Harmonic mappings and distance function

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Abstract. We prove the following theorem: every quasiconformal harmonic mapping between two plane domains with $C^{1,\alpha}$ $(\alpha < 1)$ and, respectively, $C^{1,1}$ compact boundary is bi-Lipschitz. This theorem extends a similar result of the author [10] for Jordan domains, where stronger boundary conditions for the image domain were needed. The proof uses distance function from the boundary of the image domain.

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1. Introduction and statement of the main result

We say that a function $u : D \rightarrow \mathbb{R}$ is ACL (absolutely continuous on lines) in the region $D \subset \mathbb{R}^2$, if for every closed rectangle $R \subset D$ with sides parallel to the $x$ and $y$-axes, $u$ is absolutely continuous on a.e. horizontal and a.e. vertical line in $R$. Such a function has, of course, partial derivatives $u_x$ and $u_y$ a.e. in $D$. A homeomorphism $f : D \rightarrow G$, where $D$ and $G$ are subdomains of the complex plane $\mathbb{C}$, is said to be $K$-quasiconformal ($K$-q.c), for $K \geq 1$, if $f$ is ACL and

$$|\nabla f(z)| \leq Kl(\nabla f(z)) \quad \text{a.e. on } D, \quad (1.1)$$

where

$$|\nabla f(x)| := \max_{|h|=1} |\nabla f(x)h| = |f_z| + |f_{\bar{z}}|$$

and

$$l(\nabla f(z)) := \min_{|h|=1} |\nabla f(z)h| = |f_z| - |f_{\bar{z}}|$$

(cf. [1, pages 23–24] and [22]). Note that, condition (1.1) can be written as

$$|f_{\bar{z}}| \leq k|f_z| \quad \text{a.e. on } D, \text{ where } k = \frac{K - 1}{K + 1} \quad i.e. \ K = 1 + k \frac{1}{1 - k}$$

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or in its equivalent form
\[ |\nabla f(z)|^2 \leq K J_f(z), \ z \in \mathbb{U}, \quad (1.2) \]
where \( J_f \) is the Jacobian of \( f \).

A function \( w \) is called harmonic in a region \( D \) if it has form \( w = u + iv \) where \( u \) and \( v \) are real-valued harmonic functions on \( D \). If \( D \) is simply connected, then there are two analytic functions \( g \) and \( h \) defined on \( D \) such that \( w \) has the representation
\[ w = g + \overline{h}. \]

If \( w \) is a harmonic univalent function then, by Lewy’s theorem (see [23]), \( w \) has a non-vanishing Jacobian and consequently, according to the inverse mapping theorem, \( w \) is a diffeomorphism.

Let
\[ P(r, x) = \frac{1 - r^2}{2\pi(1 - 2r \cos x + r^2)} \]
denote the Poisson kernel. Then every bounded harmonic function \( w \) defined on the unit disc \( \mathbb{U} := \{ z : |z| < 1 \} \) has the representation
\[ w(z) = P[F](z) = \int_0^{2\pi} P(r, x - \varphi) F(e^{i\varphi}) d\varphi, \quad (1.3) \]
where \( z = re^{i\varphi} \) and \( F \) is a bounded integrable function defined on the unit circle \( S^1 \).

In this paper we continue to study quasi-conformal harmonic mappings. See [25] for the pioneering work on this topic, and [8] for related earlier results. In some recent papers, a lot of work have been done on this class of mappings ([3, 10–17, 19–21, 24, 26, 28, 29]). In these papers for the Lipschitz and the co-Lipschitz character is established quasi-conformal harmonic mappings between plane domains with certain boundary conditions. In [32] the same problem is considered for hyperbolic harmonic quasi-conformal selfmappings of the unit disk. Notice that, in general, quasi-symmetric self-mappings of the unit circle do not have a quasi-conformal harmonic extension to the unit disk. In [25] an example is given of \( C^1 \) diffeomorphism of the unit circle onto itself whose Euclidean harmonic extension is not Lipschitz. Alessandrini and Nesi proved in [2] the following:

**Proposition 1.1.** Let \( F : S^1 \to \gamma \subset \mathbb{C} \) be an orientation-preserving diffeomorphism of class \( C^1 \) of \( S^1 \) onto a simple closed curve \( \gamma \). Let \( D \) be the bounded domain such that \( \partial D = \gamma \). Let \( w = P[F] \in C^1(\overline{\mathbb{U}}; \mathbb{C}) \). The mapping \( w \) is a diffeomorphism of \( \mathbb{U} \) onto \( D \) if and only if
\[ J_w > 0 \text{ everywhere on } S^1. \quad (1.4) \]

From the inequalities (1.2) and (1.4), we easily deduce the following:

**Corollary 1.2.** Under the assumption of Proposition 1.1 the harmonic mapping \( w \) is a diffeomorphism if and only if it is \( K \)-quasiconformal for some \( K \geq 1 \).
In contrast to the case of the Euclidean metric, in the case of the hyperbolic metric, if \( f : S^1 \mapsto S^1 \) is \( C^1 \) diffeomorphism, or more generally if \( f : S^{n-1} \mapsto S^{m-1} \) is a mapping with non-vanishing energy, then its hyperbolic harmonic extension is \( C^1 \) up to the boundary ([4, 5]).

To continue we need the definition of \( C^{k, \alpha} \) Jordan curves \( (k \in \mathbb{N}, 0 < \alpha \leq 1) \). Let \( \gamma \) be a rectifiable curve in the complex plane. Let \( l \) be the length of \( \gamma \). Let \( g : [0, l] \mapsto \gamma \) be an arc-length parametrization of \( \gamma \). Then \( |g'(s)| = 1 \) for all \( s \in [0, l] \). We will say that \( \gamma \in C^{k, \alpha} \), \( k \in \mathbb{N}, 0 < \alpha \leq 1 \) if \( g \in C^k \), and \( M(k, \alpha) := \sup_{t \neq s} \frac{|g^{(k)}(t) - g^{(k)}(s)|}{|t - s|^\alpha} < \infty \). Notice this important fact: if \( \gamma \in C^{1,1} \) then \( \gamma \) has a curvature \( k_z \) for a.e. \( z \in \gamma \) and \( \text{ess sup}(|k_z| : z \in \gamma) \leq M(1, 1) < \infty \).

This definition can be easily extended to an arbitrary \( C^{k, \alpha} \) compact 1-dimensional manifold (not necessarily connected).

The starting point of this paper is the following proposition.

**Proposition 1.3.** Let \( w = f(z) \) be a \( K \)-quasiconformal harmonic mapping between a Jordan domain \( \Omega_1 \) with \( C^{1, \alpha} \) boundary and a Jordan domain \( \Omega \) with \( C^{1, \alpha} \) (respectively \( C^{2, \alpha} \)) boundary. Consider in addition \( b \in \Omega_1 \) and set \( a = f(b) \). Then \( w \) is Lipschitz (respectively co-Lipschitz). Moreover there exists a positive constant \( c = c(K, \Omega, \Omega_1, a, b) \geq 1 \) such that

\[
|f(z_1) - f(z_2)| \leq c|z_1 - z_2|, \quad z_1, z_2 \in \Omega_1 \tag{1.5}
\]

and

\[
\frac{1}{c}|z_1 - z_2| \leq |f(z_1) - f(z_2)|, \quad z_1, z_2 \in \Omega_1, \tag{1.6}
\]

respectively.

See [13] for the first part of Proposition 1.3 and [10] for its second part. In [10], it was conjectured that the second part of Proposition 1.3 remains true if we assume that \( \Omega \) has \( C^{1, \alpha} \) boundary only. Notice that the proof of Proposition 1.3 relies on the Kellogg-Warschawski theorem ([6, 33, 34]) from the theory of conformal mappings, which asserts that if \( w \) is a conformal mapping of the unit disk onto a domain \( \Omega \in C^{k, \alpha} \), then \( w^{(k)} \) has a continuous extension to the boundary \( (k \in \mathbb{N}) \). It also depended on Mori’s theorem from the theory of quasiconformal mappings, which deals with the Hölder character of quasiconformal mappings between plane domains (see [1, 31]). In addition, Lemma 3.2 below is needed.

Using a different approach, we will extend here as stated in Theorem 1.4 the second part of Proposition 1.3 to the case of image domains with \( C^{1,1} \) boundary. The proof of Theorem 1.4, given in the last section, is different form the proof of second part of Proposition 1.3, and the use of the Kellogg-Warschawski theorem for the second derivative ([34]) is avoided. The distance function is used and hence a “weaker” smoothness of the boundary of image domain is needed.

**Theorem 1.4 (The main theorem).** Let \( w = f(z) \) be a \( K \)-quasiconformal harmonic mapping from the unit disk \( \mathbb{U} \) to a Jordan domain \( \Omega \) with \( C^{1,1} \) boundary. Set
\( a = f(0) \). Then \( w \) is co-Lipschitz. More precisely, there exists a positive constant \( c = c(K, \Omega, a) \geq 1 \) such that

\[
\frac{1}{c} |z_1 - z_2| \leq |f(z_1) - f(z_2)|, \quad z_1, z_2 \in \Omega.
\] (1.7)

Since the composition of a quasiconformal harmonic and a conformal mapping is itself quasiconformal harmonic, using Theorem 1.4 and Kellogg’s theorem for the first derivative we obtain:

**Corollary 1.5.** Let \( w = f(z) \) be a \( K \)-quasiconformal harmonic mapping between a plane domain \( \Omega_1 \) with \( C^{1,\alpha} \) compact boundary and a plane domain \( \Omega \) with \( C^{1,1} \) compact boundary. Consider \( a_0 \in \Omega_1 \) and set \( b_0 = f(a_0) \). Then \( w \) is bi-Lipschitz. Moreover there exists a positive constant \( c = c(K, \Omega, \Omega_1, a_0, b_0) \geq 1 \) such that

\[
\frac{1}{c} |z_1 - z_2| \leq |f(z_1) - f(z_2)| \leq c|z_1 - z_2|, \quad z_1, z_2 \in \Omega_1.
\] (1.8)

**Proof of Corollary 1.5.** Let \( b = f(a) \in \partial \Omega \). Since \( \partial \Omega \in C^{1,1} \), it follows that there exists a \( C^{1,1} \) Jordan curve \( \gamma_b \subset \overline{\Omega} \), whose interior \( D_b \) lies in \( \Omega \), and \( \partial \Omega \cap \gamma_b \) is a neighborhood of \( b \). See [13, Theorem 2.1] for an explicit construction of such a Jordan curve. Let \( D_a = f^{-1}(D_b) \), and take a conformal mapping \( g_a \) of the unit disk onto \( D_a \). Then \( f_a = f \circ g_a \) is a quasiconformal harmonic mapping from the unit disk onto the \( C^{1,1} \) domain \( D_b \). From Theorem 1.4 it follows that \( f_a \) is bi-Lipschitz, and from Kellogg’s theorem it follows that \( f = f_a \circ g_a^{-1} \) and its inverse \( f^{-1} \) are Lipschitz in some small neighborhood of \( a \) and of \( b = f(a) \) respectively. This means that \( \nabla f \) is bounded in some neighborhood of \( a \). Since \( \partial \Omega_1 \) is a compact, we deduce that \( \nabla f \) is bounded in \( \partial \Omega_1 \). The same holds for \( \nabla f^{-1} \) with respect to \( \partial \Omega \). This implies that \( f \) is bi-Lipschitz. \( \square \)

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## 2. Auxiliary results

Let \( \Omega \) be a domain in \( \mathbb{R}^2 \) having non-empty boundary \( \partial \Omega \). The distance function from the boundary is defined by

\[
d(x) = \text{dist} (x, \partial \Omega).
\] (2.1)

Let \( \Omega \) be bounded and assume \( \partial \Omega \in C^{1,1} \). These conditions on \( \Omega \) imply that \( \partial \Omega \) satisfies the following: at a.e. point \( z \in \partial \Omega \) there exists a disk \( D = D(w_z, r_z) \) depending on \( z \) such that \( \overline{D} \cap (\mathbb{C} \setminus \Omega) = \{z\} \). Moreover \( \mu := \text{ess inf} \{r_z, z \in \partial \Omega \} \).
\( \partial \Omega \) > 0. It is easy to show that \( \mu^{-1} \) bounds the curvature of \( \partial \Omega \), which means that \( \frac{1}{\mu} \geq \kappa_z \), for \( z \in \partial \Omega \). Here \( \kappa_z \) denotes the curvature of \( \partial \Omega \) at \( z \in \partial \Omega \). Under the above conditions, we have \( d \in C^{1,1}(\Gamma_\mu) \), where \( \Gamma_\mu = \{ z \in \overline{\Omega} : d(z) < \mu \} \) and for \( z \in \Gamma_\mu \) there exists \( \omega(z) \in \partial \Omega \) such that

\[
\nabla d(z) = v_{\omega(z)},
\]

(2.2)

where \( v_{\omega(z)} \) denotes the inner normal vector to the boundary \( \partial \Omega \) at the point \( \omega(z) \). See [7, Section 14.6] for the details.

**Lemma 2.1.** Let \( w : \Omega_1 \mapsto \Omega \) be a \( K \)-quasiconformal mapping and set \( \chi = -d(w(z)) \). Then

\[
|\nabla \chi| \leq |\nabla w| \leq K|\nabla \chi|
\]

(2.3)
in \( w^{-1}(\Gamma_\mu) \) for \( \mu > 0 \) such that \( 1/\mu > \kappa_0 = \text{ess sup}\{ |\kappa_z| : z \in \partial \Omega \} \).

**Proof.** Observe first that \( \nabla d \) is a unit vector. From the identity \( \nabla \chi = -\nabla d \cdot \nabla w \) it follows that

\[
|\nabla \chi| \leq |\nabla d| |\nabla w| = |\nabla w|.
\]

For a non-singular matrix \( A \) we have

\[
\inf_{|x|=1} |Ax|^2 = \inf_{|x|=1} \langle Ax, Ax \rangle = \inf_{|x|=1} \left( A^T Ax, x \right) \\
= \inf\{ \lambda : \exists x \neq 0, A^T Ax = \lambda x \} \\
= \inf\{ \lambda : \exists x \neq 0, AA^T Ax = \lambda Ax \} \\
= \inf\{ \lambda : \exists y \neq 0, AA^T y = \lambda y \} = \inf_{|x|=1} |A^T x|^2.
\]

(2.4)

We next denote that \( (\nabla \chi)^T = -(\nabla w)^T \cdot (\nabla d)^T \), therefore for \( x \in w^{-1}(\Gamma_\mu) \) we obtain

\[
|\nabla \chi| \geq \inf_{|e|=1} |(\nabla w)^T e| = \inf_{|e|=1} |\nabla w e| = l(w) \geq K^{-1}|\nabla w|.
\]

The proof of (2.3) is complete. \( \square \)

**Lemma 2.2.** Let \( \{e_1, e_2\} \) be the canonical basis of the space \( \mathbb{R}^2 \). Let \( w : \Omega_1 \mapsto \Omega \) be a twice differentiable mapping and let \( \chi = -d(w(z)) \). Then

\[
\Delta \chi(z_0) = \frac{\kappa_{w_0}}{1 - \kappa_{w_0}d(w(z_0))} \left| (O_{z_0} \nabla w(z_0))^T e_1 \right|^2 - \langle (\nabla d)(w(z_0)), \Delta w \rangle,
\]

(2.5)

where \( z_0 \in w^{-1}(\Gamma_\mu) \), \( \omega_0 \in \partial \Omega \) with \( |w(z_0) - \omega_0| = \text{dist}(w(z_0), \partial \Omega) \), \( \mu > 0 \) such that \( 1/\mu > \kappa_0 = \text{ess sup}\{ |\kappa_z| : z \in \partial \Omega \} \) and \( O_{z_0} \) is an orthogonal transformation.
Proof. Let $\nu_{\omega_0}$ be the inner unit normal vector to $\gamma$ at the point $\omega_0 \in \gamma$. Let $O_{z_0}$ be an orthogonal transformation that takes the vector $e_2$ to $\nu_{\omega_0}$. In complex notations one has:

$$O_{z_0} w = -i \nu_{\omega_0} w.$$ 

Take $\tilde{\Omega} := O_{z_0} \Omega$. Let $\tilde{d}$ be the distance function for $\tilde{\Omega}$. Then

$$d(w) = \tilde{d}(O_{z_0} w) = \text{dist} (O_{z_0} w, \partial \tilde{\Omega}).$$

Therefore $\chi(z) = -\tilde{d}(O_{z_0}(w(z)))$. Furthermore

$$\Delta \chi(z) = -\sum_{i=1}^{2} (D^2 \tilde{d})(O_{z_0}(w(z)))(O_{z_0} \nabla w(z)e_i, O_{z_0} \nabla w(z)e_i)$$

$$- \langle \nabla d(w(z)), \Delta w(z) \rangle.$$ 

(2.6)

To continue, we make use of the following proposition.

**Proposition 2.3 ([7, Lemma 14.17]).** Let $\Omega$ be bounded and assume $\partial \Omega \in C^{1,1}$. Then, with notation as in Lemma 2.2, we have

$$(D^2 \tilde{d})(O_{z_0} w(z_0)) = \text{diag} \left( \frac{-\kappa_{\omega_0}}{1 - \kappa_{\omega_0} d}, 0 \right) = \left( \begin{array}{cc} -\kappa_{\omega_0} & 0 \\ 0 & 0 \end{array} \right),$$

(2.7)

where $\kappa_{\omega_0}$ denotes the curvature of $\partial \Omega$ at $\omega_0 \in \partial \Omega$.

Applying (2.7) we have

$$\sum_{i=1}^{2} (D^2 \tilde{d})(O_{z_0}(w(z_0)))(O_{z_0}(\nabla w(z_0))e_i, O_{z_0}(\nabla w(z_0))e_i) = \sum_{i=1}^{2} \sum_{j,k=1}^{2} D_{j,k} \tilde{d}(O_{z_0}(w(z_0)) D_i(O_{z_0}w) j(z_0) \cdot D_i(O_{z_0}w) k(z_0)$$

$$= \sum_{j,k=1}^{2} D_{j,k} \tilde{d}(O_{z_0}(w(z_0))) \left[ (O_{z_0} \nabla w(z_0))^T e_j, (O_{z_0} \nabla w(z_0))^T e_k \right]$$

$$= \frac{-\kappa_{\omega_0}}{1 - \kappa_{\omega_0} d} |(O_{z_0} \nabla w(z_0))^T e_1|^2.$$

(2.8)

Finally we obtain

$$\Delta \chi(z_0) = \frac{\kappa_{\omega_0}}{1 - \kappa_{\omega_0} d} |(O_{z_0} \nabla w(z_0))^T e_1|^2 - \langle \nabla d(w(z_0)), \Delta w \rangle.$$
3. Proof of the main theorem

The main step to establish the main theorem is the following lemma.

**Lemma 3.1.** Let \( w = f(z) \) be a \( K \)-quasiconformal mapping of the unit disk onto a \( C^{1,1} \) Jordan domain \( \Omega \) satisfying the differential inequality

\[
|\Delta w| \leq B|\nabla w|^2, \quad B \geq 0
\]

(3.1)

for some \( B \geq 0 \). Assume in addition that \( w(0) = a_0 \in \Omega \). Then there exists a constant \( C(K, \Omega, B, a) > 0 \) such that

\[
\left| \frac{\partial w}{\partial r}(t) \right| \geq C(K, \Omega, B, a_0) \text{ for almost every } t \in S^1.
\]

(3.2)

**Proof.** Let us find \( A > 0 \) such that the function \( \varphi_w(z) = -\frac{1}{A} + \frac{1}{A} e^{-Ad(w(z))} \) is subharmonic on \( \{ z : d(w(z)) < \frac{1}{2\kappa_0} \} \), where

\[
\kappa_0 = \text{ess sup}\{ |\kappa_w| : w \in \gamma \}.
\]

Let \( \chi = -d(w(z)) \). Combining (2.3), (2.5) and (3.1) we get

\[
|\Delta \chi| \leq 2\kappa_0|\nabla \chi|^2 + B|\nabla \chi|^2 \leq (2\kappa_0 + B)K^2|\nabla \chi|^2.
\]

(3.3)

Take

\[
g(t) = -\frac{1}{A} + \frac{1}{A} e^{At}.
\]

Then \( \varphi_w(z) = g(\chi(z)) \). Thus

\[
\Delta \varphi_w = g''(\chi)|\nabla \chi|^2 + g'(\chi)\Delta \chi.
\]

(3.4)

Since

\[
g'(\chi) = e^{-Ad(w(z))}
\]

(3.5)

and

\[
g''(\chi) = Ae^{-Ad(w(z))},
\]

(3.6)

it follows that

\[
\Delta \varphi_w \geq (A - (2\kappa_0 + B)K^2)|\nabla \chi|^2 e^{-Ad(u(z))}.
\]

(3.7)

In order to have \( \Delta \varphi_w \geq 0 \), it is enough to take

\[
A = (2\kappa_0 + B)K^2.
\]

(3.8)

Choosing

\[
\varrho = \max \left\{ |z| : \text{dist}(w(z), \gamma) = \frac{1}{2\kappa_0} \right\},
\]

we have that \( \varphi_w \) satisfies the conditions of the following generalization of the Hopf lemma ([9]):
Lemma 3.2 ([10]). Let $\varphi$ satisfy $\Delta \varphi \geq 0$ in $R_\varrho = \{ z: \varrho \leq |z| < 1 \}$, $0 < \varrho < 1$, $\varphi$ be continuous on $\overline{R_\varrho}$, $\varphi < 0$ in $R_\varrho$, $\varphi(t) = 0$ for $t \in S^1$. Assume that the radial derivative $\frac{\partial \varphi}{\partial r}$ exists almost everywhere on $S^1$. Set $M(\varphi, \varrho) = \max_{|z|=\varrho} \varphi(z)$. Then the following inequality holds

$$\frac{\partial \varphi(t)}{\partial r} > \frac{2M(\varphi, \varrho)}{\varrho^2(1 - e^{1/\varrho^2 - 1})} \text{ for a.e. } t \in S^1. \quad (3.9)$$

We will make use of (3.9), but under some improvement for the class of quasiconformal harmonic mappings. The idea is to make the right-hand side of (3.9) independent of the mapping $w$ for $\varphi = \varphi_w$.

We will say that a quasiconformal mapping $f : \mathbb{U} \mapsto \Omega$ is normalized if $f(1) = w_0$, $f(e^{2\pi i/3}) = w_1$ and $f(e^{-2\pi i/3}) = w_2$, where $w_0 w_1$, $w_1 w_2$ and $w_2 w_0$ are arcs of $\gamma = \partial \Omega$ having the same length $|\gamma|/3$.

In what follows we will prove that, for the class $\mathcal{H}(\Omega, K, B)$ of normalized $K$-quasiconformal mappings, satisfying (3.1) for some $B \geq 0$, and mapping the unit disk onto the domain $\Omega$, the inequality (3.9) holds uniformly (see (3.10)). Let

$$\varrho := \sup \left\{ |z| : \text{dist}(w(z), \gamma) = \frac{1}{2\kappa_0}, w \in \mathcal{H}(\Omega, K, B) \right\}.$$

Then there exists a sequence $\{w_n\}, w_n \in \mathcal{H}(\Omega, K, B)$ such that

$$\varrho_n = \max \left\{ |z| : \text{dist}(w_n(z), \gamma) = \frac{1}{2\kappa_0} \right\},$$

and

$$\varrho = \lim_{n \to \infty} \varrho_n.$$ 

Now notice that if $w_n$ is a sequence of normalized $K$-quasiconformal mappings of the unit disk onto $\Omega$ then, up to taking a subsequence, $w_n$ is a locally uniformly convergent sequence converging to some quasiconformal mapping $w \in \mathcal{H}(\Omega, K, B)$. Under the condition on the boundary of $\Omega$, by [27, Theorem 4.4] this sequence is uniformly convergent on $\mathbb{U}$. Then there exists a sequence $z_n$ such that $\text{dist}(w_n(z_n), \gamma) = \frac{1}{2\kappa_0}$, $\lim_{n \to \infty} z_n = z_0$ and $\varrho = |z_0|$. Since $w_n$ converges uniformly to $w$, it follows that $\lim_{n \to \infty} w_n(z_n) = w(z_0)$, and $\text{dist}(w(z_0), \gamma) = \frac{1}{2\kappa_0}$. This implies that $\varrho < 1$. Let now

$$M(\varrho) := \sup \{ M(\varphi_w, \varrho), w \in \mathcal{H}(\Omega, K, B) \}.$$ 

Using a similar argument we obtain that there exists a uniformly convergent sequence $w_n$, converging to a mapping $w_0$, such that

$$M(\varrho) = \lim_{n \to \infty} M(\varphi_{w_n}, \varrho) = M(\varphi_{w_0}, \varrho).$$
Thus
\[ M(\varrho) < 0. \]
Placing \( M(\varrho) \) instead of \( M(\varrho, \varphi) \) and \( \varphi_w \) instead of \( \varphi \) in (3.9), we obtain
\[ \frac{\partial \varphi_w(t)}{\partial r} > \frac{2M(\varrho)}{\varrho^2(1 - e^{1/\varrho^2-1})} := C(K, \Omega, B) \text{ for a.e. } t \in S^1. \] (3.10)
To continue observe that
\[ \frac{\partial \varphi_w(t)}{\partial r} = e^{Ad(w(z))} |\nabla d| \left| \frac{\partial w}{\partial r}(t) \right| = e^{Ad(w(z))} \left| \frac{\partial w}{\partial r}(t) \right|. \]
Combining (3.8) and (3.10) we obtain for a.e. \( t \in S^1 \)
\[ \left| \frac{\partial w}{\partial r}(t) \right| = e^{-Ad(w(z))} \frac{\partial \varphi_w(t)}{\partial r} \geq e^{-K^2} \frac{2M(\varrho)}{\varrho^2(1 - e^{1/\varrho^2-1})}. \]
Lemma 3.1 is now proved for a normalized mapping \( w \). If \( w \) is not normalized then we take the composition of \( w \) and an appropriate Möbius transformation in order to obtain the desired inequality. The proof of Lemma 3.1 is complete. \( \Box \)

Conclusion of the proof of Theorem 1.4. In this setting \( w \) is harmonic, therefore \( B = 0 \). Assume first that \( w \in C^1(\overline{U}) \). Let \( l(\nabla w)(t) = ||w_z(t)|| - |w_\cdot(t)||. \) Since \( w \) is \( K \)-quasiconformal, according to (3.2) we have
\[ l(\nabla w)(t) \geq \frac{\left| \nabla w(t) \right|}{K} \geq \frac{\left| \frac{\partial w}{\partial r}(t) \right|}{K} \geq \frac{C(K, \Omega, 0, a_0)}{K} \] (3.11)
for \( t \in S^1 \). Therefore, having in mind Lewy’s theorem ([23]), which states that \( |w_\cdot| > |w_z| \) for \( z \in \mathbb{U} \), we obtain for \( t \in S^1 \) that \( |w_\cdot(t)| \neq 0 \) and hence
\[ \frac{1}{|w_\cdot|} \frac{C(K, \Omega, 0, a_0)}{K} + \frac{|w_\cdot|}{|w_{\zeta}|} \leq 1, \quad t \in S^1. \]
Since \( w \in C^1(\overline{U}) \), it follows that the functions
\[ a(z) := \frac{w_{\zeta}}{w_\cdot}, \quad b(z) := \frac{1}{w_\cdot} \frac{C(K, \Omega, 0, a_0)}{K} \]
are well-defined holomorphic functions in the unit disk having a continuous extension to the boundary. As \( |a| + |b| \) is bounded on the unit circle by 1, it follows that it is bounded on the whole unit disk by 1 because
\[ |a(z)| + |b(z)| \leq P[|a|_{S^1}](z) + P[|b|_{S^1}](z) = P[|a|_{S^1} + |b|_{S^1}](z), \quad z \in \mathbb{U}. \]
This in turn implies that for every \( z \in U \)
\[
I(\nabla w)(z) \geq \frac{C(K, \Omega, 0, a_0)}{K} =: C(\Omega, K, a_0).
\]
(3.12)

This yields that
\[
C(K, \Omega, a_0) \leq \frac{|w(z_1) - w(z_2)|}{|z_1 - z_2|}, \quad z_1, z_2 \in U.
\]

Assume now that \( w \notin C^1(\overline{U}) \). We begin with a definition.

**Definition 3.3.** Let \( G \) be a domain in \( \mathbb{C} \) and let \( a \in \partial G \). We will say that \( G_a \subset G \) is a \( \partial \)-neighborhood of \( a \) if there exists a disk \( D(a, r) := \{ z : |z - a| < r \} \) such that
\[
D(a, r) \cap G \subset G_a.
\]

Let \( t = e^{i\beta} \in S^1 \), so that \( w(t) \in \partial \Omega \). Let \( \gamma \) be an arc-length parametrization of \( \partial \Omega \) with \( \gamma(s) = w(t) \). Since \( \partial \Omega \in C^{1,1} \), there exists a \( \partial \)-neighborhood \( \Omega_t \) of \( w(t) \) with \( C^{1,1} \) Jordan boundary such that
\[
\Omega_t^\tau := \Omega_t + i\gamma'(s) \cdot \tau \subset \Omega, \quad \text{and} \quad \partial \Omega_t^\tau \subset \Omega \quad \text{for} \quad 0 < \tau \leq \tau_t \quad (\tau_t > 0).
\]
(3.13)

An example of a family \( \Omega_t^\tau \) such that \( \partial \Omega_t^\tau \in C^{1,1} \) and with the property (3.13) has been given in [13].

Let \( a_t \in \Omega_t \) be arbitrary. Then \( a_t + i\gamma'(s) \cdot \tau \in \Omega_t^\tau \). Take \( U_t = f^{-1}(\Omega_t^\tau) \). Let \( \eta_t^\tau \) be a conformal mapping of the unit disk onto \( U_t \) such that \( \eta_t^\tau(0) = f^{-1}(a_t + i\gamma'(s) \cdot \tau) \), and \( \arg \frac{d\eta_t^\tau}{dz}(0) = 0 \). Then the mapping
\[
f_t^\tau(z) := f(\eta_t^\tau(z)) - i\gamma'(s) \cdot \tau
\]
is a harmonic \( K \)-quasiconformal mapping of the unit disk onto \( \Omega_t \) satisfying the condition \( f_t^\tau(0) = a_t \). Moreover
\[
f_t^\tau \in C^1(\overline{U}).
\]

Using the case \( w \in C^1(\overline{U}) \), it follows that
\[
|\nabla f_t^\tau(z)| \geq C(K, \Omega_t, a_t).
\]

On the other hand
\[
\lim_{\tau \to 0^+} \nabla f_t^\tau(z) = \nabla (f \circ \eta_t)(z)
\]
on the compact sets of \( \overline{U} \) as well as
\[
\lim_{\tau \to 0^+} \frac{d\eta_t^\tau}{dz}(z) = \frac{d\eta_t}{dz}(z),
\]
where \( \eta_t \) is a conformal mapping of the unit disk onto \( U_0 = f^{-1}(\Omega_t) \) with \( \eta_t(0) = f^{-1}(a_t) \). It follows that
\[
|\nabla f_t(z)| \geq C(K, \Omega_t, a_t).
\]
Applying the Schwarz reflexion principle to the mapping \( \eta_t \) and using the formula
\[
\nabla (f \circ \eta_t)(z) = \nabla f \cdot \frac{d\eta_t}{dz}(z)
\]
it follows that in some \( \partial \)-neighborhood \( \tilde{U}_t \) of \( t \in S^1 \) with smooth boundary where \( (D(t, r_t) \cap \bigcup \subset \tilde{U}_t \) for some \( r_t > 0 \), the function \( f \) satisfies the inequality
\[
|\nabla f(z)| \geq \frac{C(K, \Omega_t, a_t)}{\max\{|\eta_t'(\xi)| : \xi \in \tilde{U}_t\}} =: \tilde{C}(K, \Omega_t, a_t) > 0. \tag{3.14}
\]
Since \( S^1 \) is a compact set, it can be covered by a finite family \( \partial \tilde{U}_{t_j} \cap S^1 \cap D(t, r_t/2), j = 1, \ldots, m \). It follows that the inequality
\[
|\nabla f(z)| \geq \min\{\tilde{C}(K, \Omega_{t_j}, a_{t_j}) : j = 1, \ldots, m\} =: \tilde{C}(K, \Omega, a_0) > 0 \tag{3.15}
\]
holds in the annulus
\[
\tilde{R} = \left\{ z : 1 - \frac{\sqrt{3}}{2} \min_{1 \leq j \leq m} r_{t_j} < |z| < 1 \right\} \subset \bigcup_{j=1}^{m} \tilde{U}_{t_j}.
\]
This implies that the subharmonic function \( S = |a(z)| + |b(z)| \) is bounded in \( \mathbb{U} \). According to the maximum principle, it is bounded by 1 in the whole unit disk. This in turn implies again (3.12) and consequently
\[
\frac{C(K, \Omega, a_0)}{K} |z_1 - z_2| \leq |w(z_1) - w(z_2)|, \quad z_1, z_2 \in \mathbb{U}. \quad \Box
\]

References


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