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On a stronger Lazer-McKenna conjecture for Ambrosetti-Prodi type problems

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Abstract. We consider an elliptic problem of Ambrosetti-Prodi type involving critical Sobolev exponent on a bounded smooth domain. We show that if the domain has some symmetry, the problem has infinitely many (distinct) solutions whose energy approach to infinity even *for a fixed parameter*, thereby obtaining a stronger result than the Lazer-McKenna conjecture.

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1. Introduction

Elliptic problems of Ambrosetti-Prodi type have the following form:

$$\begin{cases} -\Delta u = g(u) - \bar{s}\varphi_1(x), & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $g(t)$ satisfies $\lim_{t\to-\infty} \frac{g(t)}{t} = \nu < \lambda_1$, $\lim_{t\to+\infty} \frac{g(t)}{t} = \mu > \lambda_1$, λ_1 is the first eigenvalue of $-\Delta$ with Dirichlet boundary condition and $\varphi_1 > 0$ is the first eigenfunction. Here $\mu = +\infty$ and $\nu = -\infty$ are allowed. It is well-known that the location of μ, ν with respect to the spectrum of $(-\Delta, H_0^1(\Omega))$ plays an important role in the multiplicity of solutions for problem (1.1). See for example [3, 8, 9, 18–20, 23–26, 31–34]. In the early 1980s, Lazer and McKenna conjectured that if $\mu = +\infty$ and $g(t)$ does not grow too fast at infinity, (1.1) has an unbounded number of solutions as $\bar{s} \to +\infty$. See [24].

In this paper, we will consider the following special case:

$$\begin{cases} -\Delta u = u_+^{2^*-1} + \lambda u - \bar{s}\varphi_1, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{1.2}$$

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where Ω is a bounded domain in $\mathbb{R}^N$ with C^2 boundary, $N \geq 3$, $\lambda < \lambda_1$, $\bar{s} > 0$, $u_+ = \max(u, 0)$ and $2^* = 2N/(N-2)$.

It is easy to see that (1.2) has a negative solution

$$\underline{u}_{\bar{s}} = -\frac{\bar{s}}{\lambda_1 - \lambda}\varphi_1,$$

if $\lambda < \lambda_1$. Moreover, if $\underline{u}_{\bar{s}} + u$ is a solution of (1.2), then u satisfies

$$\begin{cases} -\Delta u = (u - s\varphi_1)_+^{2^*-1} + \lambda u, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{1.3}$$

where $s = \frac{\bar{s}}{\lambda_1 - \lambda} > 0$.

Let us recall some recent results on the Lazer-McKenna conjecture related to (1.3). Firstly, Dancer and the second author proved in [12, 13] that for $N \geq 2$ and $\lambda \in (-\infty, \lambda_1)$, the Lazer-McKenna conjecture is true if the critical exponent in (1.3) is replaced by sub-critical one. In the critical case, it was proved in [27,28,36] that if $N \geq 6$ and $\lambda \in (0, \lambda_1)$, then (1.3) has unbounded number of solutions as $s \to +\infty$. The solutions constructed for (1.3) concentrate either at the maximum points of the first eigenfunction [27], or at some boundary points of the domain [36] as $s \to +\infty$. On the other hand, Druet proves in [21] that the conditions $N \geq 6$ and $\lambda \in (0, \lambda_1)$ are necessary for the existence of the peak-solutions constructed in [27, 36]. More precisely, the result in [21] states that if $N = 3, 4, 5$, or $N \geq 6$ and $\lambda \leq 0$, then (1.3) has no solution u_s, such that the energy of u_s is bounded as $s \to +\infty$. This result suggests that it is more difficult to find solutions for (1.3) in the lower dimensional cases $N = 3, 4, 5$, or in the case $\lambda \leq 0$ and $N \geq 6$.

Note that all the results just mentioned state that (1.3) has more and more solutions as *the parameter* $s \to +\infty$. But for *fixed* $s > 0$, it is hard to estimate how many solutions (1.3) has. (In the critical case, for fixed s, it is even unknown if there is a solution.)

In this paper, we will deal with (1.3) in the lower dimensional cases $N = 4, 5, 6$, or $N \geq 7$ and $\lambda \leq 0$, assuming that the domain Ω satisfies the following symmetry condition:

$(S1)$: If $x = (x_1, \cdots, x_N) \in \Omega$,
then, for any $\theta \in [0, 2\pi]$, $(r\cos\theta, r\sin\theta, x_3, \cdots, x_N) \in \Omega$, where $r = \sqrt{x_1^2 + x_2^2}$;

$(S2)$: If $x = (x_1, \cdots, x_N) \in \Omega$,
then, for any $3 \leq i \leq N$, $(x_1, x_2, x_3, \cdots, -x_i, \cdots, x_N) \in \Omega$.

The main result of this paper is the following:

Theorem 1.1. *Suppose that* Ω *satisfies* $(S1)$ *and* $(S2)$. *Assume that one of the following conditions holds:*

(i) $N = 4, 5$, $\lambda < \lambda_1$ *and* $s > 0$*;*
(ii) $N = 6$, $\lambda < \lambda_1$ *and* $s > |\lambda| s_0$ *for some* $s_0 > 0$*, which depends on* Ω *only;*
(iii) $N \geq 7$, $\lambda = 0$ *and* $s > 0$.

Then, (1.3) *has infinitely many distinct solutions whose energy can approach to infinity.*

The result in Theorem 1.1 is stronger than the Lazer-McKenna conjecture. Note that in Theorem 1.1, the constant s is *fixed*. In fact, all the parameters are *fixed*. This gives a striking contrast to the results in [27,36], where s is regarded as a parameter which needs to tend to infinity in order to obtain the results there. As far as the authors know, this seems to be the first such result for Ambrosetti-Prodi type problems. We believe Theorem 1.1 should be true in any general domain and hence we pose the following stronger Lazer-McKenna conjecture:

Stronger Lazer-McKenna Conjecture: *Let* s *be fixed and* $\lambda < \lambda_1$. *Then problem* (1.3) *has infinitely many solutions.*

We are not able to obtain similar result for the cases $N = 3$, and $N \geq 7$ and $\lambda < 0$. But we have the following weaker result for $N \geq 7$ and $\lambda < 0$, which gives a positive answer to the Lazer–McKenna conjecture in this case:

Theorem 1.2. *Suppose that* Ω *satisfies* $(S1)$ *and* $(S2)$*, and* $N \geq 7$, $\lambda < \lambda_1$. *Then, the number of distinct solutions for* (1.3) *is unbounded as* $s \to +\infty$.

Problem (1.3) is a bit delicate in the case $N = 3$. When $s = 0$, Brezis and Nirenberg [7] proved that (1.3) has a least energy solution if $\lambda \in (0, \lambda_1)$, while for $N = 3$, this result holds only if $\lambda \in (\lambda^*, \lambda_1)$ for some $\lambda^* > 0$ (if Ω is a ball, $\lambda^* = \frac{\lambda_1}{4}$). The main reason for this difference is that the function defined in (1.4) does not decay fast enough if $N = 3$. Similarly, the main reason that we are not able to prove Theorem 1.1 for $N = 3$ is that the function defined in (1.7) does not decay fast enough.

In the Lazer and McKenna conjecture, the parameter s is large. Let us now consider the other extreme case: $s \to 0+$. Using the same argument as in [7], we can show that for $\lambda \in (\lambda^*, \lambda_1)$, $\lambda^* = 0$ if $N = 4$, $\lambda^* > 0$ if $N = 3$, (1.3) has a least energy solution if $s > 0$ is small. We can obtain more in the case $N = 3$.

Theorem 1.3. *Suppose that* Ω *satisfies* $(S1)$ *and* $(S2)$*, and* $N = 3$, $\lambda < \lambda_1$. *Then, the number of the solutions for* (1.3) *is unbounded as* $s \to 0+$.

Note that the result in Theorem 1.3 is not trivial, because if $\lambda < \lambda^*$, we can not find even one solution by using the method in [7]. Moreover, we show that (1.3) has more and more solutions as $s \to 0+$ for all $\lambda < \lambda_1$ if $N = 3$.

The readers can refer to [6, 10, 11, 17] for results on the Lazer-McKenna conjecture for other type of nonlinearities.

In Theorems 1.1-1.3, we have assumed that $N \geq 3$. When $N = 2$, M. del Pino and Munoz [17] proved the Lazer-McKenna conjecture when the right hand nonlinearity is e^u (which is still subcritical in $\mathbb{R}^2$). The authors believe that when $N = 2$, results similar to Theorems 1.1-1.3 may be true if the right hand nonlinearity is of the *critical type*, *i.e.*, $h(u)e^{u^2}$. When $N = 1$, the critical exponent is $\frac{N+2}{N-2} = -3$. In this case, some form of Lazer-McKenna conjecture may be true if the right hand nonlinearity is $-u^{-3}$. We refer to [1] and [2] for discussions on critical nonlinearities in dimensions $N = 1, 2$.

Before we close this section, let us outline the proof of Theorems 1.1 and 1.2 and discuss the conditions imposed in these two theorems.

For any $\bar{x} \in \mathbb{R}^N$, $\mu > 0$, denote

$$U_{\mu,\bar{x}}(y) = (N(N-2))^{\frac{N-2}{4}} \frac{\mu^{(N-2)/2}}{(1 + \mu^2|y - \bar{x}|^2)^{(N-2)/2}}. \tag{1.4}$$

Then, $U_{\mu,\bar{x}}$ satisfies $-\Delta U_{\mu,\bar{x}} = U_{\mu,\bar{x}}^{2^*-1}$. In this paper, we will use the following notation: $U = U_{1,0}$.

Let

$$\varepsilon = \frac{s^{\frac{2}{N-2}}}{k^2}, \quad \mu = \frac{\Lambda}{\varepsilon}, \quad \Lambda \in [\delta, \delta^{-1}]$$

and $k \geq k_0$, where $\delta > 0$ is a small constant, and $k_0 > 0$ is a large constant, which is to be determined later.

Using the transformation $u(y) \mapsto \varepsilon^{-\frac{N-2}{2}} u\left(\frac{y}{\varepsilon}\right)$, we find that (1.3) becomes

$$\begin{cases} -\Delta u = \left(u - s\varepsilon^{\frac{N-2}{2}} \varphi_1(\varepsilon y)\right)_+^{2^*-1} + \lambda \varepsilon^2 u, & \text{in} \quad \Omega_\varepsilon, \\ u = 0, & \text{on} \quad \partial\Omega_\varepsilon, \end{cases} \tag{1.5}$$

where $\Omega_\varepsilon = \{y : \ \varepsilon y \in \Omega\}$. Let

$$\Phi_\varepsilon(y) = \varepsilon^{\frac{N-2}{2}} \varphi_1(\varepsilon y).$$

For $\xi \in \Omega_\varepsilon$, we define $W_{\Lambda,\xi}$ as the unique solution of

$$\begin{cases} -\Delta W - \lambda \varepsilon^2 W = U_{\Lambda,\xi}^{2^*-1} & \text{in} \quad \Omega_\varepsilon, \\ W = 0 & \text{on} \quad \partial\Omega_\varepsilon. \end{cases} \tag{1.6}$$

Let $y = (y', y'') \in \mathbb{R}^N$, where $y' = (y_1, y_2)$, and $y'' = (y_3, \cdots, y_N)$. Define

$$H_s = \Big\{u : u \in H^1(\Omega_\varepsilon), u \text{ is even in } y_h, h = 3, \cdots, N, u(r\cos\theta, r\sin\theta, y'') = u\left(r\cos\left(\theta + \frac{2\pi j}{k}\right), r\sin\left(\theta + \frac{2\pi j}{k}\right), y''\right), j = 1, \ldots, k-1\Big\},$$

and

$$\mathbf{x}_j = \left(\frac{r}{\varepsilon}\cos\frac{2(j-1)\pi}{k}, \frac{r}{\varepsilon}\sin\frac{2(j-1)\pi}{k}, 0\right), \quad j = 1, \cdots, k,$$

where 0 is the zero vector in $\mathbb{R}^{N-2}$.

Let

$$W_{r,\Lambda}(y) = \sum_{j=1}^{k} W_{\Lambda,\mathbf{x}_j}. \tag{1.7}$$

We are going to construct a solution for (1.3), which is close to $W_{r,\Lambda}$ for some suitable Λ and r and large k.

Theorem 1.1 is a direct consequence of the following result:

Theorem 1.4. *Under the same conditions as in Theorem* 1.1*, there is an integer* $k_0 > 0$*, such that for any integer* $k \geq k_0$*,* (1.5) *has a solution* u_k *of the form*

$$u_k = W_{r_k,\Lambda_k}(y) + \omega_k,$$

where $\omega_k \in H_s$*, and as* $k \to +\infty$*,* $r_k \to r_0 > 0$*,* $\Lambda_k \to \Lambda_0 > 0$*,* $\|\omega_k\|_{L^\infty} \to 0$.

On the other hand, if $N \geq 7$ and $\lambda < 0$, we have the following weaker result:

Theorem 1.5. *Suppose that* $N \geq 7$ *and* $\lambda < \lambda_1$*. Then there is a large constant* $s_0 > 0$*, such that for any* $s > s_0$*, and integer* k *satisfying* $s^{\frac{(2-2\theta)(N-4)}{(N-6)(N-2)}} \leq k \leq s^{\frac{(2-\theta)(N-4)}{(N-6)(N-2)}}$*, where* $\theta > 0$ *is a fixed small constant,* (1.5) *has a solution* $u_{k,s}$ *of the form*

$$u_{k,s} = W_{r_k,\Lambda_k}(y) + \omega_{k,s},$$

where $\omega_{k,s} \in H_s$*, and as* $s \to +\infty$*,* $r_k \to r_0 > 0$*,* $\Lambda_k \to \Lambda_0 > 0$*,* $\|\omega_{k,s}\|_{L^\infty} \to 0$.

Since $s^{\frac{(2-\theta)(N-4)}{(N-6)(N-2)}} - s^{\frac{(2-2\theta)(N-4)}{(N-6)(N-2)}} \to +\infty$ as $s \to +\infty$, Theorem 1.2 is a direct consequence of Theorem 1.5. Let us point out that in the case $N \geq 7$ and $\lambda \in (0, \lambda_1)$, the solutions in Theorem 1.5 are different from those constructed in [27,36], where the energy of the solutions remains bounded as $s \to +\infty$.

It is easy to see that Theorem 1.3 is a direct consequence of the following result:

Theorem 1.6. *Suppose that* $N = 3$ *and* $\lambda < \lambda_1$*. Then there is a small constant* $s_1 > 0$ *and a large constant* $k_0 > 0$ *(independent of* s*), such that for any* $s \in (0, s_1)$*, and integer* k *satisfying*

$$k_0 \leq k \leq C s^{-\frac{2\tau}{1-2\tau}}, \tag{1.8}$$

for some $\tau \in (0, \frac{4}{11})$*, then* (1.5) *has a solution* $u_{k,s}$ *of the form*

$$u_{k,s} = W_{r_k,\Lambda_k}(y) + \omega_{k,s},$$

where $\omega_{k,s} \in H_s$*, and as* $s \to 0$*,* $r_k \to r_0 > 0$*,* $\Lambda_k \to \Lambda_0 > 0$*,* $\|\omega_{k,s}\|_{L^\infty} \to 0$.

Let make a few remarks on the conditions imposed on Theorems 1.1 and 1.2. It is easy to see that the first eigenfunction $\varphi_1 \in H_s$. In this paper, we denote

$$\bar{\varphi}(r) = \varphi_1(r, 0).$$

The functional corresponding to (1.5) is

$$I(u) = \frac{1}{2}\int_{\Omega_\varepsilon} \left(|Du|^2 - \lambda\varepsilon^2 u^2\right) - \frac{1}{2^*}\int_{\Omega_\varepsilon} (u - s\Phi_\varepsilon)_+^{2^*}, \quad u \in H_s.$$

Let Γ be a connected component of the set $\Omega \cap \{y_3 = \cdots = y_N = 0\}$. Then, by $(S1)$, there are $r_2 > r_1 \geq 0$, such that

$$\bar{\Gamma} = \left\{y : r_1 \leq \sqrt{y_1^2 + y_2^2} \leq r_2,\ y_3 = \cdots = y_N = 0\right\}.$$

If $N = 4, 5$, then $\frac{N-2}{2} < 2$. We obtain from Proposition A.3,

$$I(W_{r,\Lambda}) = k\left(A_0 + \frac{A_2 s\bar{\varphi}(r)\varepsilon^{\frac{N-2}{2}}}{\Lambda^{\frac{N-2}{2}}} - \frac{A_3\varepsilon^{N-2}k^{N-2}}{r^{N-2}\Lambda^{N-2}} + O\left(\varepsilon^{\frac{(N-2)(1+\sigma)}{2}}\right)\right). \quad (1.9)$$

It is easy to see that the function

$$r^{\frac{N-2}{2}}\bar{\varphi}(r), \quad r \in [r_1, r_2], \quad (1.10)$$

has a maximum point r_0, satisfying $r_0 \in (r_1, r_2)$, since $r_i^{\frac{N-2}{2}}\bar{\varphi}(r_i) = 0$, $i = 1, 2$. As a result,

$$\frac{A_2 s\bar{\varphi}(r)}{\Lambda^{\frac{N-2}{2}}} - \frac{A_3}{r^{N-2}\Lambda^{N-2}}, \quad (r, \Lambda) \in (r_1, r_2) \times (\delta, \delta^{-1}),$$

has a maximum point (r_0, Λ_0), where

$$\Lambda_0 = \left(\frac{2A_3}{A_2 s r_0^{N-2}\bar{\varphi}(r_0)}\right)^{\frac{2}{N-2}},$$

for any fixed $s > 0$. Thus, $I(W_{r,\Lambda})$ has a maximum point in $(r_1, r_2) \times (\delta, \delta^{-1})$, if $k > 0$ is large.

If $N = 6$, then $\frac{N-2}{2} = 2$. Thus, we find from Proposition A.3,

$$I(W_{r,\Lambda}) = k\left(A_0 + (-\lambda A_1 + A_2 s\bar{\varphi}(r))\frac{\varepsilon^2}{\Lambda^2} - \frac{A_3\varepsilon^4 k^4}{r^4\Lambda^4} + O\left(\varepsilon^{2+\sigma}\right)\right). \quad (1.11)$$

Let

$$g(r) = r^2(A_2 s\bar{\varphi}(r) - A_1\lambda), \quad r \in [r_1, r_2]. \quad (1.12)$$

It is easy to see that we can always choose a constant $s_0 > 0$, such that if $s > |\lambda| s_0$, then $g(r)$ has a maximum point r_0, satisfying $g(r_0) > 0$, $r_0 \in (r_1, r_2)$. As a result,

$$\frac{-\lambda A_1 + A_2 s\bar{\varphi}(r)}{\Lambda^2} - \frac{A_3}{r^4\Lambda^4}, \quad (r, \Lambda) \in (r_1, r_2) \times (\delta, \delta^{-1}),$$

has maximum point (r_0, Λ_0), where

$$\Lambda_0 = \left(\frac{2A_3}{(-\lambda A_1 + A_2 s\bar{\varphi}(r_0))r_0^4} \right)^{\frac{1}{2}},$$

for any fixed $s > 0$. Thus, $I(W_{r,\lambda})$ has a maximum point in $(r_1, r_2) \times (\delta, \delta^{-1})$, if $k > 0$ is large.

If $N \geq 7$ and $\lambda = 0$, then Proposition A.3 gives

$$I(W_{r,\Lambda}) = k\left(A_0 + \frac{A_2 s\bar{\varphi}(r)\varepsilon^{\frac{N-2}{2}}}{\Lambda^{\frac{N-2}{2}}} - \frac{A_3\varepsilon^{N-2}k^{N-2}}{r^{N-2}\Lambda^{N-2}} + O\left(\varepsilon^{\frac{(N-2)(1+\sigma)}{2}}\right)\right), \quad (1.13)$$

So, we are in the same situation as the case $N = 4, 5$.

On the other hand, if $N \geq 7$, then $\frac{N-2}{2} > 2$. Thus $\varepsilon^{\frac{N-2}{2}}$ is a higher order term of ε^2. Thus if $\lambda \neq 0$, then for each fixed $s > 0$, we have

$$I(W_{r,\Lambda}) = k\left(A_0 - \frac{\lambda A_1\varepsilon^2}{\Lambda^2} - \frac{A_3\varepsilon^{N-2}k^{N-2}}{r^{N-2}\Lambda^{N-2}} + O\left(\varepsilon^{2+\sigma}\right)\right), \quad (1.14)$$

But

$$-\frac{\lambda A_1}{\Lambda^2} - \frac{A_3}{r^{N-2}\Lambda^{N-2}}, \quad (r, \Lambda) \in (r_1, r_2) \times (\delta, \delta^{-1}),$$

does not have a critical point even if $\lambda < 0$. So, we don't know whether $I(W_{r,\Lambda})$ has a critical point. Thus, to obtain a solution for (1.3), we need to let s change so that

$$\varepsilon^2 \ll s\varepsilon^{\frac{N-2}{2}}, \quad \varepsilon \ll 1. \quad (1.15)$$

If (1.15) holds, then

$$I(W_{r,\Lambda}) = k\left(A_0 + \frac{A_2\bar{\varphi}(r)s\varepsilon^{\frac{N-2}{2}}}{\Lambda^{\frac{N-2}{2}}} - \frac{A_3\varepsilon^{N-2}k^{N-2}}{r^{N-2}\Lambda^{N-2}} + O\left((s\varepsilon^{\frac{N-2}{2}})^{1+\sigma}\right)\right). \quad (1.16)$$

So, we are in a similar situation as $\lambda = 0$. Note the (1.15) implies

$$k \ll s^{\frac{2(N-4)}{(N-2)(N-6)}}, \quad k \gg s^{\frac{1}{N-2}},$$

which gives an upper bound for k. Therefore, in this case, we are not able to obtain the existence of infinitely many solutions even if $s > 0$ is large.

In the case $N = 3$, for fixed $s > 0$, some estimates which are valid for $N \ge 4$ may not be true due to the slow decay of the function $W_{r,\Lambda}$. Under the condition $s \le Ck^{-\frac{1}{2\tau}+1}$ for some $\tau \in (0, \frac{4}{11})$, we can recover all these estimates. But the condition $s \le Ck^{-\frac{1}{2\tau}+1}$ imposes an upper bound (1.8) for the number of bubbles k.

The energy of the solutions obtained in Theorems 1.4 and 1.5 is very large because k must be large. This result is in consistence of the result in [21].

Finally, let us point out that the eigenvalue φ_1 is not essential in this paper. We can replace φ_1 by any function φ, satisfying $\varphi > 0$ in Ω, $\varphi = 0$ on $\partial\Omega$ and $\varphi \in H_s$.

We will use the reduction argument as in [4,5,14–16,29,30] and [38] to prove the main results of this paper. Unlike those papers, where a parameter always appears in some form, in Theorem 1.4, s is a fixed positive constant. To prove Theorem 1.4, *the number of the bubbles k is used as a parameter to carry out the reduction*. Similar idea has been used in [35, 37].

2. The reduction

In this section, we will reduce the problem of finding a k-peak solution for (1.3) to a finite dimension problem.

Let

$$\|u\|_* = \sup_y \left(\sum_{j=1}^{k} \frac{1}{(1 + |y - \mathbf{x}_j|)^{\frac{N-2}{2}+\tau}} \right)^{-1} |u(y)|, \tag{2.1}$$

and

$$\|f\|_{**} = \sup_y \left(\sum_{j=1}^{k} \frac{1}{(1 + |y - \mathbf{x}_j|)^{\frac{N+2}{2}+\tau}} \right)^{-1} |f(y)|, \tag{2.2}$$

where $\tau \in (0, 1)$ is a constant, such that

$$\sum_{j=2}^{k} \frac{1}{|\mathbf{x}_j - \mathbf{x}_1|^{\tau}} \le C. \tag{2.3}$$

Recall that $\varepsilon = \frac{s^{\frac{2}{N-2}}}{k^2}$, and

$$\sum_{j=2}^{k} \frac{1}{|\mathbf{x}_j - \mathbf{x}_1|^{\tau}} \le C\varepsilon^{\tau} k^{\tau} \sum_{j=2}^{k} \frac{1}{j^{\tau}} \le C\varepsilon^{\tau} k.$$

In order to achieve (2.3), we need to choose τ according to whether $s > 0$ is fixed or not. We choose τ as follows:

$$\tau = \begin{cases} \frac{1}{2}, & \text{in Theorems 1.4 and 1.5;} \\ \text{the number in (1.8)}, & \text{in Theorem 1.6.} \end{cases} \tag{2.4}$$

Let

$$Y_{i,1} = \frac{\partial W_{\Lambda,\mathbf{x}_i}}{\partial \Lambda}, \qquad Z_{i,1} = -\Delta Y_{i,1} - \lambda\varepsilon^2 Y_{i,1} = (2^* - 1) U_{\Lambda,\mathbf{x}_i}^{2^*-2} \frac{\partial U_{\Lambda,\mathbf{x}_i}}{\partial \Lambda},$$

and

$$Y_{i,2} = \frac{\partial W_{\Lambda,\mathbf{x}_i}}{\partial r}, \qquad Z_{i,2} = -\Delta Y_{i,2} - \lambda\varepsilon^2 Y_{i,2} = (2^* - 1) U_{\Lambda,\mathbf{x}_i}^{2^*-2} \frac{\partial U_{\Lambda,\mathbf{x}_i}}{\partial r}.$$

We consider

$$\begin{cases} -\Delta\phi_k - \lambda\varepsilon^2\phi_k - (2^*-1)\left(W_{r,\Lambda} - s\Phi_\varepsilon\right)_+^{2^*-2}\phi_k = h + \sum_{j=1}^{2}\sum_{i=1}^{k} c_j Z_{i,j}, \text{ in } \Omega_\varepsilon, \\ \phi_k \in H_s, \\ \left\langle \sum_{i=1}^{k} Z_{i,j}, \phi_k \right\rangle = 0, \quad j = 1, 2, \end{cases} \tag{2.5}$$

for some number c_j, where $\langle u, v\rangle = \int_{\Omega_\varepsilon} uv$.

We need the following result, whose proof is standard.

Lemma 2.1. *Let f satisfy $\|f\|_{**} < \infty$ and let u be the solution of*

$$-\Delta u - \lambda\varepsilon^2 u = f \quad in \quad \Omega_\varepsilon, \qquad u = 0 \quad on \quad \partial\Omega_\varepsilon,$$

where $\lambda < \lambda_1$. Then we have

$$|u(y)| \le C \int_{\Omega_\varepsilon} \frac{|f(z)|}{|z-y|^{N-2}} dz.$$

Next, we need the following lemma to carry out the reduction.

Lemma 2.2. *Assume that ϕ_k solves (2.5) for $h = h_k$. If $\|h_k\|_{**}$ goes to zero as k goes to infinity, so does $\|\phi_k\|_*$.*

Proof. We argue by contradiction. Suppose that there are $k \to +\infty$, $h = h_k$, $\Lambda_k \in [\delta, \delta^{-1}]$, and ϕ_k solving (2.5) for $h = h_k$, $\Lambda = \Lambda_k$, with $\|h_k\|_{**} \to 0$, and $\|\phi_k\|_* \ge c' > 0$. We may assume that $\|\phi_k\|_* = 1$. For simplicity, we drop the subscript k.

By Lemma 2.1,

$$\begin{aligned} |\phi(y)| \le & C \int_{\Omega_\varepsilon} \frac{1}{|z-y|^{N-2}} W_{r,\Lambda}^{2^*-2} |\phi(z)|\, dz \\ & + C \int_{\Omega_\varepsilon} \frac{1}{|z-y|^{N-2}} \left(|h(z)| + \left| \sum_{j=1}^{2}\sum_{i=1}^{k} c_j Z_{i,j}(z) \right| \right) dz \end{aligned} \tag{2.6}$$

Using Lemma B.4 and B.5, there is a strictly positive number θ such that

$$\left|\int_{\Omega_\varepsilon} \frac{1}{|z-y|^{N-2}} W_{r,\Lambda}^{2^*-2}\phi(z)\,dz\right| \le C\|\phi\|_* \sum_{j=1}^{k} \frac{1}{(1+|y-\mathbf{x}_j|)^{\frac{N-2}{2}+\tau+\theta}}. \tag{2.7}$$

It follows from Lemma B.3 that

$$\begin{aligned}\left|\int_{\Omega_\varepsilon} \frac{1}{|z-y|^{N-2}} h(z)dz\right| &\le C\|h\|_{**} \int_{\mathbb{R}^N} \frac{1}{|z-y|^{N-2}} \sum_{j=1}^{k} \frac{1}{(1+|z-\mathbf{x}_j|)^{\frac{N+2}{2}+\tau}} dz \\ &\le C\|h\|_{**} \sum_{j=1}^{k} \frac{1}{(1+|y-\mathbf{x}_j|)^{\frac{N-2}{2}+\tau}},\end{aligned} \tag{2.8}$$

and

$$\begin{aligned}\left|\int_{\Omega_\varepsilon} \frac{1}{|z-y|^{N-2}} \sum_{i=1}^{k} Z_{i,j}(z)dz\right| &\le C\sum_{i=1}^{k} \int_{\mathbb{R}^N} \frac{1}{|z-y|^{N-2}} \frac{1}{(1+|z-\mathbf{x}_i|)^{N+2}} dz \\ &\le C\sum_{i=1}^{k} \frac{1}{(1+|y-\mathbf{x}_i|)^{\frac{N-2}{2}+\tau}}.\end{aligned} \tag{2.9}$$

Next, we estimate c_j. Multiplying (2.5) by $Y_{1,l}$ and integrating, we see that c_j satisfies

$$\left\langle \sum_{j=1}^{2}\sum_{i=1}^{k} Z_{i,j}, Y_{1,l}\right\rangle c_j = \left\langle -\Delta\phi - \lambda\varepsilon^2\phi - (2^*-1)W_{r,\Lambda}^{2^*-2}\phi, Y_{1,l}\right\rangle - \langle h, Y_{1,l}\rangle. \tag{2.10}$$

It follows from Lemma B.2 that

$$\left|\langle h, Y_{1,l}\rangle\right| \le C\|h\|_{**} \int_{\mathbb{R}^N} \frac{1}{(1+|z-\mathbf{x}_1|)^{N-2-\beta}} \sum_{j=1}^{k} \frac{1}{(1+|z-\mathbf{x}_j|)^{\frac{N+2}{2}+\tau}} dz \le C\|h\|_{**},$$

since $\beta > 0$ can be chosen as small as desired.

On the other hand,

$$\begin{aligned}&\left\langle -\Delta\phi - \lambda\varepsilon^2\phi - (2^*-1)W_{r,\Lambda}^{2^*-2}\phi, Y_{1,l}\right\rangle\\&= \left\langle -\Delta Y_{1,l} - \lambda\varepsilon^2 Y_{1,l} - (2^*-1)W_{r,\Lambda}^{2^*-2}Y_{1,l}, \phi\right\rangle\\&= (2^*-1)\left\langle U_{\Lambda,\mathbf{x}_1}^{2^*-2}\partial_l U_{\Lambda,\mathbf{x}_1} - W_{r,\Lambda}^{2^*-2}Y_{1,l}, \phi\right\rangle,\end{aligned} \tag{2.11}$$

where $\partial_l = \partial_\Lambda$ if $l=1$, $\partial_l = \partial_r$ if $l=2$.

By Lemmas B.1,

$$|\phi(y)| \le C\|\phi\|_*.$$

We consider the cases $N \ge 6$ first. Note that $\frac{4}{N-2} \le 1$ for $N \ge 6$. Using Lemmas A.1 and B.2, noting that

$$|W_{r,\Lambda}^{2^*-2} - W_{\Lambda,\mathbf{x}_1}^{2^*-2}| \le \sum_{j=2}^{k} W_{\Lambda,\mathbf{x}_j}^{2^*-2},$$

and

$$\varepsilon \le \frac{C}{1+|z-\mathbf{x}_1|},$$

we obtain

$$\begin{aligned}&\left|\left\langle U_{\Lambda,\mathbf{x}_1}^{2^*-2}\partial_l U_{\Lambda,x_j} - W_{r,\Lambda}^{2^*-2}Y_{1,l}, \phi\right\rangle\right|\\&\le C\|\phi\|_*\int_{\Omega_\varepsilon}\frac{1}{(1+|z-\mathbf{x}_1|)^{N-2-\beta}}\sum_{i=2}^{k}\frac{1}{(1+|z-\mathbf{x}_i|)^{4-\beta}}\,dz\\&\quad + C\|\phi\|_*\int_{\Omega_\varepsilon}U_{\Lambda,\mathbf{x}_1}^{2^*-2}\left(\varepsilon^{N-2}+\frac{|\lambda|\varepsilon^2}{(1+|y-\mathbf{x}_j|)^{N-4-\beta}}\right)\sum_{j=1}^{k}\frac{1}{(1+|z-\mathbf{x}_j|)^{\frac{N-2}{2}+\tau}}\\&\quad + C\|\phi\|_*\int_{\Omega_\varepsilon}U_{\Lambda,\mathbf{x}_1}\left(\varepsilon^{N-2}+\frac{|\lambda|\varepsilon^2}{(1+|y-\mathbf{x}_j|)^{N-4-\beta}}\right)^{2^*-2}\sum_{j=1}^{k}\frac{1}{(1+|z-\mathbf{x}_j|)^{\frac{N-2}{2}+\tau}}\\&\le C\|\phi\|_*\sum_{j=2}^{k}\frac{1}{|\mathbf{x}_1-\mathbf{x}_j|^{1+\sigma}}+o(1)\|\phi\|_* = o(1)\|\phi\|_*.\end{aligned} \tag{2.12}$$

For $N = 3, 4, 5,$ we have $\frac{4}{N-2} > 1$. By Lemmas B.1, B.2,

$$
\begin{aligned}
&\Big|\Big\langle U^{2^*-2}_{\Lambda,\mathbf{x}_1}\partial_l U_{\Lambda,\mathbf{x}_j} - W^{2^*-2}_{r,\Lambda} Y_{1,l}, \phi\Big\rangle\Big| \\
\le & C\int_{\Omega_\varepsilon} W^{2^*-3}_{\Lambda,\mathbf{x}_1}\sum_{j=2}^k W_{\Lambda,\mathbf{x}_j}|Y_{1,l}\phi| + C\int_{\Omega_\varepsilon}\big(\sum_{j=2}^k W_{\Lambda,\mathbf{x}_j}\big)^{\frac{4}{N-2}}|Y_1\phi| \\
&+ C\|\phi\|_*\int_{\Omega_\varepsilon} U^{2^*-2}_{\Lambda,\mathbf{x}_1}\left(\varepsilon^{N-2}+\frac{|\lambda|\varepsilon^2}{(1+|y-\mathbf{x}_j|)^{N-4-\beta}}\right)\sum_{j=1}^k\frac{1}{(1+|z-\mathbf{x}_j|)^{\frac{N-2}{2}+\tau}} \\
&+ C\|\phi\|_*\int_{\Omega_\varepsilon} U_{\Lambda,\mathbf{x}_1}\left(\varepsilon^{N-2}+\frac{|\lambda|\varepsilon^2}{(1+|y-\mathbf{x}_j|)^{N-4-\beta}}\right)^{2^*-2}\sum_{j=1}^k\frac{1}{(1+|z-\mathbf{x}_j|)^{\frac{N-2}{2}+\tau}} \\
\le & C\|\phi\|_*\int_{\Omega_\varepsilon}\frac{1}{(1+|z-\mathbf{x}_1|)^{4-\beta}}\sum_{j=2}^k\frac{1}{(1+|z-\mathbf{x}_j|)^{N-2-\beta}} \\
&+ C\int_{\Omega_\varepsilon}\big(\sum_{j=2}^k U^{1-\beta}_{\Lambda,\mathbf{x}_j}\big)^{\frac{4}{N-2}}|Y_{1,l}\phi| + o(1)\|\phi\|_* \\
\le & C\|\phi\|_*\int_{\Omega_\varepsilon}\frac{1}{(1+|z-\mathbf{x}_1|)^{N-2-\beta}}\big(\sum_{j=2}^k U^{1-\beta}_{\Lambda,\mathbf{x}_j}\big)^{\frac{4}{N-2}}\sum_{i=1}^k\frac{1}{(1+|y-\mathbf{x}_i|)^{\frac{N-2}{2}+\tau}} \\
&+ o(1)\|\phi\|_*.
\end{aligned} \tag{2.13}
$$

Let

$$
\Omega_j = \left\{ y = (y', y'') \in \Omega_\varepsilon : \left\langle \frac{y'}{|y'|}, \frac{\mathbf{x}_j}{|\mathbf{x}_j|}\right\rangle \ge \cos\frac{\pi}{k}\right\}.
$$

If $y \in \Omega_1$, then

$$
\begin{aligned}
\sum_{j=2}^k U^{1-\beta}_{\Lambda,\mathbf{x}_j} &\le \frac{1}{(1+|y-\mathbf{x}_1|)^{N-2-\tau-(N-2)\beta-\theta}}\sum_{j=2}^k\frac{1}{|\mathbf{x}_j-\mathbf{x}_1|^{\tau+\theta}} \\
&= o(1)\frac{1}{(1+|y-\mathbf{x}_1|)^{N-2-\tau-(N-2)\beta-\theta}},
\end{aligned}
$$

and

$$
\sum_{i=1}^k\frac{1}{(1+|y-\mathbf{x}_i|)^{\frac{N-2}{2}+\tau}} \le \frac{C}{(1+|y-\mathbf{x}_1|)^{\frac{N-2}{2}}}.
$$

So, we obtain

$$\int_{\Omega_1} \frac{1}{(1+|z-\mathbf{x}_1|)^{N-2-\beta}} \left(\sum_{j=2}^{k} U_{\Lambda,\mathbf{x}_j}^{1-\beta}\right)^{\frac{4}{N-2}} \sum_{i=1}^{k} \frac{1}{(1+|y-\mathbf{x}_i|)^{\frac{N-2}{2}+\tau}}$$

$$= o(1) \int_{\Omega_1} \frac{1}{(1+|z-\mathbf{x}_1|)^{N+\frac{N+2}{2}-\frac{4(\tau+\theta)}{N-2}-4\beta}} = o(1),$$

since $\frac{N+2}{2} - \frac{4(\tau+\theta)}{N-2} - 4\beta > 0$, if $\beta > 0$ and $\theta > 0$ are small.

If $y \in \Omega_l$, $l \geq 2$, then

$$\sum_{j=2}^{k} U_{\Lambda,\mathbf{x}_j}^{1-\beta} \leq \frac{C}{(1+|y-\mathbf{x}_l|)^{N-2-\tau-(N-2)\beta}},$$

and

$$\sum_{i=1}^{k} \frac{1}{(1+|y-\mathbf{x}_i|)^{\frac{N-2}{2}+\tau}} \leq \frac{C}{(1+|y-\mathbf{x}_l|)^{\frac{N-2}{2}}}.$$

As a result,

$$\int_{\Omega_l} \frac{1}{(1+|z-\mathbf{x}_1|)^{N-2}} \left(\sum_{j=2}^{k} U_{\Lambda,\mathbf{x}_j}^{1-\beta}\right)^{\frac{4}{N-2}} \sum_{i=1}^{k} \frac{1}{(1+|y-\mathbf{x}_i|)^{\frac{N-2}{2}+\tau}}$$

$$\leq C \int_{\Omega_l} \frac{1}{(1+|z-\mathbf{x}_1|)^{N-2}} \frac{1}{(1+|y-\mathbf{x}_l|)^{4-4\beta-\frac{4\tau}{N-2}+\frac{N-2}{2}}}$$

$$\leq \frac{C}{|\mathbf{x}_l-\mathbf{x}_1|^{\frac{N+2}{2}-\frac{4\tau}{N-2}-\theta-4\beta}},$$

where $\theta > 0$ is a fixed small constant.

Since $\tau = \frac{1}{2}$ for $N \geq 4$, and $\tau < \frac{1}{2}$ for $N = 3$, we find that for $\theta > 0$ and $\beta > 0$ small, $\frac{N+2}{2} - \frac{4\tau}{N-2} - \theta - 4\beta > \tau$. Thus

$$\int_{\Omega_\varepsilon} \frac{1}{(1+|z-\mathbf{x}_1|)^{N-2}} \left(\sum_{j=2}^{k} U_{\Lambda,\mathbf{x}_j}^{1-\beta}\right)^{\frac{4}{N-2}} \sum_{i=1}^{k} \frac{1}{(1+|y-\mathbf{x}_i|)^{\frac{N-2}{2}+\tau}}$$

$$\leq o(1) + C \sum_{l=2}^{k} \frac{1}{|\mathbf{x}_l-\mathbf{x}_1|^{\frac{N+2}{2}-\frac{4\tau}{N-2}-\theta}} = o(1).$$

So, we have proved

$$\left|\left\langle U_{\Lambda,\mathbf{x}_1}^{2^*-2} \partial_l U_{\Lambda,\mathbf{x}_j} - W_{r,\Lambda}^{2^*-2} Y_1, \phi \right\rangle\right| = o(1)\|\phi\|_*.$$

But there is a constant $\bar{c} > 0$,

$$\left\langle \sum_{j=1}^{2}\sum_{i=1}^{k} Z_{i,j}, Y_{1,l} \right\rangle = \bar{c}\delta_{lj} + o(1).$$

Thus we obtain that

$$c_l = o(\|\phi\|_*) + O(\|h\|_{**}).$$

So,

$$\|\phi\|_* \le \left(o(1) + \|h_k\|_{**} + \frac{\sum_{j=1}^{k} \frac{1}{(1+|y-\mathbf{x}_j|)^{\frac{N-2}{2}+\tau+\theta}}}{\sum_{j=1}^{k} \frac{1}{(1+|y-\mathbf{x}_j|)^{\frac{N-2}{2}+\tau}}} \right). \tag{2.14}$$

Since $\|\phi\|_* = 1$, we obtain from (2.14) that there is $R > 0$, such that

$$\|\phi(y)\|_{B_R(\mathbf{x}_i)} \ge c_0 > 0, \tag{2.15}$$

for some i. But $\bar{\phi}(y) = \phi(y - \mathbf{x}_i)$ converges uniformly in any compact set of $\mathbb{R}^N_+$ to a solution u of

$$\Delta u + (2^* - 1)U_{\Lambda,0}^{2^*-2} u = 0 \tag{2.16}$$

for some $\Lambda \in [\delta, \delta^{-1}]$, and u is perpendicular to the kernel of (2.16). So, $u = 0$. This is a contradiction to (2.15). □

From Lemma 2.2, using the same argument as in the proof of [14, Proposition 4.1], we can prove the following result :

Proposition 2.3. *There exists $k_0 > 0$ and a constant $C > 0$, independent of k, such that for all $k \ge k_0$ and all $h \in L^\infty(\Omega_\varepsilon)$, problem* (2.5) *has a unique solution $\phi \equiv L_k(h)$. Besides,*

$$\|L_k(h)\|_* \le C\|h\|_{**}, \qquad |c_j| \le C\|h\|_{**}. \tag{2.17}$$

Moreover, the map $L_k(h)$ is C^1 with respect to Λ.

Now, we consider

$$\begin{cases} -\Delta\left(W_{r,\Lambda} + \phi\right) - \lambda\varepsilon^2(W_{r,\Lambda} + \phi) = \left(W_{r,\Lambda} + \phi - s\Phi_\varepsilon\right)_+^{2^*-1} \\ \qquad\qquad + \sum_{j=1}^{2}\sum_{i=1}^{k} c_j Z_{i,j}, \qquad \text{in } \Omega_\varepsilon, \\ \phi \in H_s, \\ \left\langle \sum_{i=1}^{k} Z_{i,j}, \phi \right\rangle = 0, \quad j = 1, 2. \end{cases} \tag{2.18}$$

We have:

Proposition 2.4. *There is an integer $k_0 > 0$, such that for each $k \geq k_0$, $r_1 \leq r \leq r_2$, $\delta \leq \Lambda \leq \delta^{-1}$, where δ is a fixed small constant,* (2.18) *has a unique solution ϕ, satisfying*

$$\|\phi\|_* \leq C(s\varepsilon^{\frac{N-2}{2}})^{\frac{1}{2}+\sigma} + C|\lambda|\varepsilon^{1+\sigma},$$

where $\sigma > 0$ is a fixed small constant. Moreover, $\Lambda \to \phi(\Lambda)$ is C^1.

Rewrite (2.18) as

$$\begin{cases} -\Delta\phi - \lambda\varepsilon^2\phi - (2^*-1)(W_{r,\Lambda} - s\Phi_\varepsilon)_+^{2^*-2}\phi = N(\phi) + l_k \\ \qquad\qquad\qquad + \sum\limits_{j=1}^{2}\sum\limits_{i=1}^{k} c_j Z_{i,j}, \quad \text{in } \Omega_\varepsilon, \\ \phi \in H_s, \\ \left\langle \sum\limits_{i=1}^{k} Z_{i,j}, \phi \right\rangle = 0, \ j = 1, 2, \end{cases} \tag{2.19}$$

where

$$\bar{N}(\phi) = \left(W_{r,\Lambda} - s\Phi_\varepsilon + \phi\right)_+^{2^*-1} - (W_{r,\Lambda} - s\Phi_\varepsilon)_+^{2^*-1} - (2^*-1)(W_{r,\Lambda} - s\Phi_\varepsilon)_+^{2^*-2}\phi,$$

and

$$l_k = \left(W_{r,\Lambda}^{2^*-1} - \sum_{j=1}^{k} U_{\Lambda,x_j}^{2^*-1}\right) + (W_{r,\Lambda} - s\Phi_\varepsilon)_+^{2^*-1} - W_{r,\Lambda}^{2^*-1}.$$

In order to use the contraction mapping theorem to prove that (2.19) is uniquely solvable in the set on which $\|\phi\|_*$ is small, we need to estimate $N(\phi)$ and l_k.

Lemma 2.5. *We have*

$$\|\bar{N}(\phi)\|_{**} \leq C\|\phi\|_*^{\min(2^*-1,2)}.$$

Proof. We have

$$|\bar{N}(\phi)| \leq \begin{cases} C|\phi|^{2^*-1}, & N \geq 6; \\ C\left(W_{r,\Lambda}^{\frac{6-N}{N-2}}\phi^2 + |\phi|^{2^*-1}\right), & N = 3, 4, 5. \end{cases}$$

Firstly, we consider $N \geq 6$. We have

$$\begin{aligned} |\bar{N}(\phi)| &\leq C\|\phi\|_*^{2^*-1}\left(\sum_{j=1}^{k} \frac{1}{(1+|y-\mathbf{x}_j|)^{\frac{N-2}{2}+\tau}}\right)^{2^*-1} \\ &\leq C\|\phi\|_*^{2^*-1}\sum_{j=1}^{k} \frac{1}{(1+|y-\mathbf{x}_j|)^{\frac{N+2}{2}+\tau}}\left(\sum_{j=1}^{k} \frac{1}{(1+|y-\mathbf{x}_j|)^{\tau}}\right)^{\frac{4}{N-2}} \\ &\leq C\|\phi\|_*^{2^*-1}\sum_{j=1}^{k} \frac{1}{(1+|y-\mathbf{x}_j|)^{\frac{N+2}{2}+\tau}}, \end{aligned} \tag{2.20}$$

where we use the inequality

$$\sum_{j=1}^{k} a_j b_j \le \Big(\sum_{j=1}^{k} a_j^p\Big)^{\frac{1}{p}}\Big(\sum_{j=1}^{k} b_j^q\Big)^{\frac{1}{q}}, \quad \frac{1}{p}+\frac{1}{q}=1, a_j, b_j \ge 0, j=1,\dots,k,$$

and

$$\sum_{j=1}^{k}\frac{1}{(1+|y-\mathbf{x}_j|)^{\tau}} \le C+\sum_{j=2}^{k}\frac{C}{|\mathbf{x}_1-\mathbf{x}_j|^{\tau}} \le C.$$

which follows from Lemma B.1.

For $N=3,4,5$, similarly to the case $N\ge 6$, we have

$$\begin{aligned}
&|\bar{N}(\phi)|\\
\le& C\|\phi\|_*^2\Big(\sum_{j=1}^{k}\frac{1}{(1+|y-\mathbf{x}_j|)^{N-2-\beta}}\Big)^{\frac{6-N}{N-2}}\Big(\sum_{j=1}^{k}\frac{1}{(1+|y-\mathbf{x}_j|)^{\frac{N-2}{2}+\tau}}\Big)^2\\
&+C\|\phi\|_*^{2^*-1}\sum_{j=1}^{k}\frac{1}{(1+|y-\mathbf{x}_j|)^{\frac{N+2}{2}+\tau}}\\
\le& C\|\phi\|_*^2\Big(\sum_{j=1}^{k}\frac{1}{(1+|y-\mathbf{x}_j|)^{\frac{N-2}{2}+\tau}}\Big)^{2^*-1}+C\|\phi\|_*^{2^*-1}\sum_{j=1}^{k}\frac{1}{(1+|y-\mathbf{x}_j|)^{\frac{N+2}{2}+\tau}}\\
\le& C\|\phi\|_*^2\sum_{j=1}^{k}\frac{1}{(1+|y-\mathbf{x}_j|)^{\frac{N+2}{2}+\tau}}.
\end{aligned} \tag{2.21}$$

□

Next, we estimate l_k.

Lemma 2.6. *We have*

$$\|l_k\|_{**} \le C\left(s\varepsilon^{\frac{N-2}{2}}\right)^{\frac{1}{2}+\sigma}+C|\lambda|\varepsilon^{1+\sigma},$$

where $\sigma>0$ is a fixed small constant.

Proof. Recall

$$\Omega_j=\left\{y=(y',y'')\in\Omega_\varepsilon:\left\langle\frac{y'}{|y'|},\frac{\mathbf{x}_j}{|\mathbf{x}_j|}\right\rangle\ge\cos\frac{\pi}{k}\right\}.$$

From the symmetry, we can assume that $y\in\Omega_1$. Then,

$$|y-\mathbf{x}_j|\ge|y-\mathbf{x}_1|, \quad \forall\, y\in\Omega_1.$$

Thus, for $y \in \Omega_1$, by Lemma A.1,

$$
\begin{aligned}
|l_k| \le & \frac{C}{(1+|y-\mathbf{x}_1|)^{4-\beta}} \sum_{j=2}^{k} \frac{1}{(1+|y-\mathbf{x}_j|)^{N-2-\beta}} \\
& + C\left(\sum_{j=2}^{k} \frac{1}{(1+|y-\mathbf{x}_j|)^{N-2-\beta}}\right)^{2^*-1} \\
& + C\sum_{j=1}^{k} \frac{1}{(1+|y-\mathbf{x}_j|)^{4-\beta}} \left(\varepsilon^{N-2} + \frac{|\lambda|\varepsilon^2}{(1+|y-\mathbf{x}_j|)^{N-4-\beta}}\right) \\
& + C W_{r,\Lambda}^{2^*-1-\frac{1}{2}-\frac{2\sigma}{N-2}} s^{\frac{1}{2}+\frac{2\sigma}{N-2}} \varepsilon^{\frac{N-2}{4}+\sigma}.
\end{aligned} \tag{2.22}
$$

Here, we have used the inequality: for any bounded $a > 0$ and $b > 0$, $\alpha \in (0, 1]$:

$$
|(a-b)_+^{2^*-1} - a^{2^*-1}| \le C a^{2^*-1-\alpha} b^{\alpha}.
$$

Let us estimate the first term of (2.22). Using Lemma B.2, we obtain

$$
\begin{aligned}
& \frac{1}{(1+|y-\mathbf{x}_1|)^{4-\beta}} \frac{1}{(1+|y-\mathbf{x}_j|)^{N-2-\beta}} \\
\le & C\left(\frac{1}{(1+|y-\mathbf{x}_1|)^{\frac{N+2}{2}+\tau}} + \frac{1}{(1+|y-\mathbf{x}_j|)^{\frac{N+2}{2}+\tau}}\right) \frac{1}{|\mathbf{x}_j-\mathbf{x}_1|^{\frac{N+2}{2}-\tau-2\beta}} \\
\le & C \frac{1}{(1+|y-\mathbf{x}_1|)^{\frac{N+2}{2}+\tau}} \frac{1}{|\mathbf{x}_j-\mathbf{x}_1|^{\frac{N+2}{2}-\tau-2\beta}}, \quad j > 1.
\end{aligned} \tag{2.23}
$$

Since $\frac{N+2}{2} - \tau - 2\beta > 1$, we find

$$
\begin{aligned}
& \frac{1}{(1+|y-\mathbf{x}_1|)^{4-\beta}} \sum_{j=2}^{k} \frac{1}{(1+|y-\mathbf{x}_j|)^{N-2-\beta}} \\
\le & C \frac{1}{(1+|y-\mathbf{x}_1|)^{\frac{N+2}{2}+\tau}} (k\varepsilon)^{\frac{N+2}{2}-\tau-2\beta} \\
\le & C\left(s\varepsilon^{\frac{N-2}{2}}\right)^{\frac{1}{2}+\sigma} \frac{1}{(1+|y-\mathbf{x}_1|)^{\frac{N+2}{2}+\tau}}.
\end{aligned} \tag{2.24}
$$

Here we have used

$$
(k\varepsilon)^{\frac{N+2}{2}-\tau-2\beta} = O\left(\left(s\varepsilon^{\frac{N-2}{2}}\right)^{\frac{1}{2}+\sigma}\right), \tag{2.25}
$$

for some small $\sigma > 0$.

In fact, if $s > 0$ is fixed (as in Theorem 1.4), then $k = \frac{1}{\sqrt{\varepsilon}}$ and $\tau = \frac{1}{2}$. As a result,

$$(k\varepsilon)^{\frac{N+2}{2}-\tau-2\beta} = O\left(\varepsilon^{\frac{N+2}{4}-\frac{\tau}{2}-\beta}\right) = O\left(\varepsilon^{\frac{N-2}{4}+\sigma}\right).$$

So, we obtain (2.25).

If $N \geq 7$, then $\tau = \frac{1}{2}$, and

$$s^{\frac{(2-2\theta)(N-4)}{(N-6)(N-2)}} \leq k \leq s^{\frac{(2-\theta)(N-4)}{(N-6)(N-2)}}. \tag{2.26}$$

But

$$(k\varepsilon)^{\frac{N+2}{2}-\tau-2\beta} = \left(\frac{s^{\frac{2}{N-2}}}{k}\right)^{\frac{N+2}{2}-\tau-2\beta} = \frac{s^{\frac{N+1-4\beta}{N-2}}}{k^{\frac{N+1-4\beta}{2}}}$$

and

$$\left(s\varepsilon^{\frac{N-2}{2}}\right)^{\frac{1}{2}+\sigma} = \left(\frac{s^2}{k^{N-2}}\right)^{\frac{1}{2}+\sigma}$$

Thus, we see that (2.25) is equivalent to

$$s^{\frac{3-4\beta}{N-2}-2\sigma} \leq Ck^{\frac{3}{2}-2\beta-(N-2)\sigma}. \tag{2.27}$$

Using (2.26), we find (2.27) holds.

For $N = 3$, $k = \frac{s}{\sqrt{\varepsilon}}$. Thus,

$$(k\varepsilon)^{\frac{5}{2}-\tau-2\beta} = (s\varepsilon^{\frac{1}{2}})^{\frac{5}{2}-\tau-2\beta} \leq C(s\varepsilon^{\frac{1}{2}})^{\frac{1}{2}+\sigma}.$$

So, we obtain (2.25).

Now, we estimate the second term of (2.22).

Using Lemma B.2 again, we find for $y \in \Omega_1$,

$$\begin{aligned}
\frac{1}{(1+|y-\mathbf{x}_j|)^{N-2-\beta}} &\leq \frac{1}{(1+|y-\mathbf{x}_1|)^{\frac{N-2-\beta}{2}}}\frac{1}{(1+|y-\mathbf{x}_j|)^{\frac{N-2-\beta}{2}}}\\
&\leq \frac{C}{|\mathbf{x}_j-\mathbf{x}_1|^{\frac{N-2}{2}-\beta-\frac{N-2}{N+2}\tau}}\left(\frac{1}{(1+|y-\mathbf{x}_1|)^{\frac{N-2}{2}+\frac{N-2}{N+2}\tau}}+\frac{1}{(1+|y-\mathbf{x}_j|)^{\frac{N-2}{2}+\frac{N-2}{N+2}\tau}}\right)\\
&\leq \frac{C}{|\mathbf{x}_j-\mathbf{x}_1|^{\frac{N-2}{2}-\beta-\frac{N-2}{N+2}\tau}}\frac{1}{(1+|y-\mathbf{x}_1|)^{\frac{N-2}{2}+\frac{N-2}{N+2}\tau}}.
\end{aligned} \tag{2.28}$$

Suppose that $N \geq 5$. Then $\frac{N-2}{2} - \beta - \frac{N-2}{N+2}\tau > 1$ since $\tau < 1$. Then (2.28) gives for $y \in \Omega_1$

$$\begin{aligned}
&\left(\sum_{j=2}^{k} \frac{1}{(1+|y-\mathbf{x}_j|)^{N-2-\beta}}\right)^{2^*-1} \\
\leq & C\,(k\varepsilon)^{\frac{N+2}{2}-\tau-(2^*-1)\beta} \frac{1}{(1+|y-\mathbf{x}_1|)^{\frac{N+2}{2}+\tau}} \qquad (2.29)\\
= & C\left(s\varepsilon^{\frac{N-2}{2}}\right)^{\frac{1}{2}+\sigma} \frac{1}{(1+|y-\mathbf{x}_1|)^{\frac{N+2}{2}+\tau}}.
\end{aligned}$$

If $N = 3, 4$, then (2.28) gives

$$\begin{aligned}
&\left(\sum_{j=2}^{k} \frac{1}{(1+|y-\mathbf{x}_j|)^{N-2-\beta}}\right)^{2^*-1} \\
\leq & C\left(k\varepsilon^{\frac{N-2}{2}-\frac{N-2}{N+2}\tau-\beta}\right)^{2^*-1} \frac{1}{(1+|y-\mathbf{x}_1|)^{\frac{N+2}{2}+\tau}} \qquad (2.30)\\
= & Ck^{\frac{N+2}{N-2}}\varepsilon^{\frac{N+2}{2}-\tau-(2^*-1)\beta} \frac{1}{(1+|y-\mathbf{x}_1|)^{\frac{N+2}{2}+\tau}}.
\end{aligned}$$

If $N = 4$, then

$$k^{\frac{N+2}{N-2}}\varepsilon^{\frac{N+2}{2}-\tau-(2^*-1)\beta} = k^3\varepsilon^{3-\frac{1}{2}-(2^*-1)\beta} \leq C\varepsilon^{1-(2^*-1)\beta} \leq C\varepsilon^{\frac{1}{2}+\sigma}.$$

Hence for $N = 4$,

$$\left(\sum_{j=2}^{k} \frac{1}{(1+|y-\mathbf{x}_j|)^2}\right)^{2^*-1} \leq \sum_{i=1}^{k} \frac{C\varepsilon^{\frac{N-2}{4}+\sigma}}{(1+|y-\mathbf{x}_i|)^{\frac{N+2}{2}+\tau}}.$$

For $N = 3$, we have

$$k^5\varepsilon^{\frac{5}{2}-\tau-(2^*-1)\beta} = k^{2\tau+2(2^*-1)\beta}s^{5-2\tau-2(2^*-1)\beta}.$$

But

$$(s\varepsilon^{\frac{1}{2}})^{\frac{1}{2}+\sigma} = \frac{s^{1+2\sigma}}{k^{\frac{1}{2}+\sigma}}.$$

So, $k^5\varepsilon^{\frac{5}{2}-\tau-(2^*-1)\beta} \le C(s\varepsilon^{\frac{1}{2}})^{\frac{1}{2}+\sigma}$ is equivalent to

$$k \le Cs^{-\frac{8-4\tau-4\sigma-4(2^*-1)\beta)}{1+4\tau+2\sigma+4(2^*-1)\beta)}} \tag{2.31}$$

Since $k \le Cs^{-\frac{2\tau}{1-2\tau}}$, we see that (2.31) is valid if

$$\frac{8-4\tau}{1+4\tau} > \frac{2\tau}{1-2\tau}.$$

Thus, if $\tau \in (0, \frac{4}{11})$, (2.31) holds. Hence for $N = 3$, we also have

$$\left(\sum_{j=2}^{k} \frac{1}{(1+|y-\mathbf{x}_j|)^2}\right)^{2^*-1} \le \sum_{i=1}^{k} \frac{C(s\varepsilon^{\frac{N-2}{2}})^{\frac{1}{2}+\sigma}}{(1+|y-\mathbf{x}_i|)^{\frac{N+2}{2}+\tau}}.$$

Note that for $y \in \Omega_1$,

$$W_{r,\Lambda}(y) \le \frac{C}{(1+|y-\mathbf{x}_1|)^{N-2-\tau-\beta}}.$$

We claim that

$$\left(\frac{N+2}{N-2} - \frac{1}{2} - \frac{2\sigma}{N-2}\right)(N-2-\tau) \ge \frac{N+2}{2} + \tau, \tag{2.32}$$

if $N \ge 3$.

In fact, (2.32) is equivalent to

$$\tau < \frac{4(N-2)}{3N+2},$$

which is true, since $\tau = \frac{1}{2}$ if $N \ge 4$, $\tau < \frac{4}{11}$ if $N = 3$.

Thus, we obtain

$$s^{\frac{1}{2}+\frac{2\sigma}{N-2}}\varepsilon^{\frac{N-2}{4}+\sigma} W_{r,\Lambda}^{\frac{N+2}{N-2}-\frac{1}{2}-\frac{2\sigma}{N-2}} \le Cs^{\frac{1}{2}+\frac{2\sigma}{N-2}}\varepsilon^{\frac{N-2}{4}+\sigma} \frac{C}{(1+|y-\mathbf{x}_j|)^{\frac{N+2}{2}+\tau}}.$$

Finally,

$$\begin{aligned}\sum_{j=1}^{k} \frac{1}{(1+|y-\mathbf{x}_j|)^4} \frac{|\lambda|\varepsilon^2}{(1+|y-\mathbf{x}_j|)^{N-4-\beta}} &= \sum_{j=1}^{k} \frac{|\lambda|\varepsilon^2}{(1+|y-\mathbf{x}_j|)^{N-\beta}} \\ &\le C|\lambda|\varepsilon^2 \sum_{j=1}^{k} \frac{1}{(1+|y-\mathbf{x}_j|)^{\frac{N+2}{2}+\tau}},\end{aligned}$$

and

$$\sum_{j=1}^{k}\frac{1}{(1+|y-\mathbf{x}_j|)^4}\varepsilon^{N-2}\le C\varepsilon^{N-2-\frac{N-6}{2}-\tau}\sum_{j=1}^{k}\frac{1}{(1+|y-\mathbf{x}_j|)^{\frac{N+2}{2}+\tau}}$$
$$=C\varepsilon^{\frac{N+2}{2}-\tau}\sum_{j=1}^{k}\frac{1}{(1+|y-\mathbf{x}_j|)^{\frac{N+2}{2}+\tau}}\le C(k\varepsilon)^{\frac{N+2}{2}-\tau}\sum_{j=1}^{k}\frac{1}{(1+|y-\mathbf{x}_j|)^{\frac{N+2}{2}+\tau}}$$
$$\le C(s\varepsilon^{\frac{N-2}{2}})^{\frac{1}{2}+\sigma}\sum_{j=1}^{k}\frac{1}{(1+|y-\mathbf{x}_j|)^{\frac{N+2}{2}+\tau}}.$$

Combining all the above estimates, we obtain the result. □

Now, we are ready to prove Proposition 2.4.

Proof of Proposition 2.4. Let us recall that

$$\varepsilon=\frac{s^{\frac{2}{N-2}}}{k^2}.$$

Let

$$E_N=\left\{u: u\in C(\Omega_\varepsilon),\ \|u\|_*\le\sqrt{s}\varepsilon^{\frac{N-2}{4}},\ \int_{\Omega_\varepsilon}\sum_{i=1}^{k}Z_{i,j}u=0,\ j=1,2\right\}$$

Then, (2.19) is equivalent to

$$\phi=A(\phi)=:L(\bar{N}(\phi))+L(l_k).$$

Now we prove that A is a contraction map from E_N to E_N. Using Lemma 2.5, we have

$$\begin{aligned}\|A\phi\|_*\le C\|\bar{N}(\phi)\|_{**}+C\|l_k\|_{**}&\le C\|\phi\|_*^{\min(2^*-1,2)}+C\|l_k\|_{**}\\&\le C(\sqrt{s}\varepsilon^{\frac{N-2}{4}})^{\min(2^*-1,2)}+C\|l_k\|_{**}\\&\le C(\sqrt{s}\varepsilon^{\frac{N-2}{4}})^{1+\sigma}+C\|l_k\|_{**}.\end{aligned}\tag{2.33}$$

Thus, by Lemma 2.6, we find that A maps E_N to E_N.

Next, we show that A is a contraction map.

$$\|A(\phi_1)-A(\phi_2)\|_*=\|L(\bar{N}(\phi_1))-L(\bar{N}(\phi_2))\|_*\le C\|\bar{N}(\phi_1)-\bar{N}(\phi_2)\|_{**}.$$

Using

$$|\bar{N}'(t)|\le\begin{cases}C|t|^{2^*-2}, & N\ge 6;\\ C\left(W^{\frac{6-N}{N-2}}|\phi|+|\phi|^{2^*-2}\right), & N=3,4,5,\end{cases}$$

we can prove that

$$\begin{aligned}\|A(\phi_1) - A(\phi_2)\|_* &\le C\|\bar N(\phi_1) - \bar N(\phi_2)\|_{**}\\ &\le C\left(\|\phi_1\|_*^{\min(1,2^*-2)} + \|\phi_2\|_*^{\min(1,2^*-2)}\right)\|\phi_1-\phi_2\|_*\\ &\le \frac{1}{2}\|\phi_1-\phi_2\|_*.\end{aligned}$$

Thus, A is a contraction map.

It follows from the contraction mapping theorem that there is a unique $\phi \in E_N$, such that

$$\phi = A(\phi).$$

Moreover, it follows from (2.33) that

$$\|\phi\|_* \le C(\sqrt{s}\varepsilon^{\frac{N-2}{4}})^{1+\sigma} + C\|l_k\|_{**}.$$

So, the estimate for $\|\phi\|_*$ follows from Lemma 2.6. □

3. Proof of the main results

Let

$$F(r,\Lambda) = I\left(W_{r,\Lambda} + \phi\right),$$

where ϕ is the function obtained in Proposition 2.4, and let

$$I(u) = \frac{1}{2}\int_{\Omega_\varepsilon}(|Du|^2 - \lambda\varepsilon^2u^2) - \frac{1}{2^*}\int_{\Omega_\varepsilon}(u - s\Phi_\varepsilon)_+^{2^*}.$$

Using the symmetry, we can check that if (r,Λ) is a critical point of $F(\Lambda)$, then $W_{r,\Lambda} + \phi$ is a solution of (1.3).

Proposition 3.1. *We have*

$$\begin{aligned}F(r,\Lambda) = k\Bigg(A_0 + \frac{A_2\bar\varphi(r)s\varepsilon^{\frac{N-2}{2}}}{\Lambda^{\frac{N-2}{2}}} - \frac{A_3k^{N-2}\varepsilon^{N-2}}{r^{N-2}\Lambda^{N-2}}\\ + O\left((s\varepsilon^{\frac{N-2}{2}})^{1+\sigma} + (k\varepsilon)^{(N-2)(1+\sigma)}\right)\Bigg),\quad N=3,4;\end{aligned}$$

and

$$\begin{aligned}F(r,\Lambda) = k\Bigg(A_0 - \frac{A_1\lambda\varepsilon^2}{\Lambda^2} + \frac{A_2\bar\varphi(r)s\varepsilon^{\frac{N-2}{2}}}{\Lambda^{(N-2)/2}} - \frac{A_3k^{N-2}\varepsilon^{N-2}}{r^{N-2}\Lambda^{N-2}}\\ +O\left(|\lambda|\varepsilon^{2+\sigma} + \left(s\varepsilon^{\frac{N-2}{2}}\right)^{1+\sigma} + (k\varepsilon)^{(N-2)(1+\sigma)}\right)\Bigg),\quad N\ge 5.\end{aligned}$$

where the constant $A_i > 0, i = 0,1,2$ are positive constants, which are given in Proposition A.3.

Proof. There is $t \in (0,1)$, such that

$$
\begin{aligned}
F(r,\Lambda) =& I(W_{r,\Lambda}) + \left\langle I'\left(W_{r,\Lambda}\right), \phi\right\rangle + \frac{1}{2} D^2 I\left(W_{r,\Lambda} + t\phi\right)(\phi,\phi)\\
=& I(W_{r,\Lambda}) - \int_{\Omega_\varepsilon} l_k \phi + \int_{\Omega_\varepsilon} \left(|D\phi|^2 + \varepsilon^2 \mu \phi^2 - (2^* - 1)\left(W_{r,\Lambda} - s\Phi_\varepsilon + t\phi\right)_+^{2^*-2} \phi^2\right)\\
=& I(W_{r,\Lambda}) - (2^* - 1) \int_{\Omega_\varepsilon} \left(\left(W_{r,\Lambda} - s\Phi_\varepsilon + t\phi\right)_+^{2^*-2} - \left(W_{r,\Lambda} - s\Phi_\varepsilon\right)_+^{2^*-2}\right)\phi^2\\
&+ \int_{\Omega_\varepsilon} N(\phi)\phi\\
=& I(W_{r,\Lambda}) - (2^* - 1) \int_{\Omega_\varepsilon} \left(\left(W_{r,\Lambda} - s\Phi_\varepsilon + t\phi\right)_+^{2^*-2} - \left(W_{r,\Lambda} - s\Phi_\varepsilon\right)_+^{2^*-2}\right)\phi^2\\
&+ O\left(\int_{\Omega_\varepsilon} |\bar{N}(\phi)||\phi|\right).
\end{aligned}
\tag{3.1}
$$

But

$$
\begin{aligned}
&\int_{\Omega_\varepsilon} |\bar{N}(\phi)||\phi|\\
\le& C\|\bar{N}(\phi)\|_{**}\|\phi\|_* \int_{\Omega_\varepsilon} \sum_{j=1}^k \frac{1}{(1+|y-\mathbf{x}_j|)^{\frac{N+2}{2}+\tau}} \sum_{i=1}^k \frac{1}{(1+|y-\mathbf{x}_i|)^{\frac{N-2}{2}+\tau}}.
\end{aligned}
\tag{3.2}
$$

Using Lemma B.2, we find

$$
\begin{aligned}
&\sum_{j=1}^k \frac{1}{(1+|y-\mathbf{x}_j|)^{\frac{N+2}{2}+\tau}} \sum_{i=1}^k \frac{1}{(1+|y-\mathbf{x}_i|)^{\frac{N-2}{2}+\tau}}\\
=& \sum_{j=1}^k \frac{1}{(1+|y-\mathbf{x}_j|)^{N+2\tau}} + \sum_{j=1}^k \sum_{i\ne j} \frac{1}{(1+|y-\mathbf{x}_j|)^{\frac{N+2}{2}+\tau}} \frac{1}{(1+|y-\mathbf{x}_i|)^{\frac{N-2}{2}+\tau}}\\
\le& \sum_{j=1}^k \frac{1}{(1+|y-\mathbf{x}_j|)^{N+2\tau}} + C\sum_{j=1}^k \frac{1}{(1+|y-\mathbf{x}_j|)^{N+\frac{1}{2}\tau}} \sum_{i=2}^k \frac{1}{|\mathbf{x}_i - \mathbf{x}_1|^{\frac{3}{2}\tau}}\\
\le& C\sum_{j=1}^k \frac{1}{(1+|y-\mathbf{x}_j|)^{N+\frac{1}{2}\tau}},
\end{aligned}
$$

Thus, we obtain

$$
\int_{\Omega_\varepsilon} |\bar{N}(\phi)||\phi| \le Ck\|\bar{N}(\phi)\|_{**}\|\phi\|_* \le Ck\|\phi\|_*^2 \le Ck\left(|\lambda|\varepsilon^{2+\sigma} + \left(s\varepsilon^{\frac{N-2}{2}}\right)^{1+\sigma}\right).
$$

Now

$$\begin{aligned}&\left(W_{r,\Lambda}-s\Phi_\varepsilon+t\phi\right)_+^{2^*-2}-\left(W_{r,\Lambda}-s\Phi_\varepsilon\right)_+^{2^*-2}\\&=\begin{cases}O\left(|\phi|^{2^*-2}\right), & N\ge 6;\\ O\left(W_{r,\Lambda}^{\frac{6-N}{N-2}}|\phi|+|\phi|^{2^*-2}\right), & N=3,4,5.\end{cases}\end{aligned}$$

Thus, we have

$$\begin{aligned}&\left|\int_{\Omega_\varepsilon}\left(\left(W_{r,\Lambda}-s\Phi_\varepsilon+t\phi\right)^{2^*-2}\right)-\left(\left(W_{r,\Lambda}-s\Phi_\varepsilon\right)^{2^*-2}\right)\phi^2\right|\\&\le C\|\phi\|_*^{2^*}\int_{\Omega_\varepsilon}\left(\sum_{j=1}^k\frac{1}{(1+|y-\mathbf{x}_j|)^{\frac{N-2}{2}+\tau}}\right)^{2^*},\end{aligned}$$

if $N\ge 6$. If $N=3,4,5$, noting that $N-2>\frac{N-2}{2}+\tau$, we obtain

$$\begin{aligned}&\left|\int_{\Omega_\varepsilon}\left(\left(W_{r,\Lambda}-s\Phi_\varepsilon+t\phi\right)^{2^*-2}\right)-\left(\left(W_{r,\Lambda}-s\Phi_\varepsilon\right)^{2^*-2}\right)\phi^2\right|\\&\le C\int_{\Omega_\varepsilon}W_{r,\Lambda}^{\frac{6-N}{N-2}}|\phi|^3+C\int_{\Omega_\varepsilon}|\phi|^{2^*}\le\|\phi\|_*^3\int_{\Omega_\varepsilon}\left(\sum_{j=1}^k\frac{1}{(1+|y-\mathbf{x}_j|)^{\frac{N-2}{2}+\tau}}\right)^{2^*}.\end{aligned}$$

Let $\bar\eta>0$ be small. Using Lemma B.2, if $y\in\Omega_1$, then

$$\begin{aligned}&\sum_{j=2}^k\frac{1}{(1+|y-\mathbf{x}_j|)^{\frac{N-2}{2}+\tau}}\\&\le\sum_{j=2}^k\frac{1}{(1+|y-\mathbf{x}_1|)^{\frac{N-2}{4}+\frac12\tau}}\frac{1}{(1+|y-\mathbf{x}_j|)^{\frac{N-2}{4}+\frac12\tau}}\\&\le C\frac{1}{(1+|y-\mathbf{x}_1|)^{\frac{N-2}{2}+\frac12\bar\eta}}\sum_{j=2}^k\frac{1}{|\mathbf{x}_j-\mathbf{x}_1|^{\tau-\frac12\bar\eta}}\le C\varepsilon^{-\bar\eta}\frac{1}{(1+|y-\mathbf{x}_1|)^{\frac{N-2}{2}+\frac12\bar\eta}}.\end{aligned}$$

As a result,

$$\left(\sum_{j=1}^k\frac{1}{(1+|y-\mathbf{x}_j|)^{\frac{N-2}{2}+\tau}}\right)^{2^*}\le C\varepsilon^{-2^*\bar\eta}\frac{1}{(1+|y-\mathbf{x}_1|)^{N+2^*\frac12\bar\eta}},\quad y\in\Omega_1.$$

Thus

$$\int_{\Omega_\varepsilon}\left(\sum_{j=1}^k\frac{1}{(1+|y-\mathbf{x}_j|)^{\frac{N-2}{2}+\tau}}\right)^{2^*}\le Ck\varepsilon^{-2^*\bar\eta}.$$

So, we have proved

$$\left|\int_{\Omega_\varepsilon}\left(\left(W_{r,\Lambda}-s\Phi_\varepsilon+t\phi\right)^{2^*-2}\right)-\left(\left(W_{r,\Lambda}-s\Phi_\varepsilon\right)^{2^*-2}\right)\phi^2\right|$$
$$\le Ck\varepsilon^{-2^*\bar\eta}\|\phi\|_*^{\min(3,2^*)}\le Ck\varepsilon^{-2^*\bar\eta}\left(|\lambda|\varepsilon^{1+\sigma}+\left(s\varepsilon^{\frac{N-2}{2}}\right)^{\frac12+\sigma}\right)^{\min(3,2^*)} \tag{3.3}$$
$$\le Ck\left(|\lambda|\varepsilon^{2+\sigma}+\left(s\varepsilon^{\frac{N-2}{2}}\right)^{1+\sigma}\right)$$

Combining (3.1), (3.2) and (3.3), we find

$$F(r,\Lambda)=I(W_{r,\Lambda})+kO\left(|\lambda|\varepsilon^{2+\sigma}+\left(s\varepsilon^{\frac{N-2}{2}}\right)^{1+\sigma}\right). \tag{3.4}$$

□

Proof of Theorems 1.4, 1.5 *and* 1.6. We just need to prove that $F(r,\Lambda)$ has a critical point.

Firstly, we consider the cases $N\neq 6$. It follows from (3.4) and Proposition A.3 that

$$F(r,\Lambda)=k\Bigg(A_0+\frac{A_2\bar\varphi(r)s\varepsilon^{\frac{N-2}{2}}}{\Lambda^{(N-2)/2}}-\frac{A_3k^{N-2}\varepsilon^{N-2}}{r^{N-2}\Lambda^{N-2}}$$
$$+O\left((k\varepsilon)^{(N-2)(1+\sigma)}+\left(s\varepsilon^{\frac{N-2}{2}}\right)^{1+\sigma}\right)\Bigg).$$

Let

$$\bar F(r,\Lambda)=\frac{A_2\bar\varphi(r)}{\Lambda^{(N-2)/2}}-\frac{A_3}{r^{N-2}\Lambda^{N-2}},\qquad (r,\Lambda)\in[r_1,r_2]\times[\delta,\delta^{-1}].$$

Then, $\bar F(r,\Lambda)$ has a maximum point at (r_0,Λ_0), where

$$\Lambda_0=\left(\frac{2A_3}{A_2r_0^{N-2}\bar\varphi(r_0)}\right)^{\frac{2}{N-2}},$$

and r_0 is a maximum point of $r^{\frac{N-2}{2}}\bar\varphi(r)=r^{\frac{N-2}{2}}\varphi_1(r,0)$. So, if $\delta>0$ is small, (r_0,Λ_0) is an interior point of $[r_1,r_2]\times[\delta,\delta^{-1}]$. Thus, if $k>0$ is large, $F(r,\Lambda)$ attains its maximum in the interior of $[r_1,r_2]\times[\delta,\delta^{-1}]$. As a result, $F(r,\Lambda)$ has a critical point in $[r_1,r_2]\times[\delta,\delta^{-1}]$.

If $N=6$, then

$$F(r,\Lambda)=k\Bigg(A_0+\frac{-\lambda A_1\varepsilon^2+A_2\bar\varphi(r)s\varepsilon^2}{\Lambda^2}-\frac{A_3k^4\varepsilon^4}{r^4\Lambda^4}$$
$$+O\left((k\varepsilon)^{4(1+\sigma)}+(s\varepsilon^2)^{1+\sigma}\right)\Bigg).$$

Let

$$\bar F(r,\Lambda)=\frac{-\lambda A_1 s^{-1}+A_2\bar\varphi(r)}{\Lambda^2}-\frac{A_3}{r^4\Lambda^4},\quad (r,\Lambda)\in[r_1,r_2]\times[\delta,\delta^{-1}].$$

It is easy to see that there is an $s_0>0$, such that if $s>|\lambda|s_0$, then

$$\tilde\varphi(r)=:r^{\frac{N-2}{2}}\Big(-\lambda A_1 s^{-1}+A_2\bar\varphi(r)\Big),\quad r\in[r_1,r_2]$$

has a maximum point $r_0\in(r_1.r_2)$ and $\tilde\varphi(r_0)>0$. Then, $\bar F(r,\Lambda)$ has a maximum point at (r_0,Λ_0), where

$$\Lambda_0=\left(\frac{2A_3}{r_0^4\tilde\varphi(r_0)}\right)^{\frac12}.$$

So, we can prove that $F(r,\Lambda)$ has a critical point in $[r_1,r_2]\times[\delta,\delta^{-1}]$. □

A. Appendix

In this section, we will expand $I(W_{r,\Lambda})$. We always assume that $d(\bar{\mathbf{x}}_j,\partial\Omega)\ge c_0>0$, where $\bar{\mathbf{x}}_j=\varepsilon\mathbf{x}_j$. Denote

$$\bar\varphi(r)=\varphi_1(r,0).$$

First, let us recall that $W_{\Lambda,\xi}$ is the solution of

$$\begin{cases}-\Delta W-\lambda\varepsilon^2W=U_{\Lambda,\xi}^{2^*-1} & \text{in }\ \Omega_\varepsilon,\\ W=0 & \text{on }\ \partial\Omega_\varepsilon.\end{cases}\tag{A.1}$$

Let

$$\psi_{\Lambda,\xi}=U_{\Lambda,\xi}-W_{\Lambda,\xi}.$$

Then,

$$\begin{cases}-\Delta\psi_{\Lambda,\xi}-\lambda\varepsilon^2\psi_{\Lambda,\xi}=-\lambda\varepsilon^2U_{\Lambda,\xi} & \text{in }\ \Omega_\varepsilon,\\ \psi_{\Lambda,\xi}=U_{\Lambda,\xi}, & \text{on }\ \partial\Omega_\varepsilon.\end{cases}\tag{A.2}$$

To calculate $I(W_{r,\Lambda})$, we need to estimate $\psi_{\Lambda,\xi}$.

Decompose $\psi_{\Lambda,\xi}$ as follows

$$\psi_{\Lambda,\xi}=\psi_{\Lambda,\xi,1}+\psi_{\Lambda,\xi,2},$$

where $\psi_{\Lambda,\xi,1}$ is the solution of

$$\begin{cases}-\Delta\psi_{\Lambda,\xi,1}-\lambda\varepsilon^2\psi_{\Lambda,\xi,1}=-\lambda\varepsilon^2U_{\Lambda,\xi} & \text{in }\ \Omega_\varepsilon,\\ \psi_{\Lambda,\xi}=0, & \text{on }\ \partial\Omega_\varepsilon,\end{cases}\tag{A.3}$$

and $\psi_{\Lambda,\xi,2}$ is the solution of

$$\begin{cases} -\Delta\psi_{\Lambda,\xi,2} - \lambda\varepsilon^2\psi_{\Lambda,\xi,2} = 0, & \text{in} \quad \Omega_\varepsilon, \\ \psi_{\Lambda,\xi} = U_{\Lambda,\xi}, & \text{on} \quad \partial\Omega_\varepsilon. \end{cases} \tag{A.4}$$

Since

$$U_{\Lambda,\xi} \le C\varepsilon^{N-2}, \quad \text{on } \partial\Omega_\varepsilon,$$

it is easy to see that

$$|\psi_{\Lambda,\xi,2}| \le C\varepsilon^{N-2}. \tag{A.5}$$

Let $\bar{\psi}_{\Lambda,\xi,\varepsilon}$ be the solution of

$$\begin{cases} -\Delta\psi - \lambda\varepsilon^2\psi = U_{\Lambda,\xi} & \text{in} \quad \Omega_\varepsilon, \\ \psi = 0, & \text{on} \quad \partial\Omega_\varepsilon, \end{cases} \tag{A.6}$$

Then, we can check that

$$|\bar{\psi}_{\Lambda,\xi,\varepsilon}(y)| \le \frac{C\ln^m(2+|y-\xi|)}{(1+|y-\xi|)^{N-4}}, \tag{A.7}$$

where $m = 1$ if $N = 4$, otherwise, $m = 0$. Thus, we have

Lemma A.1. *We have*

$$\psi_{\Lambda,\xi} = -\lambda\varepsilon^2\bar{\psi}_{\lambda,\xi,\varepsilon} + O(\varepsilon^{N-2}).$$

where $\bar{\psi}_{\lambda,\xi,\varepsilon}$ is the solution of (A.6). *Moreover,*

$$|W_{\Lambda,\xi}| \le C|\ln\varepsilon|^m U_{\Lambda,\xi},$$

where $m = 1$ if $N = 4$, otherwise, $m = 0$.

Proof. We only need to show

$$|W_{\Lambda,\xi}| \le C|\ln\varepsilon|^m U_{\Lambda,\xi},$$

which follows from (A.7) and $\varepsilon \le \frac{C}{1+|y-\xi|}$. □

Proposition A.2. *We have*

$$I\left(W_{\Lambda,\mathbf{x}_j}\right) = A_0 + \frac{A_2\bar{\varphi}(r)s\varepsilon^{\frac{N-2}{2}}}{\Lambda^{\frac{N-2}{2}}} + O\left((s\varepsilon^{\frac{N-2}{2}})^{1+\sigma}\right), \quad N = 3, 4,$$

and

$$I\left(W_{\Lambda,\mathbf{x}_j}\right) = A_0 - \frac{A_1\lambda\varepsilon^2}{\Lambda^2} + \frac{A_2\bar{\varphi}(r)s\varepsilon^{\frac{N-2}{2}}}{\Lambda^{\frac{N-2}{2}}} + O\left(|\lambda|\varepsilon^{2+\sigma} + \left(s\varepsilon^{\frac{N-2}{2}}\right)^{1+\sigma}\right), \quad N \ge 5;$$

where

$$A_0 = \frac{1}{2}\int_{\mathbb{R}^N} |DU|^2 - \frac{1}{2^*}\int_{\mathbb{R}^N} U^{2^*}, \quad A_2 = \int_{\mathbb{R}^N} U^{2^*-1},$$

$$A_1 = \frac{1}{2}\int_{\mathbb{R}^N} U^2, \quad N \geq 5,$$

and σ is some positive constant.

Proof. Write

$$I(u) = \tilde{I}(u) - \frac{1}{2^*}\int_{\Omega_\varepsilon} \left((u - s\Phi_\varepsilon)_+^{2^*} - |u|^{2^*} \right),$$

where

$$\tilde{I}(u) = \frac{1}{2}\int_{\Omega_\varepsilon} |Du|^2 - \frac{1}{2}\lambda\varepsilon^2 \int_{\Omega_\varepsilon} u^2 - \frac{1}{2^*}\int_{\Omega_\varepsilon} |u|^{2^*}.$$

By Lemma A.1, we have

$$\begin{aligned} \tilde{I}\left(W_{\Lambda,\mathbf{x}_j}\right) &= \frac{1}{2}\int_{\Omega_\varepsilon} U_{\Lambda,\mathbf{x}_j}^{2^*-1} W_{\Lambda,\mathbf{x}_j} - \frac{1}{2^*}\int_{\Omega_\varepsilon} W_{\Lambda,\mathbf{x}_j}^{2^*} \\ &= A_0 + \frac{1}{2}\int_{\Omega_\varepsilon} U_{\Lambda,\mathbf{x}_j}^{2^*-1}\psi_{\Lambda,\mathbf{x}_j} + O\left(\int_{\Omega_\varepsilon} U_{\Lambda,\mathbf{x}_j}^{2^*-1-\sigma}\psi_{\Lambda,\mathbf{x}_j}^{1+\sigma}\right) \\ &= A_0 + \frac{1}{2}\int_{\Omega_\varepsilon} U_{\Lambda,\mathbf{x}_j}^{2^*-1}\psi_{\Lambda,\mathbf{x}_j} + O\left(|\lambda|\varepsilon^{2(1+\sigma)} + \varepsilon^{(N-2)(1+\sigma)}\right). \end{aligned} \tag{A.8}$$

On the other hand,

$$\begin{aligned} &\int_{\Omega_\varepsilon} \left(W_{\Lambda,\mathbf{x}_j} - s\Phi_\varepsilon\right)_+^{2^*} - \int_{\Omega_\varepsilon} (W_{\Lambda,\mathbf{x}_j})^{2^*} \\ &= -2^*\int_{R^N} U^{2^*-1} s\varepsilon^{\frac{N-2}{2}}\Lambda_j^{-\frac{N-2}{2}}\bar{\varphi}(r) + O\left((s\varepsilon^{\frac{N-2}{2}})^{1+\sigma}\right). \end{aligned} \tag{A.9}$$

For $N = 3, 4$, by Lemma A.1 and (A.7),

$$\int_{\Omega_\varepsilon} U_{\Lambda,\mathbf{x}_j}^{2^*-1}\psi_{\Lambda,\mathbf{x}_j} = O(\varepsilon^{N-2} + \varepsilon^2) = O\left((s\varepsilon^{\frac{N-2}{2}})^{1+\sigma}\right). \tag{A.10}$$

Here, we have used $\varepsilon = \frac{s^2}{k^2} = \frac{1}{k}s\sqrt{\varepsilon} = (s\sqrt{\varepsilon})^{1+\sigma}$ if $N = 3$. So, the result for $N = 3, 4$ follows from (A.8)–(A.10).

Suppose that $N \geq 5$. Let $\bar{\psi}_{\Lambda,\xi}$ be the solution of

$$\begin{cases} -\Delta\psi = U_{\Lambda,\xi} & \text{in} \quad R^N, \\ \psi(|y|) \to 0, & \text{as } |y| \to +\infty. \end{cases} \tag{A.11}$$

Then,

$$|\bar{\psi}_{\Lambda,\xi}| \le \frac{C}{(1+|y-\xi|)^{N-4}},$$

and

$$|\bar{\psi}_{\Lambda,\xi} - \bar{\psi}_{\Lambda,\xi,\varepsilon}| \le \frac{C\varepsilon^2 \ln^m(2+|y-\xi|)}{(1+|y-\xi|)^{N-6}},$$

where $m = 1$ if $N = 6$, otherwise, $m = 0$. Thus,

$$\begin{aligned}\int_{\Omega_\varepsilon} U_{\Lambda,\mathbf{x}_j}^{2^*-1}\psi_{\Lambda,\mathbf{x}_j} &= -\lambda\varepsilon^2 \int_{\mathbb{R}^N} U_{\Lambda,\mathbf{x}_j}^{2^*-1}\bar{\psi}_{\Lambda,\mathbf{x}_j} + O\left(\varepsilon^{N-2} + |\lambda|\varepsilon^4|\ln\varepsilon|\right)\\ &= -\lambda\varepsilon^2\int_{\mathbb{R}^N} U^2 + O\left(\varepsilon^{N-2} + |\lambda|\varepsilon^4|\ln\varepsilon|\right).\end{aligned} \tag{A.12}$$

So we obtain the result for $N \ge 5$. □

Proposition A.3. *We have*

$$\begin{aligned}I(W_{r,\Lambda}) = k\Bigg(A_0 &+ \frac{A_2\bar{\varphi}(r)s\varepsilon^{\frac{N-2}{2}}}{\Lambda^{\frac{N-2}{2}}} - \frac{A_3k^{N-2}\varepsilon^{N-2}}{r\Lambda^{N-2}}\\ &+ O\left((k\varepsilon)^{(N-2)(1+\sigma)} + (s\varepsilon^{\frac{N-2}{2}})^{1+\sigma}\right)\Bigg), \quad N = 3, 4;\end{aligned}$$

and

$$\begin{aligned}I\left(W_{r,\lambda}\right) = k\Bigg(A_0 &- \frac{A_1\lambda\varepsilon^2}{\Lambda^2} + \frac{A_2\bar{\varphi}(r)s\varepsilon^{\frac{N-2}{2}}}{\Lambda^{(N-2)/2}} - \frac{A_3k^{N-2}\varepsilon^{N-2}}{r^{N-2}\Lambda^{N-2}}\\ &+ O\left((k\varepsilon)^{(N-2)(1+\sigma)} + |\lambda|\varepsilon^{2+\sigma} + \left(s\varepsilon^{\frac{N-2}{2}}\right)^{1+\sigma}\right)\Bigg), \quad N \ge 5.\end{aligned}$$

Proof. By using the symmetry, we have

$$\begin{aligned}\int_{\Omega_\varepsilon} |DW_{r,\Lambda}|^2 - \lambda\varepsilon^2\int_{\Omega_\varepsilon} W_{r,\Lambda}^2 &= \sum_{j=1}^k\sum_{i=1}^k \int_{\Omega_\varepsilon} U_{\Lambda,\mathbf{x}_i}^{2^*-1}W_{\Lambda,\mathbf{x}_j}\\ =&k\left(\int_{\Omega_\varepsilon} U_{\Lambda,\mathbf{x}_1}^{2^*} + \int_{\Omega_\varepsilon} U_{\Lambda,\mathbf{x}_1}^{2^*-1}\psi_{\Lambda,\mathbf{x}_1} + \sum_{i=2}^k\int_{\Omega_\varepsilon} U_{\Lambda,\mathbf{x}_1}^{2^*-1}U_{\Lambda,\mathbf{x}_i}\right.\\ &\left.+O\left(\sum_{i=2}^k \frac{1}{|\mathbf{x}_i-\mathbf{x}_1|^{N-2+\sigma}}\right)\right)\\ =&k\left(\int_{\mathbb{R}^N} U^{2^*} + \int_{\Omega_\varepsilon} U_{\Lambda,\mathbf{x}_1}^{2^*-1}\psi_{\Lambda,\mathbf{x}_1} + \sum_{i=2}^k \frac{B_0}{\Lambda^{N-2}|\mathbf{x}_i-\mathbf{x}_1|^{N-2}}\right.\\ &\left.+O\left(\sum_{i=2}^k \frac{1}{|\mathbf{x}_i-\mathbf{x}_1|^{N-2+\sigma}}\right)\right),\end{aligned} \tag{A.13}$$

where $B_0 > 0$ is a constant.

Let

$$\Omega_j = \left\{ y = (y', y'') \in \Omega_\varepsilon : \left\langle \frac{y'}{|y'|}, \frac{\mathbf{x}_j}{|\mathbf{x}_j|} \right\rangle \geq \cos\frac{\pi}{k} \right\}.$$

Then,

$$|y - \mathbf{x}_i| \geq |y - \mathbf{x}_j|, \qquad \forall y \in \Omega_j.$$

We have

$$\begin{aligned}
\frac{1}{2^*}\int_{\Omega_\varepsilon} \left(W_{r,\Lambda} - s\Phi_\varepsilon\right)_+^{2^*} &= \frac{k}{2^*}\int_{\Omega_1} \left(W_{r,\Lambda} - s\Phi_\varepsilon\right)_+^{2^*} \\
&= \frac{k}{2^*}\Bigg(\int_{\Omega_1} \left(W_{\Lambda,\mathbf{x}_1} - s\Phi_\varepsilon\right)_+^{2^*} + 2^*\int_{\Omega_1}\sum_{i=2}^k \left(W_{\Lambda,\mathbf{x}_1} - s\Phi_\varepsilon\right)_+^{2^*-1} W_{\Lambda,\mathbf{x}_i} \\
&\quad + O\Bigg(\int_{\Omega_1} W_{\Lambda,\mathbf{x}_1}^{2^*-2}\Bigg(\sum_{i=2}^k W_{\Lambda,\mathbf{x}_i}\Bigg)^2\Bigg)\Bigg) \\
&= \frac{k}{2^*}\Bigg(\int_{\mathbb{R}^N} U^{2^*} - 2^*\int_{\Omega_\varepsilon} U_{\Lambda,\mathbf{x}_1}^{2^*-1}\psi_{\Lambda,\mathbf{x}_1} - \frac{2^* A_2\bar{\varphi}(r)s\varepsilon^{\frac{N-2}{2}}}{\Lambda^{(N-2)/2}} \\
&\quad + 2^*\int_{\Omega_1}\sum_{i=2}^k U_{\Lambda,\mathbf{x}_1}^{2^*-1} U_{\Lambda,\mathbf{x}_i} + O\Bigg(\int_{\Omega_1} U_{\Lambda,\mathbf{x}_1}^{2^*-2} s\Phi_\varepsilon \sum_{i=2}^k U_{\Lambda,\mathbf{x}_i} \\
&\quad + \int_{\Omega_1} U_{\Lambda,\mathbf{x}_1}^{2^*-2}\Bigg(\sum_{i=2}^k U_{\Lambda,\mathbf{x}_i}\Bigg)^2 + (s\varepsilon^{\frac{N-2}{2}})^{1+\sigma} + |\lambda|\varepsilon^{2+\sigma}\Bigg)\Bigg) \\
&= \frac{k}{2^*}\Bigg(\int_{\mathbb{R}^N} U^{2^*} - 2^*\int_{\Omega_\varepsilon} U_{\Lambda,\mathbf{x}_1}^{2^*-1}\psi_{\Lambda,\mathbf{x}_1} - \frac{2^* A_2\bar{\varphi}(r)s\varepsilon^{\frac{N-2}{2}}}{\Lambda^{\frac{N-2}{2}}} \\
&\quad + \sum_{i=2}^k \frac{2^* B_0}{\Lambda^{N-2}|\mathbf{x}_i - \mathbf{x}_1|^{N-2}} \\
&\quad + O\left((k\varepsilon)^{(N-2)(1+\sigma)} + (s\varepsilon^{\frac{N-2}{2}})^{1+\sigma} + |\lambda|\varepsilon^{2+\sigma}\right)\Bigg).
\end{aligned} \tag{A.14}$$

Since

$$|\mathbf{x}_j - \mathbf{x}_1| = 2|\mathbf{x}_1|\sin\frac{2(j-1)\pi}{k}, \quad j = 2, \dots, k,$$

we can prove

$$\sum_{j=2}^k \frac{1}{|\mathbf{x}_j - \mathbf{x}_1|^{N-2}} = B_4(\varepsilon k)^{N-2} + O\left((k\varepsilon)^{(1+\sigma)(N-2)}\right). \tag{A.15}$$

Thus, the result follows from (A.13), (A.14) and (A.15). □

B. Appendix

Firstly, we gives a few lemmas, whose proof can be found in [35, 37].

Lemma B.1. *For any* $\alpha > 0$,

$$\sum_{j=1}^{k} \frac{1}{(1+|y-\mathbf{x}_j|)^{\alpha}} \le C\left(1+\sum_{j=2}^{k}\frac{1}{|\mathbf{x}_1-\mathbf{x}_j|^{\alpha}}\right),$$

where $C > 0$ *is a constant, independent of* k.

For each fixed i and j, $i \neq j$, consider the following function

$$g_{ij}(y) = \frac{1}{(1+|y-\mathbf{x}_j|)^{\alpha}}\frac{1}{(1+|y-\mathbf{x}_i|)^{\beta}}, \tag{B.1}$$

where $\alpha \ge 1$ and $\beta \ge 1$ are two constants. Then, we have

Lemma B.2. *For any constant* $0 \le \sigma \le \min(\alpha, \beta)$*, there is a constant* $C > 0$*, such that*

$$g_{ij}(y) \le \frac{C}{|\mathbf{x}_i-\mathbf{x}_j|^{\sigma}}\left(\frac{1}{(1+|y-\mathbf{x}_i|)^{\alpha+\beta-\sigma}}+\frac{1}{(1+|y-\mathbf{x}_j|)^{\alpha+\beta-\sigma}}\right).$$

Lemma B.3. *For any constant* $0 < \sigma < N-2$*, there is a constant* $C > 0$*, such that*

$$\int_{\mathbb{R}^N} \frac{1}{|y-z|^{N-2}}\frac{1}{(1+|z|)^{2+\sigma}}\,dz \le \frac{C}{(1+|y|)^{\sigma}}.$$

Let us recall that

$$\varepsilon = \frac{s^{\frac{2}{N-2}}}{k^2}.$$

For the constant $\tau \in (0,1)$ defined is (2.4),

$$\sum_{j=2}^{k}\frac{1}{|\mathbf{x}_j-\mathbf{x}_1|^{\tau}} \le C\varepsilon^{\tau}k^{\tau}\sum_{j=2}^{k}\frac{1}{j^{\tau}} \le C\varepsilon^{\tau}k \le C,$$

and for any $\theta > 0$,

$$\sum_{j=2}^{k}\frac{1}{|\mathbf{x}_j-\mathbf{x}_1|^{\tau+\theta}} = o(1).$$

Lemma B.4. *Suppose that $N \geq 4$. There is a small $\theta > 0$, such that*

$$\begin{aligned}\int_{\mathbb{R}^N} & \frac{1}{|y-z|^{N-2}} W_{r,\Lambda}^{\frac{4}{N-2}}(z) \sum_{j=1}^{k} \frac{1}{(1+|z-\mathbf{x}_j|)^{\frac{N-2}{2}+\tau}}\, dz \\ \leq & C \sum_{j=1}^{k} \frac{1}{(1+|y-\mathbf{x}_j|)^{\frac{N-2}{2}+\tau+\theta}},\end{aligned}$$

where $W_{r,\Lambda}$ is defined in (1.7).

Proof. Recall that

$$\Omega_j = \left\{ y = (y', y'') \in \Omega_\varepsilon : \left\langle \frac{y'}{|y'|}, \frac{\mathbf{x}_j}{|\mathbf{x}_j|} \right\rangle \geq \cos \frac{\pi}{k} \right\}.$$

For $z \in \Omega_1$, we have $|z-\mathbf{x}_j| \geq |z-\mathbf{x}_1|$. Using Lemma B.2, we obtain

$$\begin{aligned}\sum_{j=2}^{k} \frac{1}{(1+|z-\mathbf{x}_j|)^{N-2-\beta}} &\leq \frac{1}{(1+|z-\mathbf{x}_1|)^{\frac{1}{2}(N-2-\beta)}} \sum_{j=2}^{k} \frac{1}{(1+|z-\mathbf{x}_j|)^{\frac{1}{2}(N-2-\beta)}} \\ &\leq \frac{C}{(1+|z-\mathbf{x}_1|)^{N-2-\beta-\tau}} \sum_{j=2}^{k} \frac{1}{|\mathbf{x}_j-\mathbf{x}_1|^{\tau}} \\ &\leq \frac{C}{(1+|z-\mathbf{x}_1|)^{N-2-\beta-\tau}},\end{aligned}$$

Thus,

$$W_{r,\Lambda}^{\frac{4}{N-2}}(z) \leq \frac{C}{(1+|z-\mathbf{x}_1|)^{4-\frac{4(\tau+\beta)}{N-2}}}.$$

As a result, for $z \in \Omega_1$, using Lemma B.2 again, we find that for $\theta > 0$ small,

$$W_{r,\Lambda}^{\frac{4}{N-2}}(z) \sum_{j=1}^{k} \frac{1}{(1+|z-\mathbf{x}_j|)^{\frac{N-2}{2}+\tau}} \leq \frac{C}{(1+|z-\mathbf{x}_1|)^{2+\frac{N-2}{2}+\tau+2-\tau-\frac{4(\tau+\beta)}{N-2}}}.$$

Since $\theta =: 2-\tau-\frac{4(\tau+\beta)}{N-2} > 0$ if $N \geq 4$ and $\beta > 0$ is small, we obtain

$$\begin{aligned}\int_{\Omega_1} & \frac{1}{|y-z|^{N-2}} W_{r,\Lambda}^{\frac{4}{N-2}}(z) \sum_{j=1}^{k} \frac{1}{(1+|z-\mathbf{x}_j|)^{\frac{N-2}{2}+\tau}}\, dz \\ \leq & \int_{\Omega_1} \frac{1}{|y-z|^{N-2}} \frac{C}{(1+|z-\mathbf{x}_1|)^{2+\frac{N-2}{2}+\tau+\theta}}\, dz \leq \frac{C}{(1+|y-\mathbf{x}_1|)^{\frac{N-2}{2}+\tau+\theta}},\end{aligned}$$

which gives

$$\begin{aligned}
&\int_{\Omega_\varepsilon} \frac{1}{|y-z|^{N-2}} W_{r,\Lambda}^{\frac{4}{N-2}}(z) \sum_{j=1}^{k} \frac{1}{(1+|z-\mathbf{x}_j|)^{\frac{N-2}{2}+\tau}}\, dz \\
&= \sum_{i=1}^{k} \int_{\Omega_i} \frac{1}{|y-z|^{N-2}} W_{r,\Lambda}^{\frac{4}{N-2}}(z) \sum_{j=1}^{k} \frac{1}{(1+|z-\mathbf{x}_j|)^{\frac{N-2}{2}+\tau}}\, dz \\
&\le \sum_{i=1}^{k} \frac{C}{(1+|y-\mathbf{x}_i|)^{\frac{N-2}{2}+\tau+\theta}}.
\end{aligned}$$

□

The above proof does not work for $N = 3$ because

$$2-\tau-\frac{4\tau}{N-2} < 0 \tag{B.2}$$

if $N = 3$ and $\tau = \frac{1}{2}$. The choice of $\tau \in (0, 1)$ should ensure

$$\sum_{j=2}^{k} \frac{1}{|\mathbf{x}_j-\mathbf{x}_1|^\tau} \le C\varepsilon^\tau k \le C.$$

The above relation shows that τ can be chosen smaller if ε becomes smaller, which in turn will make $2-\tau-\frac{4\tau}{N-2} > 0$. Noting that $\varepsilon = \frac{s^2}{k^2}$, we find that if $s \to 0+$, then $\varepsilon = o(\frac{1}{k^2})$. We have

Lemma B.5. *Suppose that $N = 3$, the parameter $s > 0$ and the integer k satisfy*

$$s \le Ck^{-\frac{1}{2\tau}+1},$$

for some $\tau \in (0, \frac{2}{5})$. Then, there is a small $\theta > 0$, such that

$$\begin{aligned}
&\int_{\mathbb{R}^3} \frac{1}{|y-z|} W_{r,\Lambda}^{4}(z) \sum_{j=1}^{k} \frac{1}{(1+|z-\mathbf{x}_j|)^{\frac{1}{2}+\tau}}\, dz \\
&\le C \sum_{j=1}^{k} \frac{1}{(1+|y-\mathbf{x}_j|)^{\frac{1}{2}+\tau+\theta}}.
\end{aligned}$$

Proof. The proof of this lemma is similar to that of Lemma B.4. We only need to use that for $\tau < \frac{2}{5}$,

$$2-5\tau > 0,$$

and

$$\varepsilon^\tau k = s^{2\tau} k^{1-2\tau} \le C.$$

Thus, we omit the details. □

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