Ann. Scuola Norm. Sup. Pisa Cl. Sci. (5)
Vol. VII (2008), 171-180

On the existence of steady-state solutions to the Navier-Stokes system for large fluxes

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Abstract. In this paper we deal with the stationary Navier-Stokes problem in a domain Ω with compact Lipschitz boundary $\partial\Omega$ and datum $\boldsymbol{a}$ in Lebesgue spaces. We prove existence of a solution for arbitrary values of the fluxes through the connected components of $\partial\Omega$, with possible countable exceptional set, provided $\boldsymbol{a}$ is the sum of the gradient of a harmonic function and a sufficiently small field, with zero total flux for Ω bounded.

Mathematics Subject Classification (2000): 76D05 (primary); 31B10, 35Q30, 42B20 (secondary).

1. Introduction

The boundary value problem associated with the Navier-Stokes equations is to find a solution to the system

$$\begin{aligned} \nu\Delta\boldsymbol{u} - \boldsymbol{u}\cdot\nabla\boldsymbol{u} &= \nabla p \quad \text{in } \Omega, && (1.1)\\ \operatorname{div}\boldsymbol{u} &= 0 \quad \text{in } \Omega, && (1.2)\\ \boldsymbol{u} &= \boldsymbol{a} \quad \text{on } \partial\Omega, && (1.3) \end{aligned}$$

where Ω is a bounded domain (open connected set) of $\mathbb{R}^n$ $(n = 2, 3)$, $\boldsymbol{u}$, p the unknown kinetic and pressure fields, ν the kinematical viscosity and $\boldsymbol{a}$ the boundary datum which must satisfy the condition

$$\int_{\partial\Omega} \boldsymbol{a}\cdot\boldsymbol{n} = 0,$$

where $\boldsymbol{n}$ is the outward unit normal to $\partial\Omega$ (see [6]). Existence of a variational solution

$$(\boldsymbol{u}, p) \in [W^{1,2}_{\sigma}(\Omega)\cap C^{\infty}(\Omega)] \times [L^2(\Omega)\cap C^{\infty}(\Omega)] \qquad (1.4)$$

Received January 29, 2007; accepted in revised form December 13, 2007.

to system (1.1)-(1.3) is known under the hypothesis of smallness of the fluxes

$$\Phi_i = \int_{\partial\Omega_i} \boldsymbol{a} \cdot \boldsymbol{n}$$

through the connected components $\partial\Omega_i$ of $\partial\Omega$, provided $\boldsymbol{a} \in W^{1/2,2}(\partial\Omega)$ and $\partial\Omega$ is Lipschitz. Precisely, in [1, 5] (see also [6, Chapter VIII]) it is proved that there is a positive constant depending on Ω such that if $\sum |\Phi_i|$ is suitably small, then (1.1)-(1.3) has a variational solution. These results have been extended for $\boldsymbol{a}$ in Lebesgue's spaces in [10]. In particular, it is proved that if $\boldsymbol{a} \in L^q(\partial\Omega)$ ($q \geq 8/3$ for $n = 3$ and $q = 2$ for $n = 2$), then system (1.1)-(1.3) has a C^∞ solution which for $q > 4$ takes the boundary datum in the sense of nontangential convergence. Moreover, making use of some regularity results in [2, 12], it is showed that if $\boldsymbol{a}$ is more regular (say Hölder continuous, with Hölder's coefficient depending on the Lipschitz character of $\partial\Omega$) then so does $\boldsymbol{u}$.

In [3] H. Fujita and H. Morimoto considered problem (1.1)-(1.3) in a regular domain with

$$\boldsymbol{a} = \mu \boldsymbol{u}_{0|\partial\Omega} + \boldsymbol{\gamma}, \tag{1.5}$$

$\mu \in \mathbb{R}$, $\boldsymbol{u}_0 = \nabla\beta$ and β harmonic function. They proved that if Ω is regular, $\beta_{|\partial\Omega} \in W^{2,2}(\partial\Omega)$,

$$\int_{\partial\Omega} \boldsymbol{\gamma} \cdot \boldsymbol{n} = 0 \tag{1.6}$$

and $\|\boldsymbol{\gamma}\|_{W^{1/2,2}(\partial\Omega)}$ is less than a suitable constant depending on ν, μ, Ω and $\boldsymbol{u}_0$, then system (1.1)-(1.3), (1.5) admits a solution (1.4) for any $\mu \in \mathbb{R}\backslash G$, with G countable subset of $\mathbb{R}$. Moreover, if $\beta \in W^{3,2}(\Omega)$ and $\|\boldsymbol{\gamma}\|_{W^{3/2,2}(\partial\Omega)}$ is sufficiently small, then $(\boldsymbol{u}, p) \in W^{2,2}_\sigma(\Omega) \times W^{1,2}(\Omega)$. This result is remarkable in view of the fact that, even though for special boundary data, it assures the existence of a solution to system (1.1)-(1.3) in arbitrary bounded regular domains for large fluxes. It is worth to mention that for the annulus $\{x \in \mathbb{R}^2 : R_1 < |x| < R_2\}$ and $\beta = \nabla \log |x|$, H. Morimoto proved that $G = \varnothing$ [8] (see also [4, 9]).

The aim of the present paper is twofold:

(i) to extend the results of [3] under more general assumptions on the domains and on the data and for any $n \geq 2$; in particular, for $n = 3$ we show that, if Ω is a bounded Lipschitz domain, $\boldsymbol{a}$ is given by (1.5) with $\boldsymbol{u}_0 \in W^{1,q}(\Omega)$, $q > 3/2$, $\boldsymbol{\gamma} \in L^2(\partial\Omega)$ satisfies (1.6) and $\|\boldsymbol{\gamma}\|_{L^2(\partial\Omega)}$ is sufficiently small, then system (1.1)-(1.3) has a solution $(\boldsymbol{u}, p) \in [W^{1/2,2}_\sigma(\Omega) \cap C^\infty(\Omega)] \times C^\infty(\Omega)$ for any $\mu \in \mathbb{R} \setminus G$, with G countable subset of $\mathbb{R}$;

(ii) to prove existence of a solution for system (1.1)-(1.3), (1.5) in a Lispschitz exterior domain in the class $[L^\infty_\sigma(\Omega, r) \cap C^\infty(\Omega)] \times [L^\infty(\complement S_R, r^2 \log r) \cap C^\infty(\Omega)]$, provided $\boldsymbol{u}_0 \in L^\infty(\Omega, r^2)$ and $\|\boldsymbol{\gamma}\|_{L^\infty(\partial\Omega)}$ is sufficiently small.

NOTATION – We use a standard vector notation, as in [6]. Let Ω be a domain of $\mathbb{R}^n$, $n \geq 2$, and let $\{\gamma(\xi)\}_{\xi\in\partial\Omega}$ be a family of circular finite (not empty) cones with vertex at ξ such that $\gamma(\xi) \setminus \{\xi\} \subset \Omega$ (as well-known, if Ω is Lipschitz, such a family of cones certainly exists). Let χ be a function in Ω; $\chi(x)$ is said to converge nontangentially at the boundary if $\chi(\xi) = \lim_{x\to\xi\,(x\in\gamma(\xi))} \chi(x) \Leftrightarrow \chi(x) \xrightarrow{\text{nt}} \chi(\xi)$, for almost all $\xi \in \partial\Omega$. As customary, $L^q(\Omega)$, $W^{s,q}(\Omega)$ and $L^q(\partial\Omega)$, $W^{s,q}(\partial\Omega)$ ($q \in [1, +\infty]$, $s \geq 0$) denote respectively the Lebesgue and the Sobolev-Besov spaces of (scalar, vector and tensor) fields in Ω and $\partial\Omega$ endowed with their natural norms; $W^{-s,q}(\partial\Omega)$ is the dual space of $W^{s,q'}(\partial\Omega)$ and $L^\infty(\Omega, f(r))$, with $f(r)$ positive function of $r = |x|$, is the Banach space of all measurable fields χ in Ω such that $\|f(r)\chi\|_{L^\infty(\Omega)} < +\infty$; if $V(\subset L^1_{\text{loc}}(\Omega))$ is a function space, V_σ stands for the subspace of V of all (weakly) divergence free vector fields; also, the subscript ϕ in the symbol W_ϕ, where $W \subset W^{s,q}(\partial\Omega)$, $s \in \mathbb{R}$, denotes the set of all fields $\boldsymbol{a} \in W$ such that $\int_{\partial\Omega} \boldsymbol{a}\cdot\boldsymbol{n}\, d\sigma = 0$ or $\langle \boldsymbol{a}, \boldsymbol{n}\rangle = 0$ respectively for $s \geq 0$ or $s < 0$, where $\langle\cdot,\cdot\rangle$ stands for the duality pairing between $W^{s,q}(\partial\Omega)$ and its dual.

2. Some results for the Stokes system

The boundary-value problem associated with the Stokes system is to find a solution to the problem

$$\begin{aligned} \nu\Delta\boldsymbol{u} &= \nabla p && \text{in } \Omega,\\ \operatorname{div}\boldsymbol{u} &= 0 && \text{in } \Omega,\\ \boldsymbol{u} &= \boldsymbol{a} && \text{on } \partial\Omega. \end{aligned} \tag{2.1}$$

If Ω is exterior we require that $\boldsymbol{u}$ tends to zero at infinity.

The following theorems are proved in [2, 7, 10, 12].

Theorem 2.1. *Let Ω be a Lipschitz bounded domain of $\mathbb{R}^n$ and let $\boldsymbol{a} \in L^q_\phi(\partial\Omega)$, $q \geq 2$. There exists a positive constant ϵ depending on Ω such that if $q \in [2, 2+\epsilon)$, then system (2.1), admits a solution $(\boldsymbol{u}, p) \in [W^{1/q,q}_\sigma(\Omega) \cap C^\infty(\Omega)] \times C^\infty(\Omega)$, $\boldsymbol{u} \xrightarrow{\text{nt}} \boldsymbol{a}$ and*

$$\|\boldsymbol{u}\|_{W^{1/q,q}(\Omega)} \leq c\|\boldsymbol{a}\|_{L^q(\partial\Omega)}. \tag{2.2}$$

Moreover:

- (i) *if $\boldsymbol{a} \in W^{1,q}(\partial\Omega)$, $q \in [2, 2+\epsilon)$, then $\boldsymbol{u} \in W^{1+1/q,q}(\Omega)$.*
- *For $n = 3$ there are two positive constants ϵ and α_0, depending on the Lipschitz character of $\partial\Omega$ such that*:

  (ii) *if $\boldsymbol{a} \in L^q(\partial\Omega)$, $q \in [2, +\infty]$, then $\boldsymbol{u} \in W^{1/q,q}(\Omega)$ and (2.2) holds;*

  (iii) *if $\boldsymbol{a} \in W^{1-1/q,q}(\partial\Omega)$, $q \in [2, 3+\epsilon)$, then*

$$\|\boldsymbol{u}\|_{W^{1,q}(\Omega)} \leq c\|\boldsymbol{a}\|_{W^{1-1/q,q}(\partial\Omega)}; \tag{2.3}$$

  (iv) *if $\boldsymbol{a} \in C^{0,\alpha}(\partial\Omega)$, $\alpha \in [0, \alpha_0)$, then*

$$\|\boldsymbol{u}\|_{C^{0,\alpha}(\overline{\Omega})} \leq c\|\boldsymbol{a}\|_{C^{0,\alpha}(\partial\Omega)}.$$

- *If $n = 4$ and $\boldsymbol{a} \in L^3(\partial\Omega)$, then*

$$\|\boldsymbol{u}\|_{L^4(\Omega)} \le c\|\boldsymbol{a}\|_{L^3(\partial\Omega)}. \tag{2.4}$$

If Ω is of class C^1, then properties (i)-(iv) *are satisfied for all $n \ge 2$ with $q \in (1, +\infty)$, $\epsilon = +\infty$ and $\alpha_0 = 1$. In particular, if $n \ge 5$ and $\boldsymbol{a} \in L^{n-1}(\partial\Omega)$, then*

$$\|\boldsymbol{u}\|_{L^n(\Omega)} \le c\|\boldsymbol{a}\|_{L^{n-1}(\partial\Omega)}. \tag{2.5}$$

Theorem 2.2. *Let Ω be a bounded domain of $\mathbb{R}^n$ of class $C^{2,1}$ and let $\boldsymbol{a} \in W_\phi^{-1/q,q}(\partial\Omega)$, with $q \in (1, +\infty)$. Then system* (2.1), *admits a solution $(\boldsymbol{u}, p) \in C_\sigma^\infty(\Omega) \times C^\infty(\Omega)$ such that $\boldsymbol{u}$ takes the boundary value $\boldsymbol{a}$ in the sense of the space $W^{-1/q,q}(\partial\Omega)$*[1] *and*

$$\|\boldsymbol{u}\|_{L^q(\Omega)} \le c\|\boldsymbol{a}\|_{W^{-1/q,q}(\partial\Omega)}. \tag{2.6}$$

Theorem 2.3. *Let Ω be an exterior domain of $\mathbb{R}^3$. If $\boldsymbol{a} \in L^\infty(\partial\Omega)$, then system* (2.1) *admits a solution $(\boldsymbol{u}, p) \in [L_\sigma^\infty(\Omega, r) \cap C^\infty(\Omega)] \times [L^\infty(\complement S_R, r^2) \cap C^\infty(\Omega)]$, $\boldsymbol{u} \xrightarrow{\text{nt}} \boldsymbol{a}$ and*

$$\|\boldsymbol{u}\|_{L^\infty(\Omega,r)} \le c\|\boldsymbol{a}\|_{L^\infty(\partial\Omega)}. \tag{2.7}$$

Moreover, property (iv) *in Theorem* 2.1 *holds unchanged.*

3. Existence theorems for the Navier-Stokes system

Thanks to the results just recalled concerning the Stokes problem (2.1), we are in a position to extend the existence results of Fujita-Morimoto [3] to data in $L_\phi^q(\partial\Omega)$ for Lipschitz domain and in $W_\phi^{-1/q,q}(\partial\Omega)$ for domains of class $C^{2,1}$, for suitable q. Moreover, we also prove an existence theorem in Lipschitz exterior domains with data in $L^\infty(\partial\Omega)$.

Theorem 3.1. *Let Ω be a Lipschitz bounded domain of $\mathbb{R}^3$ and let*

$$\boldsymbol{a} = \mu\boldsymbol{u}_{0|\partial\Omega} + \boldsymbol{\gamma}, \tag{3.1}$$

where $\mu \in \mathbb{R}$, $\boldsymbol{u}_0 = \nabla\beta$, with $\beta \in W^{2,q}(\Omega)$ $(q > 3/2)$ harmonic function, and $\boldsymbol{\gamma} \in L_\phi^2(\partial\Omega)$. Then, for every $\mu/\nu \in \mathbb{R} \setminus G$, with G countable subset of $\mathbb{R}$, there exists a constant $\kappa = \kappa(\Omega, \nu, \boldsymbol{u}_0, \mu)$ such that, if

$$\|\boldsymbol{\gamma}\|_{L^2(\partial\Omega)} \le \kappa$$

then system (1.1)-(1.3) *has a solution*

$$(\boldsymbol{u}, p) \in [W_\sigma^{1/2,2}(\Omega) \cap C^\infty(\Omega)] \times C^\infty(\Omega).$$

[1] See [10].

Proof. Recall that the fundamental solution to the Stokes equation is expressed by

$$\mathcal{U}(x-y)=\frac{1}{8\pi\nu}\left[\frac{\mathbf{1}}{|x-y|}+\frac{(x-y)\otimes(x-y)}{|x-y|^3}\right],$$
$$\mathcal{P}(x-y)=\frac{1}{4\pi}\frac{x-y}{|x-y|^3},$$

where $\mathbf{1}$ denotes the unit second-order tensor.

Let $\boldsymbol{u}\in L^3_\sigma(\Omega)$. By classical results the linear operator

$$\hat{\mathcal{L}}[\boldsymbol{u}](x)=-\int_\Omega \mathcal{U}(x-y)[\boldsymbol{u}_0\cdot\nabla\boldsymbol{u}+\boldsymbol{u}\cdot\nabla\boldsymbol{u}_0](y)$$

maps $L^3_\sigma(\Omega)$ into $W^{1,t}(\Omega)$ for some $t>3/2$. Let $\mathcal{L}_0[\boldsymbol{u}]$ be the solution to the Stokes problem with boundary datum $-\operatorname{tr}\hat{\mathcal{L}}[\boldsymbol{u}]\in W^{1-1/t,t}(\partial\Omega)$. Since by (2.3) and the trace theorem

$$\|\mathcal{L}_0[\boldsymbol{u}]\|_{W^{1,t}(\Omega)}\le c\|\operatorname{tr}\hat{\mathcal{L}}[\boldsymbol{u}]\|_{W^{1-1/t,t}(\partial\Omega)}\le\|\hat{\mathcal{L}}[\boldsymbol{u}]\|_{W^{1,t}(\Omega)}$$

we see that also the linear operator $\mathcal{L}_0$ maps $L^3_\sigma(\Omega)$ into $W^{1,t}(\Omega)$. Therefore, by Rellich's compactness theorem, the operator

$$\mathcal{L}=\hat{\mathcal{L}}+\mathcal{L}_0$$

maps compactly $L^3_\sigma(\Omega)$ into itself and $\operatorname{tr}\mathcal{L}[\boldsymbol{u}]=\mathbf{0}$ on $\partial\Omega$.

Set
$$\mathcal{F}=\mathcal{I}-\frac{\mu}{\nu}\mathcal{L},$$

where $\mathcal{I}$ denotes the identity map. By classical results there is a countable subset G of $\mathbb{R}$, with a possible accumulation at 0, such that $\mathcal{F}$ is invertible for all $\mu/\nu\notin G$.

The nonlinear operator

$$\hat{\mathcal{N}}[\boldsymbol{u}](x)=-\frac{1}{\nu}\int_\Omega\mathcal{U}(x-y)(\boldsymbol{u}\cdot\nabla\boldsymbol{u})(y)$$

maps $L^3_\sigma(\Omega)$ into $W^{1,3/2}_\sigma(\Omega)$ and it holds

$$\left\|\hat{\mathcal{N}}[\boldsymbol{u}]\right\|_{W^{1,3/2}(\Omega)}\le c\|\boldsymbol{u}\|^2_{L^3(\Omega)}. \tag{3.2}$$

Let $\mathcal{N}_0[\boldsymbol{u}]$ be the solution to the Stokes problem with boundary datum $-\operatorname{tr}\hat{\mathcal{N}}[\boldsymbol{u}]$. By the trace theorem and (3.2) we have

$$\left\|\mathcal{N}_0[\boldsymbol{u}]\right\|_{L^3(\Omega)}\le c\|\operatorname{tr}\hat{\mathcal{N}}[\boldsymbol{u}]\|_{W^{1/3,3/2}(\partial\Omega)}\le\left\|\hat{\mathcal{N}}[\boldsymbol{u}]\right\|_{W^{1,3/2}(\Omega)}\le c\|\boldsymbol{u}\|^2_{L^3(\Omega)}.$$

Of course, the operator

$$\mathcal{N} = \hat{\mathcal{N}} + \mathcal{N}_0$$

maps $L^3_\sigma(\Omega)$ into itself and $\operatorname{tr}\mathcal{N}[\boldsymbol{u}] = \mathbf{0}$ on $\partial\Omega$.

For $\mu/\nu \notin G$ consider the map

$$\boldsymbol{u}' = \overset{-1}{\mathcal{F}}\big[\boldsymbol{u}_\gamma + \mathcal{N}[\boldsymbol{u}]\big] \tag{3.3}$$

from $L^3_\sigma(\Omega)$ into itself, where $\boldsymbol{u}_\gamma$ is the solution to the Stokes problem with boundary datum $\boldsymbol{\gamma}$. Since

$$\big\|\overset{-1}{\mathcal{F}}\big[\mathcal{N}[\boldsymbol{u}]\big]\big\|_{L^3(\Omega)} \le c_0\|\boldsymbol{u}\|^2_{L^3(\Omega)},$$

taking into account (2.2), if $\|\boldsymbol{\gamma}\|_{L^2(\partial\Omega)}$ is chosen such that

$$\big\|\overset{-1}{\mathcal{F}}[\boldsymbol{u}_\gamma]\big\|_{L^3(\Omega)} < \frac{1}{4c_0},$$

then (3.3) is a contraction in the ball

$$\mathcal{S} = \left\{\boldsymbol{u} \in L^3_\sigma(\Omega) : \|\boldsymbol{u}\|_{L^3(\Omega)} \le \frac{1}{2c_0}\right\}.$$

Therefore, by a classical theorem of S. Banach, there is a unique field $\boldsymbol{u} \in \mathcal{S}$ such that

$$\boldsymbol{u} = \overset{-1}{\mathcal{F}}\big[\boldsymbol{u}_\gamma + \mathcal{N}[\boldsymbol{u}]\big].$$

Hence it follows that $\boldsymbol{u}$ is a solution to the equation

$$\boldsymbol{u} = \boldsymbol{u}_\gamma + \frac{\mu}{\nu}\mathcal{L}[\boldsymbol{u}] + \mathcal{N}[\boldsymbol{u}].$$

Since $\boldsymbol{u}_0$ is a solution to both Stokes and Navier-Stokes equations, by taking also into account standard regularity theory we see that the field $\mu\boldsymbol{u}_0 + \boldsymbol{u} \in C^\infty(\Omega)$ is a solution to equations (1.1)-(1.2) for a suitable pressure field $p \in C^\infty(\Omega)$. This solution assumes the boundary datum in the sense that $\boldsymbol{u}_0 \xrightarrow{\text{nt}} \boldsymbol{u}_{0|\partial\Omega}$, $\boldsymbol{u}_\gamma \xrightarrow{\text{nt}} \boldsymbol{\gamma}$ and $\mathcal{N}[\boldsymbol{u}]$, $\mathcal{L}[\boldsymbol{u}]$ have zero trace on $\partial\Omega$ as elements of the Sobolev space $W^{1,3/2}_0(\Omega)$ (see also Remark 3.2). $\square$

Remark 3.2. Assume for simplicity that β is a regular harmonic function. By the regularity results for the Stokes problem we have, in particular, that if the norm of $\boldsymbol{\gamma}$ is small in the corresponding function space, then

- if $\boldsymbol{\gamma} \in L^q(\partial\Omega)$, $q > 4$, and $\boldsymbol{u}_0 \in W^{1,t}(\Omega)$, $t > 3$, then $\boldsymbol{u} \xrightarrow{\text{nt}} \boldsymbol{a}$;
- if $\boldsymbol{\gamma} \in L^\infty(\partial\Omega)$, then $\boldsymbol{u} \in L^\infty(\Omega)$

and there are two positive constants ϵ and $\alpha_0(< 1)$ depending on Ω such that

- if $\boldsymbol{\gamma} \in L^s(\partial\Omega)$, $s \in [2, 2+\epsilon)$, then $\boldsymbol{u} \in W^{1/s,s}(\Omega)$,
- if $\boldsymbol{\gamma} \in W^{1-1/s,s}(\partial\Omega)$, $s \in [3/2, 3+\epsilon)$, then $\boldsymbol{u} \in W^{1,s}(\Omega)$,
- if $\boldsymbol{\gamma} \in C^{0,\alpha}(\partial\Omega)$, $\alpha \in [0, \alpha_0)$, then $\boldsymbol{u} \in C^{0,\alpha}(\overline{\Omega})$.

If Ω is of class C^1, then the above constants ϵ and α_0 can be taken arbitrarily large and equal to 1 respectively. Standard regularity results also hold for the pressure field p.

Remark 3.3. In virtue of Theorem 2.1 and estimates (2.3), (2.4), (2.5), existence theorems like the above one can also be established for all $n \geq 2$. If $n = 2$ we can take $\boldsymbol{a} \in L^{2-\epsilon}(\partial\Omega)$ with ϵ depending on Ω ($\boldsymbol{a} \in L^q(\partial\Omega)$, $q > 1$, for Ω of class C^1); if $n = 4$, we have to require $\boldsymbol{a} \in L^3(\partial\Omega)$; if $n > 4$, Ω must be of class C^1 and $\boldsymbol{a} \in L^{n-1}(\partial\Omega)$.

Taking into account Theorem 2.2 and estimate (2.2), following the argument in the proof of the above Theorem it is not difficult to get:

Theorem 3.4. *Let Ω be a bounded domain of $\mathbb{R}^n$ of class $C^{2,1}$ and let $\boldsymbol{a}$ be given by* (3.1) *with $\mu \in \mathbb{R}$, $\boldsymbol{u}_0 = \nabla\beta$, $\beta \in W^{2,q}(\Omega)$ ($q > n$) harmonic function and $\boldsymbol{\gamma} \in W_\phi^{-1/n,n}(\partial\Omega)$. There is a countable subset $G \subset \mathbb{R}$ such that, for $\mu/\nu \notin G$, there is a constant $\kappa = \kappa(\Omega, \nu, \boldsymbol{u}_0, \mu)$ such that, if*

$$\|\boldsymbol{\gamma}\|_{W^{-1/n,n}(\partial\Omega)} \leq \kappa$$

then (1.1)-(1.3) *has a solution*

$$(\boldsymbol{u}, p) \in [L^n_\sigma(\Omega) \cap C^\infty(\Omega)] \times C^\infty(\Omega).$$

Taking into account the result of H. Morimoto recalled in the introduction, we also have:

Theorem 3.5. *Let Ω be the annulus*

$$\Omega = \{x \in \mathbb{R}^2 : R_1 < |x| < R_2\},$$

and let $\boldsymbol{a}$ be given by (3.1)*, with $\boldsymbol{u}_0 \in W^{1,q}(\Omega)$, $q > 1$, $\boldsymbol{\gamma} \in L^q_\phi(\Omega)$. If $\|\boldsymbol{\gamma}\|_{L^q(\partial\Omega)}$ is sufficiently small, then* (1.1)-(1.3) *has a solution*

$$(\boldsymbol{u}, p) \in [W^{1/q,q}_\sigma(\Omega) \cap C^\infty(\Omega)] \times C^\infty(\Omega).$$

Let us pass to treat the case of an exterior domain. The following theorem holds.

Theorem 3.6. *Let Ω be an exterior Lipschitz domain of $\mathbb{R}^3$ and let $\boldsymbol{a}$ be expressed by* (3.1) *where $\boldsymbol{u}_{0|\partial\Omega}$, $\boldsymbol{\gamma} \in L^\infty(\partial\Omega)$ and $\boldsymbol{u}_0 = O(r^{-2})$. There is a countable subset $G \subset \mathbb{R}$ such that, for $\mu/\nu \notin G$, there is a constant $\xi = \xi(\Omega, \nu, \boldsymbol{u}_0, \mu)$ such that, if $\|\boldsymbol{\gamma}\|_{L^\infty(\partial\Omega)} \leq \xi$, then the Navier-Stokes problem admits a solution*

$$(\boldsymbol{u}, p) \in [L^\infty_\sigma(\Omega, r) \cap C^\infty(\Omega)] \times [L^\infty(\complement S_R, r^2 \log r) \cap C^\infty(\Omega)]$$

and $\boldsymbol{u} \xrightarrow{\text{nt}} \boldsymbol{a}$.

Proof. By well-known results about the behavior at infinity of volume potential (see, *e.g.*, [6] Lemma II.7.2), the operator

$$\hat{\mathcal{V}}[\boldsymbol{u}](x) = -\nabla \int_\Omega \mathcal{U}(x-y)(\boldsymbol{u} \otimes \boldsymbol{u}_0 + \boldsymbol{u}_0 \otimes \boldsymbol{u})(y)$$

maps boundedly $L^\infty(\Omega, r)$ into $L^\infty(\Omega, r^{2-\eta})$ for every $\eta \in (0,1)$. Hence $\hat{\mathcal{V}}[\boldsymbol{u}] \in L^q(\Omega)$, for every $q > 3/2$. Moreover, by Calderón-Zygmund's theorem $\nabla\hat{\mathcal{V}}$ maps boundedly $L^\infty(\Omega, r)$ into $L^q(\Omega)$, for every $q > 3/2$.

Let $\{\boldsymbol{u}_k\}_{k\in\mathbb{N}}$ be a bounded sequence in $L^\infty_\sigma(\Omega, r)$. By what we said above $\hat{\mathcal{V}}[\boldsymbol{u}_k]$ is bounded in $W^{1,q}(\Omega)$ for every $q > 3/2$ so that we can extract from it a subsequence, we denote by the same symbol, which converges uniformly to a field $\boldsymbol{u}$ in $C_{\mathrm{loc}}(\overline{\Omega})$. On the other hand

$$\begin{aligned}
&\big\|\hat{\mathcal{V}}[\boldsymbol{u}_k - \boldsymbol{u}_h]\big\|_{L^\infty(\Omega,r)} \\
&\quad \le R\big\|\hat{\mathcal{V}}[\boldsymbol{u}_k - \boldsymbol{u}_h]\big\|_{C(\overline{\Omega}\cap S_R)} + R^{\eta-1}\big\|r^{2-\eta}\hat{\mathcal{V}}[\boldsymbol{u}_k - \boldsymbol{u}_h]\big\|_{L^\infty(\complement S_R)} \\
&\quad \le R\big\|\hat{\mathcal{V}}[\boldsymbol{u}_k - \boldsymbol{u}_h]\big\|_{C(\overline{\Omega}\cap S_R)} + cR^{\eta-1}.
\end{aligned}$$

Let $\epsilon > 0$ and let m be such that for every $h, k > m$, $\big\|\hat{\mathcal{V}}[\boldsymbol{u}_k - \boldsymbol{u}_h]\big\|_{C(\overline{\Omega}\cap S_R)} < \epsilon/(2R)$, with $R^{1-\eta} > 2c/\epsilon$. Therefore, from the above relation it follows that $\|\hat{\mathcal{V}}[\boldsymbol{u}_k - \boldsymbol{u}_h]\|_{L^\infty(\Omega,r)} < \epsilon$ for all $h, k > m$ so that $\hat{\mathcal{V}}[\boldsymbol{u}_k]$ is a Cauchy sequence in $L^\infty_\sigma(\Omega, r)$ and the operator $\hat{\mathcal{V}}$ is compact from $L^\infty_\sigma(\Omega, r)$ into itself. Let $\mathcal{V}_0[\boldsymbol{u}]$ be the solution to the Stokes problem with boundary datum $-\operatorname{tr}\hat{\mathcal{V}}[\boldsymbol{u}]$. It is not difficult to see that $\mathcal{V}_0$ maps compactly $L^\infty(\Omega, r)$ into itself. Set

$$\mathcal{G} = \mathcal{I} - \frac{\mu}{\nu}\mathcal{V}$$

with $\mathcal{V} = \hat{\mathcal{V}} + \mathcal{V}_0$. Since $\mathcal{G}$ is compact, there is a countable subset $G \subset \mathbb{R}$ such that $\mathcal{G}$ is invertible for all $\mu/\nu \notin G$.

The operators

$$\hat{\mathcal{W}}[\boldsymbol{u}] = -\frac{1}{\nu}\nabla \int_\Omega \mathcal{U}(x-y)(\boldsymbol{u} \otimes \boldsymbol{u})(y)$$

and $\mathcal{W}_0[\boldsymbol{u}]$, solution to the Stokes problem with datum $-\operatorname{tr}\hat{\mathcal{W}}[\boldsymbol{u}]$, map $L^\infty(\Omega, r)$ into itself. Consider the map

$$\boldsymbol{u}' = \overset{-1}{\mathcal{G}}\big[\boldsymbol{u}_\gamma + \mathcal{W}[\boldsymbol{u}]\big] \tag{3.4}$$

for $\mu \notin G$, where $\mathcal{W} = \hat{\mathcal{W}} + \mathcal{W}_0$ and $\boldsymbol{u}_\gamma$ is the solution to the Stokes problem with boundary datum $\boldsymbol{\gamma}$. It is not difficult to see that

$$\big\|\overset{-1}{\mathcal{G}}\big[\mathcal{W}[\boldsymbol{u}]\big]\big\|_{L^\infty(\Omega,r)} \le c_0'\|\boldsymbol{u}\|^2_{L^\infty(\Omega,r)}.$$

Therefore, taking into account (2.7), if $\|\boldsymbol{\gamma}\|_{L^\infty(\partial\Omega)}$ is chosen such that

$$\|\overset{-1}{\mathcal{G}}[\boldsymbol{u}_\gamma]\|_{L^3(\Omega)} < \frac{1}{4c_0'},$$

then the map (3.4) has a fixed point $\boldsymbol{u}$ and the field $\boldsymbol{u} + \mu\boldsymbol{u}_0$ is a solution to the Navier-Stokes problem. □

Remark 3.7. Of course, if $\boldsymbol{u}_0$ and $\boldsymbol{\gamma}$ are more regular, then so does the solution $(\boldsymbol{u}, p)$. In particular if $\boldsymbol{u}_0,\ \boldsymbol{\gamma} \in C(\partial\Omega)$, then is $\boldsymbol{u} \in C(\overline{\Omega})$.

Remark 3.8. In virtue of the result in [13] the derivatives of $\boldsymbol{u}$ have the following behavior at infinity

$$\underbrace{\nabla\ldots\nabla}_{k\text{ times}}\boldsymbol{u} = O(r^{-1-k})$$

and

$$p = O(r^{-2}\log r), \quad \underbrace{\nabla\ldots\nabla}_{k\text{ times}} p = O(r^{-2-k}),$$

with $k \in \mathbb{N}$.

Remark 3.9. As far as uniqueness of the solutions in the above theorems are concerned, we quote [10], and [6, Chapter IX]. The existence of a solution to system (1.1)-(1.3) in a Lipschitz exterior domain, with $\boldsymbol{a} \in L^\infty(\partial\Omega)$, which converges at infinity to an assigned nonzero constant vector has been recently proved in [11].

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