Holomorphic line bundles and divisors
on a domain of a Stein manifold

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Dedicated to Professor Yoshihiro Mizuta on his sixtieth birthday

Abstract. Let $D$ be an open set of a Stein manifold $X$ of dimension $n$ such that $H^k(D, \mathcal{O}) = 0$ for $2 \leq k \leq n - 1$. We prove that $D$ is Stein if and only if every topologically trivial holomorphic line bundle $L$ on $D$ is associated to some Cartier divisor $\mathfrak{d}$ on $D$.

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1. Introduction

For every holomorphic line bundle $L$ on a reduced Stein space $X$ there exists a global holomorphic section $\sigma \in \Gamma(X, \mathcal{O}(L))$ such that the zero set $\{\sigma = 0\}$ is nowhere dense in $X$. Therefore $L$ is associated to the positive Cartier divisor $\text{div}(\sigma)$ on $X$ (see Gunning [9, pages 122–125]).

Conversely the author [1, Theorem 3] proved that an open set $D$ of a Stein manifold $X$ of dimension two is Stein if every holomorphic line bundle $L$ on $D$ is associated to some (not necessarily positive) Cartier divisor $\mathfrak{d}$ on $D$. Moreover Ballico [4, Theorem 1] proved that an open set $D$ of a Stein manifold $X$ of dimension more than two of the form $D = \{\varphi < c\}$, where $\varphi : X \to \mathbb{R}$ is a weakly 2-convex function of class $C^2$ in the sense of Andreotti-Grauert, is Stein if every holomorphic line bundle $L$ on $D$ is associated to some Cartier divisor $\mathfrak{d}$ on $D$.

In this paper we prove that an open set $D$ of a Stein manifold $X$ of dimension $n$ such that $H^k(D, \mathcal{O}) = 0$ for $2 \leq k \leq n - 1$ is Stein if every topologically trivial holomorphic line bundle $L$ on $D$ is associated to some Cartier divisor $\mathfrak{d}$ on $D$ (see Theorem 4.3). This generalizes both results above (see Corollaries 4.4 and 4.5).

The proof is by induction on $n = \dim X$ and the induction hypothesis is applied to the complex subspace $(Y, (\mathcal{O}_X/f \mathcal{O}_X)|_Y)$, where $f \in \mathcal{O}_X(X)$ and

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Y := \{[f] = 0\}. Therefore it is inevitable to consider complex spaces which are not necessarily reduced (see Theorem 4.1).

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2. Preliminaries

Throughout this paper complex spaces are always assumed to be second countable. We always denote by \(\mathcal{O}\) without subscript the reduced complex structure sheaf of an arbitrary complex space. In other words we always set \(\mathcal{O} := \mathcal{O}_X/\mathcal{N}_X\) for a complex space \((X, \mathcal{O}_X)\), where \(\mathcal{N}_X\) is the nilradical of the complex structure sheaf \(\mathcal{O}_X\).

Let \((X, \mathcal{O}_X)\) be a (not necessarily reduced) complex space and \((\text{red}, \text{red}) : (X, \mathcal{O}) \to (X, \mathcal{O}_X)\) the reduction map. We denote by \([f]\) the valuation \(x \mapsto f_x + m_x \in \mathcal{O}_X(x)\), for every \(f \in \mathcal{O}_X(U)\), where \(U\) is an open set of \(X\). Then the assignment \(f \mapsto [f]\) is identified with \(\text{red} : \mathcal{O}_X(U) \to \mathcal{O}(U)\) (see Grauert-Remmert [8, page 87]).

Let \(X\) be a reduced complex space and \(e : \mathcal{O} \to \mathcal{O}^*\) the homomorphism of sheaves on \(X\) defined by \(e_x(f_x) := \exp(2\pi \sqrt{-1} f_x)\) for every \(f_x \in \mathcal{O}_X(x)\). Then \(e\) induces the homomorphism \(e^* : H^1(X, \mathcal{O}) \to H^1(X, \mathcal{O}^*)\). As usual we identify the cohomology group \(H^1(X, \mathcal{O}^*)\) with the set of holomorphic line bundles on \(X\).

Let \(\partial\) be a Cartier divisor on a reduced complex space \(X\) defined by the meromorphic Cousin-II distribution \(\{(U_i, m_i)\}_{i \in I}\) on \(X\) (see Gunning [9, page 121]). We denote by \([\partial]\) the holomorphic line bundle on \(X\) defined by the cocycle \(\{m_i/m_j\} \in Z^1(\{(U_i)_{i \in I}, \mathcal{O}^*\})\) and we say that \([\partial]\) is the holomorphic line bundle associated to \(\partial\). We say that \(\partial\) is positive (or effective) if \(\partial\) can be defined by a holomorphic Cousin-II distribution.

Let \((X, \mathcal{O}_X)\) be a (not necessarily reduced) complex space and \(D\) an open set of \(X\). Then \(D\) is said to be locally Stein at a point \(x \in \partial D\) if there exists a neighborhood \(U\) of \(x\) in \(X\) such that the open subspace \((D \cap U, \mathcal{O}_X|_{D \cap U})\) is Stein.

Throughout this paper we use the following notation:

\[
\Delta(r) := \{t \in \mathbb{C} \mid |t| < r\} \quad \text{for} \quad r > 0, \quad \Delta := \Delta(1),
\]
\[
P(n, \varepsilon) := \Delta(1 + \varepsilon)^n, \quad \text{and}
\]
\[
H(n, \varepsilon) := \Delta^n \cup \left\{ (z_1, z_2, \ldots, z_n) \in \mathbb{C}^n \mid 1 - \varepsilon < |z_1| < 1 + \varepsilon, \right\}
\]
\[
|z_2| < 1 + \varepsilon, \quad |z_3| < 1 + \varepsilon, \quad \ldots, \quad |z_n| < 1 + \varepsilon
\]

for \(n \geq 2\) and \(0 < \varepsilon < 1\).
The pair \((P(n, \varepsilon), H(n, \varepsilon))\) is said to be a Hartogs figure. We have the following lemma which characterizes a Stein open set of \(\mathbb{C}^n\).

**Lemma 2.1 (Kajiwara-Kazama [10, Lemmas 1 and 2]).** Let \(D\) be an open set of \(\mathbb{C}^n\). Then the following two conditions are equivalent.

1. \(D\) is Stein.
2. There do not exist a biholomorphic map \(\varphi : \mathbb{C}^n \to \mathbb{C}^n\), \(\varepsilon \in (0, 1)\) and \(b = (b_1, b_2, \ldots, b_n) \in \mathbb{C}^n\) such that \(\varphi(H(n, \varepsilon)) \subset D\), \(|b_1| \leq 1 - \varepsilon\), \(\max_{2 \leq v \leq n} |b_v| = 1\) and \(\varphi(b) \in \partial D\).

**3. Lemmas**

In this section we denote by \(\Phi(X, \mathcal{O}_X)\) the composition of the induced homomorphisms

\[ H^1(X, \mathcal{O}_X) \xrightarrow{\text{red}} H^1(X, \mathcal{O}) \xrightarrow{\text{red}} H^1(X, \mathcal{O}^*) \]

for every complex space \((X, \mathcal{O}_X)\).

**Lemma 3.1.** Let \((X, \mathcal{O}_X)\) be a Stein space of pure dimension 2 and \(D\) an open set of \(X\). Let \((\theta, \tilde{\theta}) : (X, \mathcal{O}_X) \to \mathbb{C}^2\) be a holomorphic map, \(R\) and \(W\) open sets of \(X, \varepsilon \in (0, 1)\) and \(b = (b_1, b_2) \in \mathbb{C}^2\) such that \(R \subseteq W \subseteq X \setminus \text{Sing}(X, \mathcal{O}_X)\), \(\theta(W)\) is an open set of \(\mathbb{C}^2\), the restriction \(\theta|_W : W \to \theta(W)\) is biholomorphic, \(\theta(R) = \varphi(H(n, \varepsilon)) \subset D\), \(|b_1| \leq 1 - \varepsilon\), \(|b_2| = 1\) and \((\theta|_W)^{-1}(b) \in \partial D\). Then there exists a cohomology class \(\alpha \in H^1(D, \mathcal{O}_X|_D)\) such that the holomorphic line bundle \(\Phi(D, \mathcal{O}_X|_D)(\alpha)|_{D \cap R}\) on \(D \cap R\) is not associated to any Cartier divisor on \(D \cap R\).

**Proof.** Let \(\theta_v := \tilde{\theta}z_v^v\) for \(v = 1, 2\), where \(z_1\) and \(z_2\) are the coordinates of \(\mathbb{C}^2\). Let \(E_v := \{[\theta_v] \neq b_v\}\) for \(v = 1, 2\). Since \((E_v, \mathcal{O}_X|_{E_v})\) is Stein and \(1/([\theta_v] - b_v) \in \mathcal{O}_X(E_v)\), there exists \(u_v \in \mathcal{O}_X(E_v)\) such that \([u_v] = 1/([\theta_v] - b_v)\) on \(E_v\) for \(v = 1, 2\). Let \(T := \{[\theta_1] < 1 + \varepsilon\}\) and \(T_1 := \{[\theta_1] < 1 + \varepsilon, [\theta_2] > 1 + \varepsilon/2\} \cup (T \setminus \tilde{R})\). Then \((T, \mathcal{O}_X|_T)\) is Stein and \(\{R, T_1\}\) is an open covering of \(T\). Since \(H^1([R, T_1], \mathcal{O}_X|_T) = 0\) and \(R \cap T_1 \subset E_2\), there exist \(u_0 \in \mathcal{O}_X(R) = \mathcal{O}(R)\) and \(u_1 \in \mathcal{O}_X(T_1)\) such that \(u_2 = u_1 - v_0\) on \(R \cap T_1\). Let \(F := (E_2 \cap R) \cap T_1\). Let \(v \in \mathcal{O}_X(F)\) be defined by \(v = u_0 + u_2\) on \(E_2 \cap R\) and \(v = v_1\) on \(T_1\). Let \(D_1 := D \cap E_1\) and \(D_2 := D \cap ((E_2 \cap T) \cup (T \setminus \tilde{R}))\). Then \(\{D_1, D_2\}\) is an open covering of \(D\) and \(D_1 \cap D_2 \subset E_1 \cap F\). Let \(\alpha \in H^1([D_1, D_2], \mathcal{O}_X|_D)\) be the cohomology class defined by \((u_1v_1)|_{D_1 \cap D_2} \in \mathcal{O}_X(D_1 \cap D_2) = Z^1([D_1, D_2], \mathcal{O}_X|_D)\). Then by

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1 An open set \(D\) of \(\mathbb{C}^n\) satisfies condition (2) in Lemma 2.1 if and only if \(D\) is \(p\)-convex in the sense of Kajiwara-Kazama [10, page 2]. Note that the sentence “\(\varphi(D)\) is a subset of \(\Omega \ldots\)” should be “\(\varphi(D)\) is a subset of \(\Omega \ldots\)” in the definition of a boundary mapping in Kajiwara-Kazama [10, page 2].
the argument in Abe [1, page 271] the holomorphic line bundle $\Phi_{(D, \mathcal{O}_X|_D)}(\alpha)|_{D \cap R}$ is not associated to any Cartier divisor on $D \cap R$.

A zero set $N(l)$ of a linear function $l(z_1, z_2, \ldots, z_n) = \sum_{k=1}^{n} a_k z_k + b$ on $\mathbb{C}^n$, where $a_1, a_2, \ldots, a_n, b \in \mathbb{C}$ and $(a_1, a_2, \ldots, a_n) \neq (0, 0, \ldots, 0)$, is said to be a hyperplane of $\mathbb{C}^n$.

**Lemma 3.2.** Let $D$ be an open set of $\mathbb{C}^n$ and $H$ a hyperplane of $\mathbb{C}^n$. Let $Z := D \cap H$. Then for every Cartier divisor $\mathcal{D}$ on $D$ there exists a Cartier divisor $\mathcal{D}'$ on $D$ such that the support $|\mathcal{D}'|$ of $\mathcal{D}'$ is nowhere dense in $Z$ and $|\mathcal{D}'|_Z = [\mathcal{D}'|_Z]$.

**Proof.** As usual we identify a Cartier divisor on a complex manifold with a Weil divisor. Let $\mathcal{D} = \sum_{\lambda \in \Lambda} \alpha_\lambda A_\lambda$, where $A_\lambda$ is an irreducible analytic set of dimension $n - 1$ and $\alpha_\lambda \in \mathbb{Z}$ for every $\lambda \in \Lambda$, be the expression of $\mathcal{D}$ as a Weil divisor. Let $\Lambda''$ be the set of $\lambda \in \Lambda$ such that $A_\lambda$ is a connected component of $Z$. Let $\Lambda' := \Lambda \setminus \Lambda''$, $\mathcal{D}' := \sum_{\lambda \in \Lambda'} \alpha_\lambda A_\lambda$ and $\mathcal{D}'' := \sum_{\lambda \in \Lambda''} \alpha_\lambda A_\lambda$. Then the support $|\mathcal{D}'| = \bigcup_{\lambda \in \Lambda'} A_\lambda$ of $\mathcal{D}'$ is nowhere dense in $Z$. Let $\{Z_\mu\}_{\mu \in M}$, where $M \subset \mathbb{N}$, be the set of connected components of $Z$. There exists a system $\{U_\mu\}_{\mu \in M}$ of mutually disjoint open sets of $D$ such that $U_\mu$ is a neighborhood of $Z_\mu$ for every $\mu \in M$. Let $U_0 := D \setminus Z$. We choose a non-constant linear function $l$ on $\mathbb{C}^n$ such that $\{l = 0\} = H$. If there exists $\lambda \in \Lambda''$ such that $Z_\mu = A_\lambda$, then let $\beta_\mu := \alpha_\lambda$. Otherwise let $\beta_\mu := 0$. Then $\mathcal{D}''$ as a Cartier divisor is defined by the system $\{(U_\mu, 1)\} \cup \{(U_\mu, l^{\beta_\mu})\}_{\mu \in M}$. It follows that $[\mathcal{D}''|_Z]$ is holomorphically trivial on $U := \bigcup_{\mu \in M} U_\mu$. Since $U$ is a neighborhood of $Z$ in $D$, the restriction $|\mathcal{D}''|_Z$ is also holomorphically trivial. Then we have that $[\mathcal{D}]_Z = [\mathcal{D}' + \mathcal{D}'']|_Z = (|\mathcal{D}'| \otimes |\mathcal{D}''|)|_Z = [\mathcal{D}'|_Z] \otimes [\mathcal{D}''|_Z] = [\mathcal{D}'|_Z]$. □

A complex space $(X, \mathcal{O}_X)$ is said to be Cohen-Macaulay if the local $\mathbb{C}$-algebra $\mathcal{O}_{X,x}$ is Cohen-Macaulay for every $x \in X$ (see Raimondo-Silva [12]).

**Lemma 3.3.** Let $(X, \mathcal{O}_X)$ be a Cohen-Macaulay Stein space of pure dimension $n \geq 2$. Let $D$ be an open set of $X$ such that $H^k(D, \mathcal{O}_X|_D) = 0$ for $2 \leq k \leq n - 1$. Let $(\theta, \tilde{\theta}) : (X, \mathcal{O}_X) \to \mathbb{C}^n$ be a holomorphic map, $R$ and $W$ open sets of $X$, $\varepsilon \in (0, 1)$ and $b = (b_1, b_2, \ldots, b_n) \in \mathbb{C}^n$ such that $R \Subset W \subset X \setminus \text{Sing}(X, \mathcal{O}_X)$, $\theta(W)$ is an open set of $\mathbb{C}^n$, the restriction $\theta|_W : W \to \theta(W)$ is biholomorphic, $\theta(R) = P(n, \varepsilon)$, $(\theta|_W)^{-1}(H(n, \varepsilon)) \subset D$, $|b_1| \leq 1 - \varepsilon$, $\max_{2 \leq v \leq n} |b_v| = 1$ and $(\theta|_W)^{-1}(b) \in \partial D$. Then there exists a cohomology class $\alpha \in H^1(D, \mathcal{O}_X|_D)$ such that the holomorphic line bundle $\Phi_{(D, \mathcal{O}_X|_D)}(\alpha)|_{D \cap R}$ on $D \cap R$ is not associated to any Cartier divisor on $D \cap R$.

**Proof.** The proof proceed by induction on $n = \dim X$. By Lemma 3.1 the assertion is true if $n = 2$. We consider the case when $n \geq 3$. Let $\theta_\nu := \tilde{\theta} z_\nu$ for $\nu = 1, 2, \ldots, n$, where $z_1, z_2, \ldots, z_n$ are the coordinates of $\mathbb{C}^n$. We replace $W$ by the connected component of $W$ which contains $\tilde{R}$. Let $X_0$ be the irreducible component of $X$ which contains $W$. Since $(X, \mathcal{O}_X)$ is Stein, there exists $f \in \mathcal{O}_X(X)$ such that
Let $[f] = [\theta_n] - b_n$ on $X_0$ and $[f] \neq 0$ on any irreducible component of $X$. Let $Y := \{[f] = 0\} = \text{Supp} \left( \mathcal{O}_X / \mathcal{O}_X \mathcal{O}_Y \right)$ and $\mathcal{O}_Y := \left( \mathcal{O}_X / \mathcal{O}_X \mathcal{O}_Y \right) |_Y$. By the active lemma (see Grauert-Remmert [8, page 100]) we have that

$$\dim \mathcal{O}_{X,x} / f_x \mathcal{O}_{X,x} = \dim Y = n - 1 = \dim \mathcal{O}_{X,x} - 1$$

for every $x \in Y$. Therefore $f_x$ is not a zero divisor of $\mathcal{O}_{X,x}$ for every $x \in Y$ and $(Y, \mathcal{O}_Y)$ is a Cohen-Macaulay Stein space of pure dimension $n - 1$ (see Grauert-Remmert [6, page 141] or Serre [15, page 85]). Let $m : \mathcal{O}_X \to \mathcal{O}_X$ be the homomorphism defined by $m_x(h_x) := f_x h_x$ for every $h_x \in \mathcal{O}_{X,x}$ and $x \in X$. Since the sequence

$$0 \to \mathcal{O}_X \xrightarrow{m} \mathcal{O}_X \to \mathcal{O}_X / f \mathcal{O}_X \to 0$$

is exact, we have the long exact sequence of cohomology groups

$$\cdots \to H^k(D, \mathcal{O}_X |_D) \to H^k(D \cap Y, \mathcal{O}_Y |_{D \cap Y}) \to H^{k+1}(D, \mathcal{O}_X |_D) \to \cdots.$$ 

Since by assumption $H^k(D, \mathcal{O}_X |_D) = 0$ for $2 \leq k \leq n - 1$, we have that $H^k(D \cap Y, \mathcal{O}_X |_{D \cap Y}) = 0$ for $2 \leq k \leq n - 2$ and that the homomorphism $\tilde{\iota}^* : H^1(D, \mathcal{O}_X |_D) \to H^1(D \cap Y, \mathcal{O}_Y |_{D \cap Y})$ is surjective. Let $(\theta', \tilde{\theta}') : (Y, \mathcal{O}_Y) \to \mathbb{C}^{n-1}$ be the holomorphic map such that $\tilde{\theta}'z_v = (\tilde{\theta}_v) |_Y$ for $v = 1, 2, \ldots, n - 1$ (see Grauert-Remmert [8, page 22]). Let $R' := R \cap Y$, $W' := W \cap Y$ and $b' := (b_1, b_2, \ldots, b_{n-1})$. Then $\theta(x) = (\theta'(x), b_n)$ for every $x \in W'$, $R' \subset W' \subset Y \setminus \text{Sing}(Y, \mathcal{O}_Y)$, $\theta'(W')$ is an open set of $\mathbb{C}^{n-1}$ and the restriction $\theta'|_{W'} : W' \to \theta'(W')$ is biholomorphic. We have that

$$\theta'(R') \times \{b_n\} = \theta(R') = P(n, \varepsilon) \cap \{z_n = b_n\} = P(n - 1, \varepsilon) \times \{b_n\},$$

$$\left(\theta'|_{W'}\right)^{-1}(H(n - 1, \varepsilon)) = \left(\theta|_{W}\right)^{-1}(H(n - 1, \varepsilon) \times \{z_n = b_n\})$$

$$= \left(\theta|_{W}\right)^{-1}(H(n, \varepsilon)) \cap W' \subset D \cap Y,$$ 

and

$$\left(\theta'|_{W'}\right)^{-1}(b') = \left(\theta|_{W}\right)^{-1}(b) \in \partial (D \cap Y),$$

where $\partial (D \cap Y)$ denotes the boundary of $D \cap Y$ in $Y$. By induction hypothesis there exists $\alpha' \in H^1(D \cap Y, \mathcal{O}_Y |_{D \cap Y})$ such that the holomorphic line bundle $\Phi_{(D \cap Y, \mathcal{O}_Y |_{D \cap Y})}(\alpha') |_{D \cap R'}$ on $D \cap R'$ is not associated to any Cartier divisor on $D \cap R'$. Since $\tilde{\iota}^*$ is surjective, there exists $\alpha \in H^1(D, \mathcal{O}_X |_D)$ such that $\tilde{\iota}^*(\alpha) = \alpha'$. Assume that there exists a Cartier divisor $\mathcal{D}$ on $D \cap R$ such that $\Phi_{(D, \mathcal{O}_X |_D)}(\alpha) |_{D \cap R} = [\emptyset]$. By Lemma 3.2 there exists a Cartier divisor $\mathcal{C}$ on $D \cap R$ such that the support $|\mathcal{C}|$ is nowhere dense in $D \cap R'$ and $[\mathcal{C}] |_{D \cap R'} = [\mathcal{C}] |_{D \cap R'}$. Then we have that

$$\Phi_{(D \cap Y, \mathcal{O}_Y |_{D \cap Y})}(\alpha') |_{D \cap R'} = \Phi_{(D, \mathcal{O}_X |_D)}(\alpha) |_{D \cap R'} = [\emptyset] |_{D \cap R'} = [\mathcal{C}] |_{D \cap R'}$$

and it is a contradiction. It follows that $\Phi_{(D, \mathcal{O}_X |_D)}(\alpha) |_{D \cap R}$ is not associated to any Cartier divisor on $D \cap R$. □
4. Theorems

**Theorem 4.1.** Let \((X, \mathscr{O}_X)\) be a (not necessarily reduced) Cohen-Macaulay Stein space of pure dimension \(n\) and \(D\) an open set of \(X\). Assume that the following two conditions are satisfied.

i) \(H^k(D, \mathscr{O}_X|_D) = 0\) for \(2 \leq k \leq n - 1\).

ii) For every holomorphic line bundle \(L\) on \(D\) which is an element of the image of the composition \(\Phi\) of the homomorphisms

\[
H^1(D, \mathscr{O}_X|_D) \xrightarrow{\text{red}^*} H^1(D, \mathscr{O}) \xrightarrow{\text{red}^*} H^1(D, \mathscr{O}^*)
\]

there exists a biholomorphic map \(f : (X, \mathscr{O}_X) \to \mathbb{C}^n\) such that \(f(V)\) is a Stein neighborhood of \(f(W)\) and \(f(W)\) is biholomorphic (see Grauert-Remmert [7, page 151]). Take a Stein open set \(W \subset \mathbb{C}^n\) such that \(f(x_0) \in V \subset f(W)\). Then \(U := (f|_W)^{-1}(V)\). Since \(U\) is Stein, \(x_0 \in U \subset W\) and \(f(U) = V\). Since \(D\) is not locally Stein at \(x_0\), the open set \(f(D \cap U)\) of \(\mathbb{C}^n\) is not Stein. By Lemma 2.1 there exist a biholomorphic map \(\varphi : \mathbb{C}^n \to \mathbb{C}^n, \varepsilon \in (0, 1)\) and \(b = (b_1, b_2, \ldots, b_n) \in \mathbb{C}^n\) such that \(\varphi(H(n, \varepsilon)) \subset f(D \cap U), |b_1| \leq 1 - \varepsilon, \max_{2 \leq i \leq n} |b_i| = 1\) and \(\varphi(b) \in \partial(f(D \cap U))\). Let \((\theta, \tilde{\theta}) := \varphi^{-1} \circ (f, f) : (X, \mathscr{O}_X) \to \mathbb{C}^n\). We have that \(\theta(W) = \varphi^{-1}(f(W))\) is an open set of \(\mathbb{C}^n\) and \(\theta|_W : W \to \partial(W)\) is biholomorphic. Let \(P := P(n, \varepsilon)\) and \(H := H(n, \varepsilon)\). Since \(V\) is Stein and \(\varphi(H) \subset f(D \cap U) \subset V\), we have that \(\varphi(P) \subset V \subset f(W)\). Let \(R := (\tilde{\theta}|_W)^{-1}(P)\). Then we have that \(\tilde{\theta}(R) = P, (\tilde{\theta}|_W)^{-1}(H) \subset U\) and \((\tilde{\theta}|_W)^{-1}(b) \in \partial D\). By Lemma 3.3 there exists a biholomorphic line bundle \(L \in \text{im } \Phi\) such that \(L|_{D \cap R}\) is not associated to any Cartier divisor on \(D \cap R\). On the other hand by assumption there exists a Cartier divisor \(\mathfrak{d}\) on \(D\) such that \(L = [\mathfrak{d}]\) and therefore \(L|_{D \cap R} = [\mathfrak{d}]|_{D \cap R}\), which is a contradiction. It follows that \(D\) is locally Stein at every point \(x \in \partial D \setminus \text{Sing}(X, \mathscr{O}_X)\).

**Remark 4.2.** Condition i) in Theorem 4.1 can be replaced by the following weaker one:

i’ The dimension of \(H^k(D, \mathscr{O}_X|_D)\) is at most countably infinite for every integer \(k\) such that \(2 \leq k \leq n - 1\).

**Proof.** If condition i’ is satisfied, then by Ballico [3, Proposizione 7], which generalizes Siu [16, Theorem A], we have that \(\dim H^k(D, \mathscr{O}_X|_D) < +\infty\) for \(2 \leq k \leq n - 1\). We also have that \(H^k(D, \mathscr{O}_X|_D) = 0\) for \(k \geq n\) by Siu [17] and by Reiffen [13, page 277]. It follows that \(H^k(D, \mathscr{O}_S|_D) = 0\) for \(k \geq 2\) by Raimondo-Silva [12].

\(\square\)
Every complex manifold is Cohen-Macaulay (see Grauert-Remmert [6, page 142]). The image of $H^1(D, \mathcal{O}) \rightarrow H^1(D, \mathcal{O}^*)$ coincides with the set of topologically trivial holomorphic line bundles on $D$. Therefore by Theorem 4.1 and by Docquier-Grauert [5] we obtain the following theorem.

**Theorem 4.3.** Let $X$ be a Stein manifold of dimension $n$ and $D$ an open set of $X$ such that $H^k(D, \mathcal{O}) = 0$ for $2 \leq k \leq n - 1$. Then the following four conditions are equivalent.

1. $D$ is Stein.
2. For every holomorphic line bundle $L$ on $D$ there exists a positive Cartier divisor $\mathfrak{d}$ on $D$ such that $L = [\mathfrak{d}]$.
3. For every holomorphic line bundle $L$ on $D$ there exists a Cartier divisor $\mathfrak{d}$ on $D$ such that $L = [\mathfrak{d}]$.
4. For every topologically trivial holomorphic line bundle $L$ on $D$ there exists a Cartier divisor $\mathfrak{d}$ on $D$ such that $L = [\mathfrak{d}]$.

**Corollary 4.4 (Abe [1, Theorem 3]).** Let $X$ be a Stein manifold of dimension 2 and $D$ an open set of $X$. Then the four conditions in Theorem 4.3 are equivalent.

Let $X$ be a complex manifold of dimension $n$ and $\varphi : X \rightarrow \mathbb{R}$ a function of class $\mathcal{C}^2$. Then $\varphi$ is said to be weakly 2-convex if for every $x \in X$ the Levi form of $\varphi$ at $x$ has at most one negative eigenvalue. By the theorem of Andreotti-Grauert [2] we have the following corollary.

**Corollary 4.5 (Ballico [4, Theorem 1]).** Let $X$ be a Stein manifold and $\varphi : X \rightarrow \mathbb{R}$ a weakly 2-convex function of class $\mathcal{C}^2$. Let $D := \{\varphi < c\}$, where $c \in \mathbb{R}$ is a constant. Then the four conditions in Theorem 4.3 are equivalent.

We also have the following corollary (see Serre [14, page 65]).

**Corollary 4.6 (Laufer [11, Theorem 4.1]).** Let $X$ be a Stein manifold of dimension $n$ and $D$ an open set of $X$. Then the following two conditions are equivalent.

1. $D$ is Stein.
2. $H^k(D, \mathcal{O}) = 0$ for $1 \leq k \leq n - 1$.

**References**


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