

## Addendum to: On volumes of arithmetic quotients of $\mathrm{SO}(1, n)$

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**Abstract.** There are errors in the proof of uniqueness of arithmetic subgroups of the smallest covolume. In this note we correct the proof, obtain certain results which were stated as a conjecture, and we give several remarks on further developments.

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**1.1.** Let us recall some notation and basic notions. Following [1] we will assume that  $n$  is even and  $n \geq 4$ . The group of orientation preserving isometries of hyperbolic  $n$ -space is isomorphic to  $\mathrm{SO}(1, n)^o$ , the connected component of the identity of the special orthogonal group of signature  $(1, n)$ , which can be identified with  $\mathrm{SO}_0(1, n)$ , the subgroup of  $\mathrm{SO}(1, n)$  preserving the upper half space. This group is not Zariski closed in  $\mathrm{SL}_{n+1}$  thus in order to construct arithmetically defined subgroups of  $\mathrm{SO}(1, n)^o$  we consider arithmetic subgroups of the orthogonal group  $\mathrm{SO}(1, n)$  or, more precisely, of groups  $G = \mathrm{SO}(f)$  where  $f$  is an admissible quadratic form defined over a totally real number field  $k$  (see [1, Section 2.1]).

We have an exact sequence of  $k$ -isogenies:

$$1 \rightarrow C \rightarrow \tilde{G} \xrightarrow{\phi} G \rightarrow 1, \quad (1.1)$$

where  $\tilde{G}(k) \simeq \mathrm{Spin}(f)$  is the simply connected cover of  $G$  and  $C \simeq \mu_2$  is the center of  $\tilde{G}$ . This induces an exact sequence in Galois cohomology (see [5, Section 2.2.3])

$$\tilde{G}(k) \xrightarrow{\phi} G(k) \xrightarrow{\delta} H^1(k, C) \rightarrow H^1(k, \tilde{G}). \quad (1.2)$$

The main idea of this note is that by using (1.2) certain questions about arithmetic subgroups of  $G$  can be reduced to questions about the Galois cohomology group  $H^1(k, C)$ .

A coherent collection of parahoric subgroups  $P = (P_v)_{v \in V_f}$  of  $\tilde{G}$  ( $V_f = V_f(k)$  denotes the set of finite places of the field  $k$ ) defines a principal arithmetic subgroup

$\Lambda = \tilde{G}(k) \cap \prod_{v \in V_f} P_v \subset \tilde{G}(k)$  (see [2]). We fix an infinite place  $v$  of  $k$  for which  $G(k_v) \simeq \text{SO}(1, n)$  and denote it by  $Id$ . The image of  $\Lambda$  under the central  $k$ -isogeny  $\phi$  is an arithmetic subgroup of  $G$  and every maximal arithmetic subgroup of  $G(k_{Id})$  can be obtained as a normalizer of some  $\phi(\Lambda)$  [2, Proposition 1.4]. We will also consider the local stabilizers of  $P$  in the adjoint group  $G(=\bar{G})$ , defining  $\bar{P}_v$  to be the stabilizer of  $P_v$  in  $G(k_v)$  and  $\bar{P} = (\bar{P}_v)_{v \in V_f}$ . Clearly,  $\bar{P}_v \supset \phi(P_v)$ . In the notation of [1] the subgroups  $\phi(P_v)$  are called parahoric subgroups of  $G$ , however this terminology is non-standard and we will avoid using it here.

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**1.2.** Given a totally real number field  $k$  with the group of units  $U$ , let

$$k_\infty^* = \{a \in k^* \mid a_v > 0 \text{ for } v \in V_\infty \setminus Id\}, \quad U_\infty = U \cap k_\infty^*.$$

**Lemma 1.1.**  $\text{Im}(\delta) \simeq k_\infty^*/(k^*)^2$ .

*Proof.* From (1.2) we have  $\text{Im}(\delta : G(k) \rightarrow H^1(k, \mu_2)) = \text{Ker}(H^1(k, \mu_2) \rightarrow H^1(k, \tilde{G}))$ . The Hasse principle for simply connected  $k$ -groups implies that  $H^1(k, \tilde{G})$  is isomorphic to  $\prod_{v \in V_\infty} H^1(k_v, \tilde{G})$  [5, Theorem 6.6], and hence

$$\text{Im}(\delta) = \text{Ker}(H^1(k, \mu_2) \rightarrow \prod_{v \in V_\infty} H^1(k_v, \tilde{G})).$$

The group  $H^1(k, \mu_2)$  is canonically isomorphic to  $k^*/(k^*)^2$  [5, Lemma 2.6]. It is well known that for all  $v \in V_\infty$  such that the group  $G(k_v)$  is anisotropic, the map  $\phi$  in (1.2) is surjective and hence for all such  $v$ ,  $\text{Im}(\delta_v) = \text{Ker}(H^1(k_v, \mu_2) \rightarrow H^1(k_v, \tilde{G}))$  is trivial. For the remaining one infinite place  $v(= Id) \in V_\infty$ ,  $\phi(\tilde{G}(k_v))$  is a subgroup of index 2 in  $G(k_v)$  which consists of the orthogonal transformations with the trivial spinor norm. Collecting this information together we obtain the required isomorphism. □

**1.3** The proof of the uniqueness part in [1, Theorem 4.1] contains errors but the result is correct. We will now give another argument for it. In order to do so we first establish a more general fact and then apply it to the cases considered in [1].

Let  $P = (P_v)_{v \in V_f}$  and  $P' = (P'_v)_{v \in V_f}$  be two coherent collections of parahoric subgroups of  $\tilde{G}$  such that for all  $v \in V_f$ ,  $P'_v$  is conjugate to  $P_v$  under an element of  $G(k_v)$ . For all but finitely many  $v$ ,  $P_v = P'_v$  hence there is an element  $g \in G(\mathbb{A}_f)$  ( $\mathbb{A}_f$  denotes the ring of finite adèles of  $k$ ) such that  $P'$  is the conjugate of  $P$  under  $g$ . We have  $\bar{P} = \prod_{v \in V_f} \bar{P}_v$  is the stabilizer of  $P$  in  $G(\mathbb{A}_f)$ . The number of distinct

$G(k)$ -conjugacy classes of coherent collections  $P'$  as above is the cardinality  $c(\overline{P})$  of  $\mathcal{C}(\overline{P}) = G(k) \backslash G(\mathbb{A}_f) / \overline{P}$ , which is called the class group of  $G$  relative to  $\overline{P}$ . The class number  $c(\overline{P})$  is known to be finite (see e.g. [2, Proposition 3.9]). The following result can be used for obtaining further information about its value.

**Proposition 1.2.** *Let  $G = SO(f)$ ,  $\tilde{G} = Spin(f)$  for an admissible quadratic form  $f$  defined over  $k$  and let  $P = (P_v)_{v \in V_f}$  a coherent collection of parahoric subgroups of  $\tilde{G}$ . The class number  $c(P)$  divides the order  $h_{\infty, 2}$  of a restricted 2-class group of  $k$  given by*

$$h_{\infty, 2} = \frac{2^{[k:\mathbb{Q}]-1} h_2}{[U : U_{\infty}]},$$

where  $h_2$  is the order of the 2-class group of  $k$ .

*Proof.* Recall two isomorphisms (see [5, Proposition 8.8], a minor modification is needed in order to adjust the statement to our setting but the argument remains the same):

$$G(k) \backslash G(\mathbb{A}_f) / \overline{P} \simeq G(\mathbb{A}_f) / \overline{P} G(k);$$

$$G(\mathbb{A}_f) / \overline{P} G(k) \simeq \delta_{\mathbb{A}_f}(G(\mathbb{A}_f)) / \delta_{\mathbb{A}_f}(\overline{P} G(k)),$$

where  $\delta_{\mathbb{A}_f}$  is the restriction of the product map  $\prod_v G(k_v) \rightarrow \prod_v H^1(k_v, \mathbb{C})$  to  $G(\mathbb{A}_f)$ .

For every finite place  $v$ ,  $H^1(k_v, \tilde{G})$  is trivial (see [5, Theorem 6.4]) which implies  $\delta_v : G(k_v) \rightarrow H^1(k_v, \mathbb{C})$  is surjective. Thus the image of  $\delta_{\mathbb{A}_f}(G(\mathbb{A}_f))$  can be identified with the restricted direct product  $\prod' H^1(k_v, \mathbb{C})$  with respect to the subgroups  $\delta_v(\overline{P}_v)$ . Also  $\delta_{\mathbb{A}_f}(G(k))$  naturally identifies with the image of  $\delta(G(k))$  in  $H^1(k, \mathbb{C})$  under the embedding  $\psi : H^1(k, \mathbb{C}) \rightarrow \prod' H^1(k_v, \mathbb{C})$ . Hence we have an isomorphism

$$\delta_{\mathbb{A}_f}(G(\mathbb{A}_f)) / \delta_{\mathbb{A}_f}(\overline{P} G(k)) \simeq \prod' H^1(k_v, \mathbb{C}) / (\prod_v \delta_v(\overline{P}_v) \cdot \psi(\text{Im } \delta(G(k)))).$$

The group  $H^1(k_v, \mu_2)$  is canonically isomorphic to  $k_v^* / (k_v^*)^2$ , by Lemma 1.1  $\text{Im } \delta(G(k)) \simeq k_{\infty}^* / (k^*)^2$ , so we obtain

$$\frac{\prod' H^1(k_v, \mathbb{C})}{\prod_v \delta_v(\overline{P}_v) \cdot \psi(\text{Im } \delta(G(k)))} \simeq \frac{\prod' k_v^* / (k_v^*)^2}{\delta_P \cdot k_{\infty}^* / (k^*)^2} \simeq \frac{J_f}{\delta_P \cdot J_f^2 k^*} \cdot \frac{k^*}{k_{\infty}^*},$$

where  $J_f$  is the ring of finite idèles of  $k$  and  $\delta_P$  denotes  $\prod_v \delta_v(\overline{P}_v)$ .

Now,  $\#(J_f / J_f^2 k^*) = h_2$ , the group  $k^* / k_{\infty}^*$  splits as a product of local factors and  $\#(k^* / k_{\infty}^*) = 2^{[k:\mathbb{Q}]-1} / [U : U_{\infty}]$  (see [4, Chapter 6]). This implies the proposition.  $\square$

In order to give a precise formula for the class number  $c(P)$  one has to analyze the image of  $\prod_v \delta_v(\overline{P}_v)$  in  $\prod' H^1(k_v, \mathbb{C})$ . Still in many practical cases this appears to be unnecessary. Thus in order to prove the uniqueness of the minimal hyperbolic orbifolds we need to consider  $k = \mathbb{Q}[\sqrt{5}]$  (in the compact case) and  $k = \mathbb{Q}$  (for the non-compact orbifolds). In both cases  $h_2 = h = 1$ . For  $k = \mathbb{Q}[\sqrt{5}]$ ,  $U_\infty = \{1, \frac{1-\sqrt{5}}{2}\}$  and thus  $[U : U_\infty] = 2$  which implies  $h_{\infty,2} = 1$ . For  $k = \mathbb{Q}$ , clearly,  $h_{\infty,2} = 1$  as well. So in all the cases  $c(P) = 1$  which implies that the corresponding arithmetic subgroups are defined uniquely up to a conjugation by  $g \in \text{SO}(1, n)$ . It is clear that we can always chose  $g \in \text{SO}_0(1, n)$  and therefore the smallest orbifolds constructed in [1] are unique up to an (orientation preserving) isometry.

**1.4.** We now turn to Conjecture 4.1 and its analogue for the non-cocompact orbifolds in [1, Section 4.4]. Recall that in [1] the numbers  $N(r)$ ,  $N'(r)$  were defined for every  $r \geq 2$  and estimated from above. These numbers are related to the index of the principal arithmetic subgroups in their normalizers. We now prove

**Proposition 1.3.** *For every  $r \geq 2$ ,  $N(r) = N'(r) = 1$ .*

*Proof.* Let  $\Lambda$  be a principal arithmetic subgroup of  $\tilde{G}$  which corresponds to a compact or non-compact hyperbolic  $n$ -orbifold of the minimal volume,  $\Lambda' = \phi(\Lambda)$  and  $\Gamma = N_G(\Lambda')$ .

From [2, Proposition 2.9], which in turn follows from the work of J. Rohlfs, using the fact that the center of our group  $G$  is trivial, we obtain:

$$[\Gamma : \Lambda'] = \#(H^1(k, \mu_2)_\Theta \cap \delta(G(k))) = \#\text{Im}(\delta : G(k) \rightarrow H^1(k, \mu_2)).$$

By Lemma 1.1 we can identify the image of  $\delta$ . The cases we are interested in are

$$k = \mathbb{Q} : \text{Im}(\delta) = \{k^{*2}, (-1)k^{*2}\};$$

$$k = \mathbb{Q}[\sqrt{5}] : \text{Im}(\delta) = \left\{ k^{*2}, \frac{1 - \sqrt{5}}{2} k^{*2} \right\}.$$

In both cases  $[\Gamma : \Lambda'] = \#\text{Im}(\delta) = 2$ . Now it is easy to see that  $\Lambda' = \phi(\Lambda) \subset \text{SO}_0(1, n)$ . From the other side there always exists  $g \in \text{SO}(1, n) \setminus \text{SO}_0(1, n)$  which normalizes  $\phi(\Lambda)$ . For example take  $g = \text{diag}(-1, -1, 1, \dots, 1)$ . As in all the cases under consideration the quadratic form associated to  $\Lambda$  is diagonal [1, Sections 4.3, 4.4],  $g$  stabilizes  $\Lambda$  and clearly  $g \in \text{SO}(1, n) \setminus \text{SO}_0(1, n)$ . From these facts it follows that  $\Lambda'$  is a maximal arithmetic subgroup in  $\text{SO}_0(1, n)$  and thus  $N(r)$  (or  $N'(r)$ ) = 1. □

This proposition makes *precise* the statements of Theorem 4.1 and 4.4 of [1]. It also implies that Table 2 of *loc. cit.* gives the precise values of the covolumes of the smallest  $n$ -dimensional hyperbolic orbifolds in even dimensions up to 18.

One other corollary is that cocompact and non-cocompact arithmetic subgroups of  $SO(1, 2r)^\circ$  of the smallest covolumes can be obtained as the stabilizers of certain lattices described in [1, Section 4.3]. We remark that since the fields of definition of the groups have class number 1, the lattices in both cases are free as  $\mathcal{O}_k$ -modules.

**1.5.** Correction: on p. 765, l. 9 one should read “grow super-exponentially” instead of “grow exponentially”. (It follows from [1] that the Euler characteristic is bounded from below by  $\text{const} \cdot \left(\prod_{i=1}^r \frac{(2i-1)!}{(2\pi)^{2i}}\right)^{[k:\mathbb{Q}]}$  which for large enough  $r$  is  $\geq \text{const} \cdot (2r - 1)!$ )

We conclude this addendum with a few remarks on related results which appeared after the paper was published.

**1.6.** In [1, Section 4.5] we observed that for  $r > 2$  the minimal covolume among the arithmetic lattices in  $SO(1, 2r)$  is attained on a non-uniform lattice. This interesting phenomenon was first discovered by A. Lubotzky for  $SL_2$  over local fields of positive characteristic. Recently, in [6] A. Salehi Golsefidy proved that lattices of minimal covolume in classical Chevalley groups over local fields of characteristic  $p > 7$  are all non-uniform. This result gives further support to a **conjecture** that *generically (i.e. for groups of high enough rank or fields of large enough positive characteristic) the minimal covolume is always attained on a non-uniform lattice.*

**1.7.** In [3] M. Conder and C. Maclachlan constructed a compact orientable hyperbolic 4-manifold which has Euler characteristic 16. The previously known smallest example which was used in order to formulate the main result in [1, Section 5] had  $\chi = 26$ . The construction of [3] agrees with our Theorem 5.5 and it also allows us to give a more precise formulation of the theorem:

**Theorem 5.5'.** *If there exists a compact orientable arithmetic hyperbolic 4-manifold  $M$  with  $\chi(M) \leq 16$ , then  $M$  is defined over  $\mathbb{Q}[\sqrt{5}]$  and has the form  $\Gamma_M \backslash \mathcal{H}^4$  with  $\Gamma_M$  being a torsion-free subgroup of index  $7200\chi(M)$  of the group  $\Gamma_1$  of the smallest arithmetic hyperbolic 4-orbifold.*

## References

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