Throughout this paper, unless stated otherwise, we will make the assumption that the measure σ is not identically 0.

Theorem 2.2. a) *Let* $1 < p < n$, $x_0 \in \mathbb{R}^n$, *and suppose* $u(\,\cdot\,, x_0)$ *is a fundamental solution of* $\mathcal{L}$ *with pole at* x_0. *Then* (1.10) *holds with* $C = 1$. *In addition, there is a constant* $c > 0$, *depending on* n, p *such that the bound* (1.5) *holds. In other words, for all* $x \in \mathbb{R}^n$

$$u(x, x_0) \geq c\,|x - x_0|^{\frac{p-n}{p-1}} \exp\left(c\mathbf{W}^{|x-x_0|}_{1,p}(\sigma)(x) + c\mathbf{I}^{|x-x_0|}_p(\sigma)(x_0)\right).$$

b) *If* $p \geq n$, *and* u *is a nonnegative* p*-superharmonic function satisfying the differential inequality:*

$$\mathcal{L}u \geq 0, \quad \textit{in } \mathbb{R}^n$$

then $u \equiv 0$.

Remark 2.3. Part b) of Theorem 2.2 is a Liouville theorem, and when $p > n$ it is related to the important recent works of Serrin and Zou (see [57, Theorem II′]), and Bidaut-Véron and Pohozaev [6]. When $p = n$ the result is a straightforward consequence of well known local estimates of the Riesz measure of a p-superharmonic function, for instance one may use [32, Lemma 3.5]. For several special cases the result follows from those in [6].

Remark 2.4. As we shall see below (in Lemma 4.3), the condition (1.10) is in fact necessary for the existence of a positive p-superharmonic function satisfying the inequality $\mathcal{L}u \geq 0$ in the p-superharmonic sense.

In the case when $1 < p \leq n$, it is a nontrivial fact that when $\sigma \equiv 0$ that the fundamental solution is in fact unique; this was proved in [28]. An alternative method is outlined in [60], where uniqueness of the fundamental solution to the fully nonlinear k-Hessian operators when $1 \le k \le n/2$ is treated. However, when σ is not trivial, it is known even in the linear case ($p = 2$, or $k = 1$) that solutions of $\mathcal{L}$ are not necessarily unique for a general measure σ (see [47]). It is therefore desirable to single out a distinguished class of fundamental solutions. We are interested in fundamental solutions of $\mathcal{L}$ which behave like the lower bound (1.5). The existence of such fundamental solutions, called *minimal fundamental solutions*, is the content of the next theorem.

Theorem 2.5. *Let* $1 < p < n$, $x_0 \in \mathbb{R}^n$ *and suppose* σ *is a nonnegative Borel measure so that* (1.10) *holds. There is a constant* $C_0 = C_0(n, p) > 0$ *such that if* (1.10) *holds with constant* $C < C_0$*, then there exists a fundamental solution* $u(\cdot, x_0)$ *of* $\mathcal{L}$ *with pole at* x_0*, together with a constant* $c = c(n, p, C) > 0$*, so that the upper bound* (1.6) *holds for all* $x \in \mathbb{R}^n$*, i.e.*

$$u(x, x_0) \le c\,|x - x_0|^{\frac{p-n}{p-1}} \exp\left(c\mathbf{W}_{1,p}^{|x-x_0|}(\sigma)(x) + c\mathbf{I}_p^{|x-x_0|}(\sigma)(x_0) \right).$$

Remark 2.6. As a corollary of Proposition 3.8 - which states that *whenever there exists a fundamental solution of* $\mathcal{L}$ *with pole at* x_0*, then there exists a unique minimal fundamental solution of* $\mathcal{L}$ *with pole at* x_0 - we assert the existence of a unique minimal fundamental solution of (1.1) obeying the bounds (1.5) and (1.6). See Corollary 3.10 below.

When $p = 2$, the p-Laplacian reduces to the Laplacian operator and Theorems 2.2 and 2.5 are contained in some very recent work of M. Frazier and the second author [16]. In fact when $p = 2$ the lower bound, Theorem 2.2, has been known for some time, under various restrictions on σ (see [19]). The corresponding upper bound seems to be much deeper. In [16,17] such bounds for the Green function of Schrödinger type equations with the fractional Laplacian operator are discussed.

Remark 2.7. From our method it is clear that Theorems 2.2 and 2.5 continue to hold if we replace the p-Laplacian operator by the general quasilinear $\mathcal{A}$-Laplacian operator div $\mathcal{A}$(x, ∇u) (see, *e.g.*, [22], and Section 3 below). The constants appearing in the theorems will then in addition depend on the structural constants of $\mathcal{A}$.

Having constructed a fundamental solution, we now turn to considering how regular it is away from the pole x_0. This is the content of the next theorem.

Theorem 2.8. *Suppose the hypothesis of Theorem* 2.5 *are satisfied, and that* $u(x, x_0) \not\equiv \infty$*, with* $u(x, x_0)$ *the fundamental solution constructed in Theorem* 2.5*. Then, there exists* $C_0 = C_0(n, p) > 0$ *so that if* (1.10) *holds with* $C < C_0$*, then:*

$$u(\cdot, x_0) \in W^{1,p}_{\mathrm{loc}}(\mathbb{R}^n \backslash \{x_0\}).$$

Remark 2.9. The local Sobolev regularity $W^{1,p}_{\mathrm{loc}}(\mathbb{R}^n \backslash \{x_0\})$ is optimal for solutions of $\mathcal{L}(u) = 0$ under the assumption (1.10) on σ, see [24]. Theorem 2.8 seems to be new in the linear case $p = 2$. In this case the proof, given in Section 8, can clearly be easily adapted to deduce the local regularity of the minimal Green's function of the Schrödinger operator in a bounded domain Ω, as was constructed recently in [16,17].

2.2. We now move onto a fully nonlinear analogue of Theorems 2.2 and 2.5. Let $1 \le k \le n$ be an integer. Then the second operator we consider, denoted by $\mathcal{G}$, is the fully nonlinear operator defined by:

$$\mathcal{G}(u) = F_k(-u) - \sigma\,|u|^{k-1}\,u. \tag{2.4}$$

Here σ is again a nonnegative Borel measure, and F_k is the k-Hessian operator, introduced by Caffarelli, Nirenberg and Spruck [8], and defined for smooth functions u by:

$$F_k(u) = \sum_{1\le i_1<\cdots<i_k\le n} \lambda_{i_1}\ldots\lambda_{i_k}$$

with $\lambda_1,\ldots\lambda_n$ denoting the eigenvalues of the Hessian matrix D^2u. We will use the notion of k-convex functions, introduced by Trudinger and Wang [59], to state our results. See Section 3 for a brief discussion and definitions.

Definition 2.10. A fundamental solution (with pole at x_0) $u(\,\cdot\,,x_0)$ of $\mathcal{G}$ is a function such that $-u(\,\cdot,x_0)$ is a k-convex function so that $u(\,\cdot,x_0)\in L^k_{\text{loc}}(\sigma)$ satisfying $\mathcal{G}u(\,\cdot,x_0)=\delta_{x_0}$ in the viscosity sense, and $\inf_{x\in\mathbb{R}^n} u(x,x_0)=0$.

The necessary condition on σ is now considered in terms of the k-Hessian capacity, introduced in [61];

$$\text{cap}_k(E) = \sup\{\ \mu_k[u](E)\ :\ u \text{ is } k\text{-convex in } \mathbb{R}^n,\ -1<u<0\ \}, \tag{2.5}$$

for a compact set E. Here $\mu_k[u]$ is the k-Hessian measure of u; see Theorem 3.6 below.

Theorem 2.11. a) *Let $1\le k<n/2$, and let $x_0\in\mathbb{R}^n$. If $u(\,\cdot,x_0)$ is a fundamental solution of $\mathcal{G}$, then there is a constant $C>0$, $C=C(n,k)$, such that*

$$\sigma(E) \le C\,\text{cap}_k(E) \quad \textit{for all compact sets } E\subset\mathbb{R}^n. \tag{2.6}$$

In addition, there is a constant $c>0$, $c=c(n,k,C)$, such that

$$\begin{aligned} u(x,x_0) \ge c\,|x-x_0|^{2-\frac{n}{k}} \exp\left(c\int_0^{|x-x_0|}\left(\frac{\sigma(B(x,r)}{r^{n-2k}}\right)^{1/k}\frac{dr}{r}\right) \\ \cdot \exp\left(c\int_0^{|x-x_0|}\frac{\sigma(B(x_0,r))}{r^{n-2k}}\frac{dr}{r}\right). \end{aligned} \tag{2.7}$$

b) *Let $k\ge n/2$. Then if u is a nonnegative function so that $-u$ is a k-convex function satisfying the inequality:*

$$\mathcal{G}(u)\ge 0 \quad \textit{in } \mathbb{R}^n$$

then $u\equiv 0$.

Theorem 2.12. *Let $1 \leq k < n/2$, and suppose σ is a nonnegative Borel measure satisfying* (2.6). *There is a constant $C_0 = C_0(n,k)$, such that if $C < C_0$ and* (2.6) *holds with constant C, then there exists a fundamental solution $u(\cdot, x_0)$ of $\mathcal{G}$, together with a constant $c = c(n,k,C)$ so that*

$$u(x,x_0) \leq c\,|x-x_0|^{2-\frac{n}{k}} \exp\left(c \int_0^{|x-x_0|} \left(\frac{\sigma(B(x,r))}{r^{n-2k}} \right)^{1/k} \frac{dr}{r} \right) \cdot \exp\left(c \int_0^{|x-x_0|} \frac{\sigma(B(x_0,r))}{r^{n-2k}} \frac{dr}{r} \right). \tag{2.8}$$

Remark 2.13. Part b) of Theorem 2.11 is easy to see using well known local estimates. For instance, one can readily deduce the result from [59, Theorem 3.1], along with a routine approximation argument using weak convergence of Hessian measures.

3. Preliminaries

3.1. Notation. For an open set Ω, we denote by $L^p_{\mathrm{loc}}(\Omega)$ to be the space of functions locally integrable to the p-th power with respect to Lebesgue measure. Similarly, $L^p_{\mathrm{loc}}(\Omega, d\sigma)$ then denotes the space of functions which are locally integrable to the p-th power with respect to σ measure. $W^{1,p}_{\mathrm{loc}}(\Omega)$ is the space of functions $u \in L^p_{\mathrm{loc}}(\Omega)$, with weak derivative $\nabla u \in L^p_{\mathrm{loc}}(\Omega; \mathbb{R}^n)$. From time to time, we will use the symbol $A \lesssim B$ to mean that $A \leq CB$ with the constant $C > 0$ depending on the allowed parameters of the particular result being proved.

For a set E, we will write either $\sigma(E)$ of $|E|_\sigma$ to denote the σ-measure of E.

3.2. In this section we will introduce some fundamental results from the potential theory of nonlinear elliptic equations. Two results will be key to our study: a potential estimate; and a weak continuity result. The potential which the estimates will involve is called the Wolff potential [21]. For $s > 1$ and $0 < \alpha s < n$, we define the Wolff potential of a nonnegative Borel measure μ by:

$$\mathbf{W}_{\alpha,s}\mu(x) = \int_0^\infty \left(\frac{\mu(B(x,r))}{r^{n-\alpha s}} \right)^{1/(s-1)} \frac{dr}{r} \tag{3.1}$$

We first will discuss quasilinear equations. The material regarding these equations is drawn from [22,32,33,42,50,51,61].

Let us assume that $\mathcal{A} : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ satisfies:

$$x \to \mathcal{A}(x,\xi) \text{ is measurable for all } \xi \in \mathbb{R}^n, \text{ and}$$

$$\xi \to \mathcal{A}(x,\xi) \text{ is continuous for a.e. } x \in \mathbb{R}^n.$$

In addition suppose that there are constants $0 < \alpha \leq \beta < \infty$ so that for a.e. $x \in \mathbb{R}^n$:

$$\alpha\,|\xi|^p \leq \mathcal{A}(x,\xi)\cdot\xi, \quad \text{and} \quad |\mathcal{A}(x,\xi)| \leq \beta\,|\xi|^{p-1}.$$

We will also assume that:

$$(\mathcal{A}(x,\xi_1) - \mathcal{A}(x,\xi_2))\cdot(\xi_1 - \xi_2) > 0$$

whenever $\xi_1 \neq \xi_2$.

Now, let Ω be an open subset of $\mathbb{R}^n$, (we will be most interested in the case $\Omega = \mathbb{R}^n$). Whenever $u \in \mathrm{W}^{1,p}_{\mathrm{loc}}(\Omega)$, we define the divergence of $\mathcal{A}(x, \nabla u)$ in the distributional sense. As follows from the classical regularity theory of de Giorgi, Nash and Moser, any $u \in W^{1,p}_{\mathrm{loc}}(\Omega)$ solution of $-\mathrm{div}\mathcal{A}(x, \nabla u) = 0$ in the distributional sense has a locally Hölder continuous representative, and we call these representatives $\mathcal{A}$-harmonic functions. Here and in the following the p-Laplacian operator corresponds to the choice of $\mathcal{A}(x,\xi) = |\xi|^{p-2}\xi$, in this case $\mathcal{A}$-harmonic functions are called p-harmonic functions, and similarly p-superharmonic functions are $\mathcal{A}$-superharmonic functions (as defined below) in this special case.

In analogy with classical superharmonic functions, we define the $\mathcal{A}$-superharmonic functions via a comparison principle. We say that $u : \Omega \to (-\infty, \infty]$ is $\mathcal{A}$-superharmonic if u is lower semicontinuous, is not identically infinite in any component of Ω, and satisfies the following comparison principle: Whenever $D \subset\subset \Omega$ and $h \in C(\bar{D})$ is $\mathcal{A}$-harmonic in D, with $h \leq u$ on ∂D, then $h \leq u$ in D.

An $\mathcal{A}$-superharmonic function u does not necessarily have to belong to $\mathrm{W}^{1,p}_{\mathrm{loc}}(\Omega)$, but its truncates $T_k(u) = \min(u,k) \in \mathrm{W}^{1,p}_{\mathrm{loc}}(\Omega)$ for all $k > 0$. In addition $T_k(u)$ are supersolutions, *i.e.* $-\mathrm{div}\mathcal{A}(\,\cdot\,, \nabla T_k(u)) \geq 0$, in the distributional sense (see [22]).

The above paragraph leads us to the definition of the ***generalized gradient*** of an $\mathcal{A}$-superharmonic function u as the unique Lebesgue measurable function Du so that:

$$Du(x) = \nabla(T_k(u))(x) \text{ whenever } x \in \{u < k\}$$

see ***e.g.*** [32, page 595]. The function Du is then necessarily the distributional gradient of u.

Let u be $\mathcal{A}$-superharmonic and let $1 \leq q < n/(n-1)$. Then it is proved in [32] that $|Du|^{p-1}$ and $\mathcal{A}(\cdot, Du)$ belong to $\mathrm{L}^q_{\mathrm{loc}}(\Omega)$. This allows us to define a nonnegative distribution for each $\mathcal{A}$-superharmonic function u by:

$$-\mathrm{div}\mathcal{A}(x,\nabla u)(\psi) = \int_\Omega \mathcal{A}(x, Du)\cdot\nabla\psi\, dx \tag{3.2}$$

for $\psi \in C_0^\infty(\Omega)$. So, the Riesz representation theorem yields the existence of a unique nonnegative Borel measure $\mu[u]$ so that $-\mathrm{div}\mathcal{A}(x,\nabla u) = \mu[u]$. Furthermore, by the integrability of the gradient, it follows that for any $r > n$:

$$\int_\Omega \mathcal{A}(\,\cdot\,, Du)\cdot\nabla\phi dx = \int_\Omega \phi d\mu, \text{ for all } \phi \in W^{1,r}(\Omega) \text{ with compact support.} \tag{3.3}$$

We now formally define solutions involving the perturbed operator $\mathcal{L}$ in the p-superharmonic sense:

Definition 3.1. For a nonnegative measure ω we will say that $-\text{div}\mathcal{A}(\cdot, \nabla u) = \omega$ in the p-superharmonic sense if u is p-superharmonic, and $\mu[u] = \omega$. Thus $\mathcal{L}(u) = \omega$ in the p-superharmonic sense if $\mu[u] = \sigma u^{p-1} + \omega$.

Remark 3.2. A major open problem in the theory of the quasilinear operators modeled by the p-Laplacian is to find the correct notion of solution in order to guarantee both existence and uniqueness. As a result there are many notions of solution which have been developed, of which p-superharmonicity is the weakest. There are alternative notions of solutions which we could have introduced to obtain our results, for instance either *renormalized solutions* or *supersolutions up to all levels*, see [12] and [42] respectively. We chose to use the language of $\mathcal{A}$-superharmonic functions because the potential estimates (Theorems 3.4 and 3.5) were developed in this framework. It was shown in [26] that $\mathcal{A}$-superharmonic functions coincide with the notion of viscosity supersolutions for the operator $\mathcal{A}$. Moreover, it has very recently been shown that p-superharmonic functions are *locally* renormalized solutions, see [31]. For more information on these competing notions of solution, we refer the reader to T. Kilpeläinen's survey article [30].

We now state a very useful convergence result, contained in Kileplainen and Maly [32, Theorem 1.17].

Theorem 3.3 ([32]). *Suppose $\{u_j\}_j$ is a sequence of nonnegative $\mathcal{A}$-superharmonic functions in an open set Ω. Then there is a subsequence $\{u_{j_k}\}_k$ which converges almost everywhere to a nonnegative function u which is either p-superharmonic or identically infinite in each component of Ω.*

The next result, first stated explicitly in [61], shows that $\mathcal{A}$-Laplace operator is weakly continuous.

Theorem 3.4 ([61]). *Suppose $\{u_j\}_j$ is a sequence of nonnegative $\mathcal{A}$-superharmonic functions which converge almost everywhere to an $\mathcal{A}$-superharmonic function u. Then $\mu[u_j]$ converges weakly to $\mu[u]$.*

The second major result we need is the Wolff's potential estimates of Kilpeläinen and Maly [33] (see also [42,50]).

Theorem 3.5 ([33]). *Let u be a nonnegative $\mathcal{A}$-superharmonic function in $\mathbb{R}^n$ so that $\inf_{x\in\mathbb{R}^n} u(x) = 0$. If $\mu = -\text{div}\,\mathcal{A}(\cdot\,, \nabla \text{u})$, then there is a constant $K = K(n, p, \alpha, \beta)$, so that for all $x \in \mathbb{R}^n$,*

$$\frac{1}{K}\,\mathbf{W}_{1,p}\mu(x) \leq u(x) \leq K\,\mathbf{W}_{1,p}\mu(x). \tag{3.4}$$

3.3. We now turn to the fully nonlinear counterpart of these results. A very recent and comprehensive account of the k-Hessian equation is [65]. Here k-convex functions associated to the k-Hessian operator, introduced by Trudinger and Wang [59],

We are now in a position to prove Theorem 2.8.

Proof of Theorem 2.8. Let us assume that C_0 has been chosen so that Lemmas 8.2 and 8.3 are both valid. To prove the theorem, we will aim to construct the sequence $\{u_m\}_m$ as in (7.3) from the proof of Theorem 2.5 with the additional property that $u_m \in W^{1,p}_{\text{loc}}(\mathbb{R}^n \backslash \{x_0\})$, with constants independent on m. We will do this inductively, as in the proof of Theorem 2.5. Let $u_0 = G(\cdot, x_0)$, with $G(x, x_0)$ as in (1.4). Note $G(\cdot, x_0) \in C^{\infty}_{\text{loc}}(\mathbb{R}^n \backslash \{x_0\})$. Suppose that we have constructed $u_1, \ldots, u_{m-1}$ so that:

$$-\Delta_p u_j = \epsilon \sigma u_{j-1}^{p-1} + \delta_{x_0},$$

with $u_j \leq v$, and $u_{j-1} \in W^{1,p}_{\text{loc}}(\mathbb{R}^n \backslash \{x_0\})$. Let K be a compact subset of $\mathbb{R}^n \backslash \{0\}$, then we claim that $u_{m-1}^{p-1} d\sigma \in W^{-1,p'}(K)$. This will follow from the capacity strong type inequality. Indeed, since σ satisfies (1.10) with constant $C(\sigma) < C_0$, it follows [44], that:

$$\int |h|^p d\sigma \leq C(\sigma) \left(\frac{p}{p-1} \right)^p \int |\nabla h|^p dx, \text{ for all } h \in C_0^{\infty}(\mathbb{R}^n),$$

and this can be extended by continuity to functions $h \in W_0^{1,p}(\mathbb{R}^n)$. Now, let $h \in C_0^{\infty}(K)$, and K' be a subset $K \subset\subset K' \subset\subset \mathbb{R}^n \backslash \{x_0\}$ along with a function $g \in C_0^{\infty}(K')$, $g \equiv 1$ on K, $g \geq 0$. Then:

$$\int h u_{m-1}^{p-1} d\sigma = \int h u_{m-1}^{p-1} g^{p-1} d\sigma \leq \left(\int |h|^p d\sigma \right)^{1/p} \left(\int u_{m-1}^p g^p d\sigma \right)^{\frac{p-1}{p}}$$

$$\lesssim ||\nabla h||_p ||\nabla (u_{m-1} g)||_p^{p-1} \leq C_K ||\nabla h||_p,$$

and hence $u_{m-1}^{p-1} d\sigma \in W^{-1,p'}(K)$, as claimed. Now let ν_j be the measure:

$$\nu_j = \frac{\chi_{B(x_0, 2^{-j})}}{|B(x_0, 2^{-j})|},$$

from Poincaré's inequality it follows that $\nu_j \in W^{-1,p'}(B(x_0, 2^j))$. Note in addition that $\nu_j \to \delta_{x_0}$ weakly as measures. Invoking the theory of monotone operators, see *e.g.* [39,58], we assert the existence of a unique solution $w_m^j \in W_0^{1,p}(B(x_0, 2^j))$ of:

$$\begin{cases} -\Delta_p w_m^j = \epsilon \sigma u_{m-1}^{p-1} \chi_{B(x_0, 2^j) \backslash B(x_0, 2^{-j})} + \nu_j \text{ in } B(x_0, 2^j), \\ w_m^j \in W_0^{1,p}(B(x_0, 2^j)). \end{cases} \tag{8.11}$$

Furthermore, by the global potential estimate for renormalized solutions, [50, Theorem 2.1], it follows:

$$w_m^j(x) \leq K \epsilon \mathbf{W}_{1,p}(u_{m-1}^{p-1} d\sigma)(x) + K \mathbf{W}_{1,p}(\nu_k)(x),$$

where the constant $K > 0$ can be assumed to be the same as the constant appearing in Theorem 3.5. But, for $x \notin B(x_0, 2 \cdot 2^{-j})$, a simple computation yields:

$$\mathbf{W}_{1,p}(\nu_k)(x) \leq \frac{n-p}{p-1} 2^{\frac{n-p}{p-1}} |x - x_0|^{\frac{p-n}{p-1}}. \tag{8.12}$$

Using the hypothesis $u_{m-1} \leq v$, it follows for $x \in B(x_0, 2^j) \backslash B(x_0, 2^{1-j})$ that:

$$w_m^j(x) \leq K\epsilon \mathbf{W}_{1,p}(v^{p-1} d\sigma)(x) + K \frac{n-p}{p-1} 2^{\frac{n-p}{p-1}} |x - x_0|^{\frac{p-n}{p-1}}.$$

Let us now choose the constant $B(n, p)$ appearing in (8.3) as $B(n, p) = 2K(n - p)/(p-1)2^{\frac{n-p}{p-1}}$. Then, by construction of v, it follows as in the argument around display (7.2), that we can choose $\epsilon > 0$ and $C_0 > 0$ so that if $C(\sigma) < C_0$, then:

$$K\epsilon \mathbf{W}_{1,p}(v^{p-1} d\sigma)(x) + K \frac{n-p}{p-1} 2^{\frac{n-p}{p-1}} |x - x_0|^{\frac{p-n}{p-1}} \leq v(x),$$

and hence,

$$w_m^j(x) \leq v(x), \text{ for all } x \in B(x_0, 2^j) \backslash B(x_0, 2 \cdot 2^{-j}). \tag{8.13}$$

We are now in a position to derive the uniform gradient estimate.

Let $\phi \in C_0^\infty(B(x_0, 2^j) \backslash B(x_0, 2 \cdot 2^{-j})$. Then test the weak formulation of w_m^j with the valid test function function $\phi^p \cdot w_m^j \in W_0^{1,p}(B(x_0, 2^j))$. It follows:

$$\int |\nabla w_m^j|^p \phi^p dx = p \int |\nabla w_m^j|^{p-2} \nabla w_m^j \cdot \nabla \phi w_m^j \phi^{p-1} + \int \phi^p u_m^j u_{m-1}^{p-1} d\sigma$$

Using Young's inequality in the first term, and utilizing the bounds (8.13) and $u_{m-1} \leq v$, we find that:

$$\frac{1}{p} \int |\nabla w_m^j|^p dx \leq \int v^p \phi^p d\sigma + \frac{1}{p} \int v^p |\nabla \phi|^p dx = C(n, p, C(\sigma), \operatorname{supp}(\phi)) < \infty,$$

where Lemmas 8.2 and 8.3 have been used. Using Theorems 3.3 and 3.4, we let $j \to \infty$ to find a solution u_m of (7.3). Furthermore, by weak compactness in $W^{1,p}$, we deduce that $u_m \in W_{\text{loc}}^{1,p}(\mathbb{R}^n \backslash \{x_0\})$ with the local bound on the gradient independent of m. We now follow the rest of the proof of Theorem 2.5 from display (7.3), using weak compactness again to deduce a fundamental solution $u \in W_{\text{loc}}^{1,p}(\mathbb{R}^n \backslash \{x_0\})$, so that $u \leq v$. $\square$

References

[1] D. R. ADAMS and L. I. HEDBERG, "Function Spaces and Potential Theory", Grundlehren der mathematischen Wissenschaften, Vol. 314, Springer, Berlin–Heidelberg–New York, 1996.

[2] H. ABDUL-HAMID and M.-F. BIDAUT-VÉRON, *On the connection between two quasilinear elliptic problems with source terms of order 0 or 1,* Commun. Contemp. Math. **12** (2010), 727–788.

[3] D. H. ARMITAGE and S. J. GARDINER, "Classical Potential Theory", Springer Monographs in Math., Springer, London–Berlin–Heidelberg, 2001.

[4] M.-F. BIDAUT-VÉRON, *Local and global behavior of solutions of quasilinear equations of Emden-Fowler type,* Arch. Ration. Mech. Anal. **107** (1989), 293–324.

[5] M.-F. BIDAUT-VÉRON, R. BORGHOL and L. VÉRON, *Boundary Harnack inequality and a priori estimates of singular solutions of quasilinear elliptic equations*, Calc. Var. Partial Differential Equations **27** (2006), 159–177.

[6] M.-F. BIDAUT-VÉRON and S. POHOZAEV, *Nonexistence results and estimates for some nonlinear elliptic problems,* J. Anal. Math. **84** (2001), 1–49.

[7] M. BIROLI, *Schrödinger type and relaxed Dirichlet problems for the subelliptic p-Laplacian*, Potential Anal. **15** (2001), 1–16.

[8] L. CAFFARELLI, L. NIRENBERG and J. SPRUCK, *The Dirichlet problem for nonlinear second-order elliptic equations III. Functions of the eigenvalues of the Hessian*, Acta Math. **155** (1985), 261–301.

[9] C. CASCANTE, J. M. ORTEGA and I. E. VERBITSKY, *Nonlinear potentials and two weight trace inequalities for general dyadic and radial kernels*, Indiana Univ. Math. J. **53** (2004), 845–882.

[10] G. DAL MASO and A. GARRONI, *New results on the asymptotic behavior of Dirichlet problems in perforated domains*, Math. Models Methods Appl. Sci. **4** (1994), 373–407.

[11] G. DAL MASO and A. MALUSA, *Some properties of reachable solutions of nonlinear elliptic equations with measure data,* Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) **25** (1997), 375–396 (1998).

[12] G. DAL MASO, F. MURAT, L. ORSINA and A. PRIGNET, *Renormalized solutions of elliptic equations with general measure data*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) **28** (1999), 741–808.

[13] C. FEFFERMAN and E. M. STEIN, *Some maximal inequalities*, Amer. J. Math. **93** (1971), 107–115.

[14] V. FERONE and F. MURAT, *Nonlinear problems having natural growth in the gradient: an existence result when the source terms are small*, Nonlinear Anal. **42** (2000), 1309–1326.

[15] M. FRAZIER and I. E. VERBITSKY, *Solvability conditions for a discrete model of Schrödinger's equation*, In: "Analysis, Partial Differential Equations and Applications", The Vladimir Maz'ya Anniversary Volume, A. Cialdea, F. Lanzara and P. Ricci (eds.), Operator Theory: Adv. Appl. **179**, Birkhäuser (2010), 65–80.

[16] M. FRAZIER and I. E. VERBITSKY, *Global Green's function estimates*, In: "Around the Research of Vladimir Maz'ya III, Analysis and Applications", Ari Laptev (ed.), Intern. Math. Series **13**, Springer (2010), 105–152.

[17] M. FRAZIER, F. NAZAROV and I. E. VERBITSKY, *Global estimates for kernels of Neumann series, Green's functions, and the conditional gauge*, preprint 2013.

[18] J. GARNETT and P. JONES, *BMO from dyadic BMO*, Pacific J. Math. **99** (1982), 351–371.

[19] A. GRIGOR'YAN and W. HANSEN, *Lower estimates for perturbed Green function*, J. Anal. Math. **104** (2008), 25–58.

[20] K. HANSSON, V. G. MAZ'YA and I. E. VERBITSKY, *Criteria of solvability for multidimensional Riccati equations*, Ark. Mat. **37** (1999), 87–120.

[21] L. I. HEDBERG and T. WOLFF, *Thin sets in nonlinear potential theory*, Ann. Inst. Fourier (Grenoble) **33** (1983), 161–187.

[22] J. HEINONEN, T. KILPELÄINEN and O. MARTIO, "Nonlinear Potential Theory of Degenerate Elliptic Equations", Dover Publications, 2006 (unabridged republication of 1993 edition, Oxford Universiy Press).
[23] P. HONZÍK and B. J. JAYE, *On the good-λ inequality for nonlinear potentials*, Proc. Amer. Math. Soc. **140** (2012), 4167–4180.
[24] B. J. JAYE, V. G. MAZ'YA and I. E. VERBITSKY, *Existence and regularity of positive solutions of elliptic equations of Schrödinger type*, J. Anal. Math. **118** (2012), 577–621.
[25] B. J. JAYE and I. E. VERBITSKY, *Local and global behaviour of solutions to nonlinear equations with natural growth terms*, Arch. Ration. Mech. Anal. **204** (2012), 627–681.
[26] P. JUUTINEN, P. LINDQVIST and J. J. MANFREDI, *On the equivalence of viscosity solutions and weak solutions for a quasi-linear equation,* SIAM J. Math. Anal. **33** (2001), 699–717.
[27] J. KAZDAN and R. KRAMER, *Invariant criteria for existence of second-order quasi-linear elliptic equations*, Comm. Pure. Appl. Math. **31** (1978), 619–645.
[28] S. KICHENASSAMY and L. VÉRON, *Singular solutions to the p-Laplace equation*, Math. Ann. **275** (1986), 599–615.
[29] T. KILPELÄINEN, *Singular solutions of p-Laplacian type equations*, Ark. Mat. **37** (1999), 275–289.
[30] T. KILPELÄINEN, *p-Laplacian type equations involving measures,* In: "Proc. ICM", Vol. III, Beijing, 2002, Higher Ed. Press, Beijing, 2002, 167–176.
[31] T. KILPELÄINEN, T. KUUSI and A. TUHOLA-KUJANPÄÄ, *Superharmonic functions are locally renormalized solutions*, Ann. Inst. Henri Poincaré, Anal. Non Linéare **28** (2011), 775–795.
[32] T. KILPELÄINEN and J. MALY, *Degenerate elliptic equations with measure data and nonlinear potentials*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) **19** (1992), 591–613.
[33] T. KILPELÄINEN and J. MALY, *The Wiener test and potential estimates for quasilinear elliptic equations*, Acta Math. **172** (1994), 137–161.
[34] T. KILPELÄINEN and X. XU, *On the uniqueness problem for quasilinear elliptic equations involving measures,* Rev. Mat. Iberoamericana **12** (1996), 461–475.
[35] D. LABUTIN, *Isolated singularities of fully nonlinear elliptic equations*, J. Differential Equations **177** (2001), 49–76.
[36] D. LABUTIN, *Potential estimates for a class of fully nonlinear elliptic equations*, Duke Math. J. **111** (2002), 1–49.
[37] J. LEWIS and K. NYSTRÖM, *Boundary behaviour for p harmonic functions in Lipschitz and starlike Lipschitz ring domains,* Ann. Sci. École Norm. Sup. **40** (2007), 765–813.
[38] Y. Y. LI, *Conformally invariant fully nonlinear elliptic equations and isolated singularities*, J. Funct. Anal. **233** (2006), 380–425.
[39] J.-L. LIONS, "Quelques Méthodes de Résolution des Problèmes aux Limites Non Linéaires", Dunod; Gauthier-Villars, Paris, 1969.
[40] V. LISKEVICH and I. I. SKRYPNIK, *Isolated singularities of solutions to quasilinear elliptic equations*, Potential Anal. **28** (2008), 1–16.
[41] W. LITTMAN, G. STAMPACCHIA and H. F. WEINBERGER, *Regular points for elliptic equations with discontinuous coefficients.* Ann. Scuola Norm. Sup. Pisa Cl. Sci. (3) **17** (1963), 43–77.
[42] J. MALY and W. ZIEMER, "Fine Regularity of Solutions of Elliptic Partial Differential Equations", Math. Surveys and Monographs, Vol. 51, Amer. Math. Soc., Providence, RI, 1997.
[43] V. MAZ'YA, *The continuity at a boundary point of the solutions of quasi-linear elliptic equations,* Vestnik Leningrad. Univ. **25** (1970), 42–55.
[44] V. MAZ'YA, "Sobolev Spaces", Springer Series in Soviet Math., Springer, 1985 (new edition in press).
[45] G. MINGIONE, *The Calderón-Zygmund theory for elliptic problems with measure data*. Ann. Sc. Norm. Super. Pisa Cl. Sci. (5) **6** (2007), 195–261.