

On surfaces with $p_g = q = 1$ and non-ruled bicanonical involution

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Abstract. This paper classifies surfaces S of general type with $p_g = q = 1$ having an involution i such that S/i has non-negative Kodaira dimension and that the bicanonical map of S factors through the double cover induced by i .

It is shown that S/i is regular and either: a) the Albanese fibration of S is of genus 2 or b) S has no genus 2 fibration and S/i is birational to a $K3$ surface. For case a) a list of possibilities and examples are given. An example for case b) with $K^2 = 6$ is also constructed.

Mathematics Subject Classification (2000): 14J29.

1. Introduction

Let S be a smooth irreducible projective surface of general type. The *pluricanonical map* ϕ_n of S is the map given by the linear system $|nK_S|$, where K_S is the canonical divisor of S . For minimal surfaces S , ϕ_n is a birational morphism if $n \geq 5$ (cf. [4, Chapter VII, Theorem (5.2)]). The *bicanonical map*

$$\phi_2 : S \longrightarrow \mathbb{P}^{K_S^2 + \chi(S) - 1}$$

is a morphism if $p_g(S) \geq 1$ (this result is due to various authors, see [7] for more details). This paper focuses on the study of surfaces S with $p_g(S) = q(S) = 1$ having an involution i such that the Kodaira dimension of S/i is non-negative and ϕ_2 is composed with i , *i.e.* it factors through the double cover $p : S \rightarrow S/i$.

There is an instance where the bicanonical map is necessarily composed with an involution: suppose that S has a fibration of genus 2, *i.e.* it has a morphism f from S to a curve such that a general fibre F of f is irreducible of genus 2. The

Part of this work began during a one month stay in Warwick, in July 2004, which was partially supported by the european contract EAGER no. HPRN-CT-2000-00099 and the University of Warwick. The author is a collaborator of the Center for Mathematical Analysis, Geometry and Dynamical Systems of Instituto Superior Técnico, Universidade Técnica de Lisboa and was partially supported by FCT (Portugal) through program POCTI/FEDER and Project POCTI/MAT/44068/2002. The author also thanks the Departamento de Matemática da Universidade de Trás-os-Montes e Alto Douro.

Received October 18, 2005; accepted in revised form December 18, 2006.

system $|2K_S|$ cuts out on F a subseries of the bicanonical series of F , which is composed with the hyperelliptic involution of F , and then ϕ_2 is composed with an involution. This is the so called *standard case* of non-birationality of the bicanonical map.

By the results of Bombieri, [2], improved later by Reider, [22], a minimal surface S satisfying $K^2 > 9$ and ϕ_2 non-birational necessarily presents the standard case of non-birationality of the bicanonical map.

The non-standard case of non-birationality of the bicanonical map, *i.e.* the case where ϕ_2 is non-birational and the surface has no genus 2 fibration, has been studied by several authors.

Du Val, [16], classified the regular surfaces S of general type with $p_g \geq 3$, whose general canonical curve is smooth and hyperelliptic. Of course, for these surfaces, the bicanonical map is composed with an involution i such that S/i is rational. The families of surfaces exhibited by Du Val, presenting the non-standard case, are nowadays called the *Du Val examples*.

Other authors have later studied the non-standard case: the articles [8, 10, 12, 13, 25] and [3] treat the cases $\chi(\mathcal{O}_S) > 1$ or $q(S) \geq 2$ (*cf.* the expository paper [11] for more information on this problem).

Xiao Gang, [25], presented a list of possibilities for the non-standard case of non-birationality of the bicanonical morphism ϕ_2 . For the case when ϕ_2 has degree 2 and the bicanonical image is a ruled surface, Theorem 2 of [25] extended Du Val's list to $p_g(S) \geq 1$ and added two extra families (this result is still valid assuming only that ϕ_2 is composed with an involution such that the quotient surface is a ruled surface). Recently G. Borrelli [3] excluded these two families, confirming that the only possibilities for this instance are the Du Val examples.

For irregular surfaces the following holds (see [25, Theorems 1, 3], [8, Theorem A], [12, Theorem 1.1], [13]):

Suppose that S is a smooth minimal irregular surface of general type having non-birational bicanonical map. If $p_g(S) \geq 2$ and S has no genus 2 fibration, then only the following (effective) possibilities occur:

- $p_g(S) = q(S) = 2$, $K_S^2 = 4$;
- $p_g(S) = q(S) = 3$, $K_S^2 = 6$.

In both cases ϕ_2 is composed with an involution i such that $\text{Kod}(S/i) = 2$.

This paper completes this result classifying the minimal surfaces S with $p_g(S) = q(S) = 1$ such that ϕ_2 is composed with an involution i satisfying $\text{Kod}(S/i) \geq 0$.

The main result is the following:

Theorem 1.1. *Let S be a smooth minimal irregular surface of general type with an involution i such that $\text{Kod}(S/i) \geq 0$ and the bicanonical map ϕ_2 of S is composed with i . If $p_g(S) = q(S) = 1$, then only the following possibilities can occur:*

- a) S/i is regular, the Albanese fibration of S has genus 2 and
 - (i) $\text{Kod}(S/i) = 2$, $\chi(S/i) = 2$, $K_S^2 = 2$, $\deg(\phi_2) = 8$, or
 - (ii) $\text{Kod}(S/i) = 1$, $\chi(S/i) = 2$, $2 \leq K_S^2 \leq 4$, $\deg(\phi_2) \geq 4$, or
 - (iii) S/i is birational to a K3 surface, $3 \leq K_S^2 \leq 6$, $\deg(\phi_2) = 4$;
- b) S has no genus 2 fibration and S/i is birational to a K3 surface.

Moreover, there are examples for (i), (ii) with $K_S^2 = 4$, (iii) with $K_S^2 = 3, 4$ or 5 and for b) with $K_S^2 = 6$ and ϕ_2 of degree 2.

Remark 1.2. Examples for (iii) were given by Catanese in [6]. The other examples will be presented in Section 5.

Note that surfaces of general type with $p_g = q = 1$ and $K^2 = 3$ or 8 were also studied by Polizzi in [19] and [20].

In the example in Section 5 for case b) of Theorem 1.1, S has $p_g = q = 1$ and $K^2 = 6$. This seems to be the first construction of a surface with these invariants. This example contradicts a result of Xiao Gang. More precisely, the list of possibilities in [25] rules out the case where S has no genus 2 fibration, $p_g(S) = q(S) = 1$ and S/i is birational to a K3 surface. In Lemma 7 of [25] it is written that R has only negligible singularities, but the possibility $\chi(K_{\tilde{P}} + \tilde{\delta}) < 0$ in formula (3) of page 727 was overlooked. In fact we will see that R (\overline{B} in our notation) can have a non-negligible singularity.

An important technical tool that will be used several times is the *canonical resolution* of singularities of a surface. This is a resolution of singularities as described in [4].

The paper is organized as follows. Section 2 studies some general properties of a surface of general type S with an involution i . Section 3 states some properties of surfaces with $p_g = q = 1$. Section 4 contains the proof of Theorem 1.1. Crucial ingredients for this proof are the existence of the Albanese fibration of S and the formulas of Section 2. In Section 5 examples for Theorem 1.1 are obtained, via the construction of branch curves with appropriate singularities. The Computational Algebra System *Magma* is used to perform the necessary calculations (visit <http://magma.maths.usyd.edu.au/magma> for more information about Magma).

Notation and conventions. We work over the complex numbers; all varieties are assumed to be projective algebraic. We do not distinguish between line bundles and divisors on a smooth variety. Linear equivalence is denoted by \equiv . A *nodal curve* or *(-2)-curve* C on a surface is a curve isomorphic to \mathbb{P}^1 such that $C^2 = -2$. Given a surface X , $\text{Kod}(X)$ means the *Kodaira dimension* of X . We say that a curve singularity is *negligible* if it is either a double point or a triple point which resolves to at most a double point after one blow-up. A *(n, n) point*, or *point of type (n, n)*, is a point of multiplicity n with an infinitely near point also of multiplicity n . An *involution* of a surface S is an automorphism of S of order 2. We say that a map is composed with an involution i of S if it factors through the map $S \rightarrow S/i$. The rest of the notation is standard in Algebraic Geometry.

ACKNOWLEDGEMENTS. First of all I thank my supervisor Margarida Mendes Lopes for all her support. I would also like to thank Miles Reid for the useful conversations and the hospitality in Warwick where I started learning the Computational System Magma.

2. Generalities on involutions

Let S be a smooth minimal surface of general type with an involution i . As S is minimal of general type, this involution is biregular. The fixed locus of i is the union of a smooth curve R'' (possibly empty) and of $t \geq 0$ isolated points P_1, \dots, P_t . Let S/i be the quotient of S by i and $p : S \rightarrow S/i$ be the projection onto the quotient. The surface S/i has nodes at the points $Q_i := p(P_i)$, $i = 1, \dots, t$, and is smooth elsewhere. If $R'' \neq \emptyset$, the image via p of R'' is a smooth curve B'' not containing the singular points Q_i , $i = 1, \dots, t$. Let now $h : V \rightarrow S$ be the blow-up of S at P_1, \dots, P_t and set $R' = h^*R''$. The involution i induces a biregular involution \tilde{i} on V whose fixed locus is $R := R' + \sum_1^t h^{-1}(P_i)$. The quotient $W := V/\tilde{i}$ is smooth and one has a commutative diagram:

$$\begin{array}{ccc} V & \xrightarrow{h} & S \\ \pi \downarrow & & \downarrow p \\ W & \xrightarrow{g} & S/i \end{array}$$

where $\pi : V \rightarrow W$ is the projection onto the quotient and $g : W \rightarrow S/i$ is the minimal desingularization map. Notice that

$$A_i := g^{-1}(Q_i), \quad i = 1, \dots, t,$$

are (-2) -curves and $\pi^*(A_i) = 2 \cdot h^{-1}(P_i)$. Set $B' := g^*(B'')$. Because π is a double cover with branch locus $B' + \sum_1^t A_i$, there exists a line bundle L on W such that

$$2L \equiv B := B' + \sum_1^t A_i.$$

It is well known that (cf. [4, Chapter V, Section 22]):

$$p_g(S) = p_g(V) = p_g(W) + h^0(W, \mathcal{O}_W(K_W + L)),$$

$$q(S) = q(V) = q(W) + h^1(W, \mathcal{O}_W(K_W + L))$$

and

$$K_S^2 - t = K_V^2 = 2(K_W + L)^2, \tag{2.1}$$

$$\chi(\mathcal{O}_S) = \chi(\mathcal{O}_V) = 2\chi(\mathcal{O}_W) + \frac{1}{2}L(K_W + L).$$

Furthermore, from the papers [12] and [9], if S is a smooth minimal surface of general type with an involution i , then

$$\chi(\mathcal{O}_W(2K_W + L)) = h^0(W, \mathcal{O}_W(2K_W + L)), \quad (2.2)$$

$$\chi(\mathcal{O}_W) - \chi(\mathcal{O}_S) = K_W(K_W + L) - h^0(W, \mathcal{O}_W(2K_W + L)) \quad (2.3)$$

and the bicanonical map

$$\phi_2 \text{ is composed with } i \text{ if and only if } h^0(W, \mathcal{O}_W(2K_W + L)) = 0. \quad (2.4)$$

From formulas (2.1) and (2.3) one obtains the number t of nodes of S/i :

$$t = K_S^2 + 6\chi(\mathcal{O}_W) - 2\chi(\mathcal{O}_S) - 2h^0(W, \mathcal{O}_W(2K_W + L)). \quad (2.5)$$

Let P be a minimal model of the resolution W of S/i and $\rho : W \rightarrow P$ be the natural projection. Denote by \bar{B} the projection $\rho(B)$ and by δ the “projection” of L .

Remark 2.1. Resolving the singularities of \bar{B} we obtain exceptional divisors E_i and numbers $r_i \in 2\mathbb{N}^+$ such that $E_i^2 = -1$, $K_W = \rho^*(K_P) + \sum E_i$ and $B = \rho^*(\bar{B}) - \sum r_i E_i$.

Proposition 2.2. *With the previous notations, the bicanonical map ϕ_2 is composed with i if and only if*

$$\chi(\mathcal{O}_P) - \chi(\mathcal{O}_S) = K_P(K_P + \delta) + \frac{1}{2} \sum (r_i - 2).$$

Proof. From formulas (2.3), (2.4) and Remark 2.1 we get

$$\begin{aligned} \chi(\mathcal{O}_P) - \chi(\mathcal{O}_S) &= \frac{1}{2} K_W(2K_W + 2L) \\ &= \frac{1}{2} \left(\rho^*(K_P) + \sum E_i \right) \left(2\rho^*(K_P + \delta) + \sum (2 - r_i) E_i \right) \\ &= K_P(K_P + \delta) + \frac{1}{2} \sum (r_i - 2). \quad \square \end{aligned}$$

3. Surfaces with $p_g = q = 1$ and an involution

Let S be a minimal smooth projective surface of general type satisfying $p_g(S) = q(S) = 1$.

Note that then $2 \leq K_S^2 \leq 9$: we have $K_S^2 \leq 9\chi(\mathcal{O}_S)$ by the Myiaoka-Yau inequality (see [4, Chapter VII, Theorem (4.1)]) and $K_S^2 \geq 2p_g$ for an irregular surface (see [14]).

Furthermore, if the bicanonical map of S is not birational, then $K_S^2 \neq 9$. In fact, by [12], if $K_S^2 = 9$ and ϕ_2 is not birational, then S has a genus 2 fibration, while Théorème 2.2 of [24] implies that if S has a genus 2 fibration and $p_g(S) = q(S) = 1$, then $K_S^2 \leq 6$.

Since $q(S) = 1$ the Albanese variety of S is an elliptic curve E and the Albanese map is a connected fibration (see e. g. [1] or [4]).

Suppose that S has an involution i . Then i preserves the Albanese fibration (because $q(S) = 1$) and so we have a commutative diagram

$$\begin{array}{ccccc}
 V & \xrightarrow{h} & S & \longrightarrow & E \\
 \pi \downarrow & & \downarrow p & & \downarrow \\
 W & \longrightarrow & S/i & \longrightarrow & \Delta
 \end{array} \tag{3.1}$$

where Δ is a curve of genus ≤ 1 . Denote by

$$f_A : W \rightarrow \Delta$$

the fibration induced by the Albanese fibration of S .

Recall that

$$\rho : W \rightarrow P$$

is the projection of W onto its minimal model P and

$$\overline{B} := \rho(B),$$

where $B := B' + \sum_1^t A_i \subset W$ is the branch locus of π .

Let

$$\overline{B'} := \rho(B'), \quad \overline{A_i} = \rho(A_i).$$

When \overline{B} has only negligible singularities, the map ρ contracts only exceptional curves contained in fibres of f_A . In fact, there exists otherwise a (-1) -curve $J \subset W$ such that $JB = 2$ and so $\pi^*(J)$ is a rational curve transverse to the fibres of the (genus 1 base) Albanese fibration of S , which is impossible. Moreover, ρ contracts no curve meeting $\sum A_i$, because $h : V \rightarrow S$ is the contraction of isolated (-1) -curves. Therefore the singularities of \overline{B} are exactly the singularities of $\overline{B'}$, i.e. $\overline{B'} \cap \sum \overline{A_i} = \emptyset$. In this case the image of f_A on P will be denoted by $\overline{f_A}$.

If $\Delta \cong \mathbb{P}^1$, then the double cover $E \rightarrow \Delta$ is ramified over 4 points p_j of Δ , thus the branch locus $B' + \sum_1^t A_i$ is contained in 4 fibres

$$F_A^j := f_A^*(p_j), \quad j = 1, \dots, 4,$$

of the fibration f_A . Hence, by Zariski's Lemma (see e. g. [4]), the irreducible components B'_i of B' satisfy $B_i'^2 \leq 0$. If \overline{B} has only negligible singularities, then also $\overline{B'}^2 \leq 0$. As $\pi^*(F_A^j)$ is of multiplicity 2, each component of F_A^j which is not a component of the branch locus $B' + \sum_1^t A_i$ must be of even multiplicity.

4. The classification theorem

In this section we will prove Theorem 1.1. We will freely use the notation and results of Sections 2 and 3.

Proof of Theorem 1.1. Since $p_g(P) \leq p_g(S) = 1$, then $\chi(\mathcal{O}_P) \leq 2 - q(P) \leq 2$. Proposition 2.2 gives $\chi(\mathcal{O}_P) \geq 1$, because K_P is nef (i.e. $K_P C \geq 0$ for every curve C). So from Proposition 2.2 and the classification of surfaces (see e. g. [1] or [4]) only the following cases can occur:

1. P is of general type;
2. P is a surface with Kodaira dimension 1;
3. P is an Enriques surface, \overline{B} has only negligible singularities;
4. P is a K3 surface, \overline{B} has a 4-uple or (3, 3) point, and possibly negligible singularities.

We will show that: case 3 does not occur, in cases 1 and 2 the Albanese fibration has genus 2 and only in case 4 the Albanese fibration can have genus $\neq 2$.

Each of cases 1,..., 4 will be studied separately. We start by considering:

Case 1. As P is of general type, $K_P^2 \geq 1$ and K_P is nef, Proposition 2.2 gives $\chi(\mathcal{O}_P) = 2$, $K_P^2 = 1$, $K_P \delta = 0$ and \overline{B} has only negligible singularities. The equality $K_P \overline{B}' = K_P 2\delta = 0$ implies $\overline{B}'^2 < 0$ when $B' \not\equiv 0$. In the notation of Remark 2.1 one has $K_W \equiv \rho^*(K_P) + \sum E_i$ and $B' = \rho^*(\overline{B}') - 2 \sum E_i$. So

$$\begin{aligned} K_S^2 &= K_V^2 + t = \frac{1}{4}(2K_V)^2 + t = \frac{1}{4}\pi^*(2K_W + B)^2 + t \\ &= \frac{1}{2}(2K_W + B)^2 + t = \frac{1}{2}(2K_W + B')^2 = \frac{1}{2}(2K_P + \overline{B}')^2 = \frac{1}{2}(4 + \overline{B}'^2). \end{aligned}$$

Since $K_S^2 \geq 2p_g(S)$ for an irregular surface (see [14]), $\overline{B}'^2 < 0$ is impossible, hence $B' = 0$ and $K_S^2 = 2$. By [5] minimal surfaces of general type with $p_g = q = 1$ and $K^2 = 2$ have Albanese fibration of genus 2. This is case (i) of Theorem 1.1. We will see in Section 5 an example for this case.

Finally the fact that $\deg(\phi_2) = 8$ follows immediately because ϕ_2 is a morphism onto \mathbb{P}^2 and $(2K_S)^2 = 8$.

Next we exclude:

Case 3. Using the notation of Remark 2.1 of Section 2, we can write $K_W \equiv \rho^*(K_P) + \sum E_i$ and $2L \equiv \rho^*(2\delta) - 2 \sum E_i$, for some exceptional divisors E_i . Hence

$$\begin{aligned} L(K_W + L) &= \frac{1}{2}L(2K_W + 2L) \\ &= \frac{1}{2}(\rho^*(\delta) - \sum E_i)(2\rho^*(K_P) + \rho^*(2\delta)) = \frac{1}{2}\delta(2K_P + 2\delta) = \delta^2 \end{aligned}$$

and then, from (2.1), $\delta^2 = -2$. Now (2.4) and (2.5) imply $t = K_S^2 + 4$, thus

$$\overline{B'}^2 = \overline{B}^2 + 2t = (2\delta)^2 + 2t = -8 + 2t = 2K_S^2 > 0.$$

This is a contradiction because we have seen that $\overline{B'}^2 \leq 0$ when $\overline{B'}$ has only negligible singularities. Thus case 3) does not occur.

Now we focus on:

Case 2. Since we are assuming that $\text{Kod}(P) = 1$, P has an elliptic fibration (*i.e.* a morphism $f_e : P \rightarrow C$ where C is a curve and the general fibre of f_e is a smooth connected elliptic curve). Then K_P is numerically equivalent to a rational multiple of a fibre of f_e (see e. g. [1] or [4]). As $K_P\delta \geq 0$, Proposition 2.2, together with $\chi(\mathcal{O}_P) \leq 2$, yield $K_P\delta = 0$ or 1.

Denote by F_e (respectively F_A) a general fibre of f_e (respectively f_A) and let $\overline{F_A} := \rho(F_A)$. If $K_P\delta = 0$, then $F_e\overline{B} = 0$, which implies that the fibration f_e lifts to an elliptic fibration on S . This is impossible because S is a surface of general type. So $K_P\delta = 1$ and, since $p_g(P) \leq p_g(S) = 1$, the only possibility allowed by Proposition 2.2 is

$$p_g(P) = 1, q(P) = 0 \text{ and } \overline{B} \text{ has only negligible singularities.}$$

Now $q(P) = 0$ implies that the elliptic fibration f_e has a rational base, thus the canonical bundle formula (see e. g. [4, Chapter V, Section 12]) gives $K_P \equiv \sum (m_i - 1)F_i$, where $m_i F_i$ are the multiple fibres of f_e . From

$$2 = 2\delta K_P = \overline{B'}K_P = \overline{B'} \sum (m_i - 1)F_i, \quad \overline{B'}F_i \equiv 0 \pmod{2}$$

we get

$$K_P \equiv \frac{1}{2}F_e.$$

Since \overline{B} has only negligible singularities, $\overline{B'}^2 \leq 0$ and then

$$2K_S^2 = (2K_W + B')^2 = \rho^* \left(2K_P + \overline{B'} \right)^2 = 8 + \overline{B'}^2 \leq 8. \quad (4.1)$$

Therefore $2 \leq K_S^2 \leq 4$. If $K_S^2 = 2$, then the Albanese fibration of S is of genus 2, by [5]. So, to prove statement a), (ii) of Theorem 1.1, we must show that for $K_S^2 = 3$ or 4 the Albanese fibration of S has genus 2. We will study each of these cases separately.

First we consider

• $K_S^2 = 4$

Let $\overline{F_A^i} := \rho(F_A^i)$, $i = 1, \dots, 4$.

Claim 4.1. If f_A is not a genus 2 fibration then

$$\overline{F_A^j} = 2\overline{B'},$$

for some $j \in \{1, \dots, 4\}$.

Proof. By formula (4.1) $\overline{B'}^2 = 0$, and so $\overline{B'}$ contains the support of $x \geq 1$ of the $\overline{F_A^i}$'s. The facts $K_P \overline{F_A} > 0$ (because $g(\overline{F_A}) \geq 2$) and $K_P \overline{B'} = 2$ imply $x = 1$, i.e. $\overline{F_A^j} = k\overline{B'}$, for some $j \in \{1, \dots, 4\}$ and $k \in \mathbb{N}^+$. If $k = 1$ then $\overline{F_A} K_P = 2$, thus $\overline{F_A}$ is of genus 2 and S is as in case (ii) of Theorem 1.1.

Suppose now $k \geq 2$. Then each irreducible component of the divisor

$$D := \overline{F_A^1} + \dots + \overline{F_A^4}$$

whose support is not in $\sum_1^4 \overline{A_i}$ is of multiplicity greater than 1. The fibration $\overline{f_A}$ gives a cover $F_e \rightarrow \mathbb{P}^1$ of degree $\overline{F_A} F_e$, for a general fibre F_e of the elliptic fibration f_e . The Hurwitz formula (see e. g. [17]) says that the ramification degree r of this cover is $2\overline{F_A} F_e$. Let p_1, \dots, p_n be the points in $F_e \cap D$ and α_i be the intersection number of F_e and D at p_i . Of course $F_e D = 4\overline{F_A} F_e = \sum_1^n \alpha_i$ and then $F_e \cap \sum \overline{A_i} = \emptyset$ implies $\alpha_i \geq 2, i = 1, \dots, n$. We have

$$2\overline{F_A} F_e = r \geq \sum_1^n (\alpha_i - 1) = \sum_1^n \alpha_i - n = 4\overline{F_A} F_e - n,$$

i.e. $n \geq 2\overline{F_A} F_e$. The only possibility is $n = 2\overline{F_A} F_e$ and $\alpha_i = 2 \forall i$, which means that every component Γ of D such that $\Gamma F_e \neq 0$ is exactly of multiplicity 2. In particular an irreducible component of $\overline{B'}$ is of multiplicity 2, thus $k = 2$, i.e. $\overline{F_A^j} = 2\overline{B'}$. □

Claim 4.2. There is a smooth rational curve C contained in a fibre F_C of the elliptic fibration f_e , and not contained in fibres of $\overline{f_A}$, such that

$$m := \widehat{C} \sum_1^t A_i \leq 3, \tag{4.2}$$

where \widehat{C} is the strict transform of C in W .

Proof. Since $\overline{A_i} F_e = \overline{A_i} 2K_P = 0$, then each $\overline{A_i}$ is contained in a fibre of f_e , and in particular the elliptic fibration f_e has reducible fibres. Denote by C an irreducible component of a reducible fibre F_C of f_e , by ξ the multiplicity of C in F_C and by \widehat{C} the strict transform of C in W . If the intersection number of C and the support of $F_C - \xi C$ is greater than 3 then, from the configurations of singular fibres of an elliptic fibration (see e. g. [4, Chapter V, Section 7]), F_C must be of type I_0^* ,

i.e. it has the following configuration: it is the union of four disjoint (-2) -curves $\theta_1, \dots, \theta_4$ with a (-2) -curve θ , with multiplicity 2, such that $\theta\theta_i = 1, i = 1, \dots, 4$.

So if $\widehat{C} \sum_1^t A_i > 3$, the fibre F_C containing C is of type I_0^* with $\widehat{C} \sum_1^t A_i = 4$. Since the number of nodes of S/i is $t = K_S^2 + 10 = 14 \not\equiv 0 \pmod{4}$, there must be a reducible fibre such that for every component $C \not\subset \sum_1^t \overline{A}_i$, $\widehat{C} \sum_1^t A_i \leq 3$. As $f_e \neq \overline{f}_A$ and the \overline{A}_i 's are contained in fibres of f_e and in fibres of \overline{f}_A , we can choose C not contained in fibres of \overline{f}_A . \square

Let C be as in Claim 4.2 and consider the resolution $\widetilde{V} \rightarrow V$ of the singularities of $\pi^*(\widehat{C})$. Let $G \subset \widetilde{V}$ be the strict transform of $\pi^*(\widehat{C})$. Notice that G has multiplicity 1, because C transverse to the fibres of \overline{f}_A implies $C \not\subset \overline{B}$. Recall that E denotes the basis of the Albanese fibration of S .

Claim 4.3. The Albanese fibration of \widetilde{V} induces a cover $G \rightarrow E$ with ramification degree

$$r := K_{\widetilde{V}}G + G^2.$$

Proof. Let G_1, \dots, G_h be the connected (hence smooth) components of G . The curve C is not contained in fibres of \overline{f}_A , thus G is not contained in fibres of the Albanese fibration of \widetilde{V} . This fibration induces a cover $G_i \rightarrow E$ with ramification degree, from the Hurwitz formula,

$$r_i = 2g(G_i) - 2 = K_{\widetilde{V}}G_i + G_i^2.$$

This way we have a cover $G \rightarrow E$ with ramification degree

$$r = \sum r_i = K_{\widetilde{V}}(G_1 + \dots + G_h) + (G_1^2 + \dots + G_h^2) = K_{\widetilde{V}}G + G^2. \quad \square$$

We are finally in position to show that $g(F_A) = 2$.

Let $n := \widehat{C}B'$. We have

$$\begin{aligned} 2K_V\pi^*(\widehat{C}) &= \pi^*(2K_W + B' + \sum A_i)\pi^*(\widehat{C}) \\ &= 2(2K_W + B' + \sum A_i)\widehat{C} = 4K_W\widehat{C} + 2(B' + \sum A_i)\widehat{C} \\ &= 4(-2 - \widehat{C}^2) + 2(n + m) = -8 - 2\pi^*(\widehat{C})^2 + 2(n + m), \end{aligned}$$

i.e.

$$K_V\pi^*(\widehat{C}) + \pi^*(\widehat{C})^2 = n + m - 4.$$

Suppose that $g(F_A) \neq 2$. Let $\Lambda \subset V$ be the double Albanese fibre induced by $\overline{F}_A^j = 2\overline{B}'$ (as in Claim 4.1) and $\widetilde{\Lambda} \subset \widetilde{V}$ be the total transform of Λ . From

$$G\widetilde{\Lambda} = \pi^*(\widehat{C})\Lambda \geq \pi^*(\widehat{C})\pi^*(B') = 2n$$

one has $r \geq n$. Then

$$n + m - 4 = K_V \pi^*(\widehat{C}) + \pi^*(\widehat{C})^2 \geq K_{\widehat{V}} G + G^2 = r \geq n$$

and so $m \geq 4$, which contradicts Claim 4.2.

So if $K_S^2 = 4$, then the Albanese fibration of S is of genus 2.

We will now consider the possibility

• $K_S^2 = 3$

In this case a general Albanese fibre Λ has genus 2 or 3 (see [7]). Suppose then $g(\Lambda) = 3$. Surfaces S with $K_S^2 = g(\Lambda) = 3$ are studied in detail in [7]. There (see also [18]) it is shown that the relative canonical map γ , given by $|K_S + n\Lambda|$ for some n , is a morphism.

We know that $K_P \overline{B}' = 2$ and $\overline{B}'^2 = -2$, by (4.1). We have already seen that \overline{B} has only negligible singularities (which means $r_i = 2 \forall i$, in the notation of Remark 2.1) and then ρ contracts no curve meeting $\sum A_i$. Let R' be the support of $\pi^*(B')$.

Claim 4.4. We have

$$K_V R' = 1.$$

Proof.

$$\begin{aligned} 2K_V \cdot 2R' &= \pi^*(2K_W + B)\pi^*(B') = 2(2K_W + B)B' \\ &= 2(2K_W + B')B' = 2\left(2\rho^*(K_P) + \rho^*(\overline{B}')\right)\left(\rho^*(\overline{B}') - \sum 2E_i\right) \\ &= 2(2K_P + \overline{B}')\overline{B}' = 2(4 - 2) = 4, \end{aligned}$$

thus $K_V R' = 1$. □

As the map

$$\gamma \circ h : V \longrightarrow \gamma(S)$$

is a birational morphism, $\gamma \circ h(R')$ is a line (plus possibly some isolated points). This way there exists a smooth rational curve $\beta \subset B'$ such that

$$K_V \widetilde{\beta} = 1,$$

where $\widetilde{\beta} \subset R'$ is the support of $\pi^*(\beta)$. The adjunction formula gives $\widetilde{\beta}^2 = -3$, thus $\beta^2 = -6$. Notice that $\widetilde{\beta}$ is the only component of R' which is not contracted by the map $\gamma \circ h$.

Let

$$\begin{aligned} \alpha &:= B' - \beta \subset W, \\ \overline{\beta} &:= \rho(\beta), \quad \overline{\alpha} := \rho(\alpha) \subset P. \end{aligned}$$

When α is non-empty, the support of $\pi^*(\alpha)$ is an union of (-2) -curves, since it is contracted by $\gamma \circ h$. Equivalently α is a disjoint union of (-4) -curves.

Claim 4.5. We have

$$K_W^2 \geq -2.$$

Proof. Consider the Chern number c_2 and the second Betti number b_2 . It is well known that, for a surface X ,

$$c_2(X) = 12\chi(\mathcal{O}_X) - K_X^2, \quad b_2(X) = c_2(X) - 2 + 4q(X).$$

Therefore

$$b_2(W) = 22 - K_W^2, \quad b_2(V) = b_2(S) + t = 11 + 13 = 24.$$

The inequality $K_W^2 \geq -2$ follows from the fact $b_2(V) \geq b_2(W)$. \square

From Claim 4.5, we conclude that the resolution of $\overline{B'}$ blows-up at most two double points, thus

$$B'^2 \geq -2 + 2(-4) = -10 = \beta^2 + (-4).$$

This implies that α is a smooth (-4) -curve when $\alpha \neq 0$.

Claim 4.6. Only the following possibilities can occur:

- $\overline{\beta}$ has one double point and no other singularity, or
- $\overline{\alpha}, \overline{\beta}$ are smooth, $\overline{\alpha}\overline{\beta} = 2$.

Proof. Recall that $B' = \alpha + \beta$ is contained in fibres of f_A and, since $\overline{B'}$ has only negligible singularities, then also $\overline{B'} = \overline{\alpha} + \overline{\beta}$ is contained in fibres of $\overline{f_A}$. In particular $\overline{\alpha}^2, \overline{\beta}^2 \leq 0$.

If $\overline{\alpha}$ is singular, then it has arithmetic genus $p_a(\overline{\alpha}) = 1$ and $\overline{\alpha}^2 = 0$. But then $\overline{\alpha}$ has the same support of a fibre of $\overline{f_A}$, which is a contradiction because $\overline{f_A}$ is not elliptic. Therefore $\overline{\alpha}$ is smooth.

Since $K_P\overline{\alpha} \geq 0$, $K_P\overline{B'} = 2$ implies $K_P\overline{\beta} \leq 2$. We know that β is a smooth rational curve and $\beta^2 = -6$, thus $K_W\beta = 4$. If $\overline{\beta}$ is smooth, then one must have $\overline{\alpha}\overline{\beta} > 1$. From Claim 4.5 the only possibility in this case is $\overline{\alpha}\overline{\beta} = 2$. If $\overline{\beta}$ is singular, then $\overline{\beta}^2 \leq 0$ implies that $\overline{\beta}$ has one ordinary double point and no other singularity. \square

Let $D := \overline{\beta}$ if $\overline{\beta}$ is singular. Otherwise let $D := \overline{\alpha} + \overline{\beta}$.

The 2-connected divisor $\tilde{D} := \frac{1}{2}(\rho \circ \pi)^*(D)$ has arithmetic genus $p_a(\tilde{D}) = 1$. We know that $(K_V + n\Lambda)\tilde{D} = 1$ (because $K_V R' = 1$) and that \tilde{D} contains a component A such that $(K_V + n\Lambda)A = 0$ (because D has at least one negligible singularity). These two facts imply, from [10, Proposition A.5, (ii)], that the relative canonical map γ has a base point in \tilde{D} . As mentioned above, γ is a morphism, which is a contradiction.

Finally the assertion about $\deg(\phi_2)$ in Case 2.: we have proved that S has a genus 2 fibration, so it has an hyperelliptic involution j . The bicanonical map ϕ_2 factors through both i and j , thus $\deg(\phi_2) \geq 4$.

This finishes the proof of case a), (ii) of Theorem 1.1.

We end the proof of Theorem 1.1 with Case a), (iii): A surface of general type with a genus 2 fibration and $p_g = q = 1$ satisfies $K^2 \leq 6$ (see [24]). Denote by j the map such that $\phi_2 = j \circ i$. The quotient S/i is a $K3$ surface thus, from [15], $\deg(j) \leq 2$. Analogously to Case 2, $\deg(\phi_2) \geq 4$, thus $\deg(j) = 2$, $\deg(\phi_2) = 4$ and then $K_S^2 \neq 2$ (see Case 1).

It follows from [24, page 66] that, if the genus 2 fibration of S has a rational basis, then $K_S^2 = 3$. It is shown in [19] that, in these conditions, $\deg(\phi_2) = 2$. We then conclude that the genus 2 fibration of S is the Albanese fibration.

Examples for case a), (iii) with $K_S^2 = 3, 4$ or 5 were given by Catanese in [6]. The existence of the other cases is proved in the next section. \square

5. Examples

In this section we will construct smooth minimal surfaces of general type S with $p_g(S) = q(S) = 1$ having an involution i such that the bicanonical map ϕ_2 of S is composed with i and:

- 1) $K_S^2 = 6, g = 3, \deg(\phi_2) = 2, S/i$ is birational to a $K3$ surface;
- 2) $K_S^2 = 4, g = 2, \text{Kod}(S/i) = 1$;
- 3) $K_S^2 = 2, g = 2, \text{Kod}(S/i) = 2$,

where g denotes the genus of the Albanese fibration of S .

Example 5.1. In [23] Todorov gives the following construction of a surface of general type S with $p_g(S) = 1, q(S) = 0$ and $K_S^2 = 8$. Consider a Kummer surface Q in \mathbb{P}^3 , i.e. a quartic having as singularities only 16 nodes (ordinary double points). Let $G \subset Q$ be the intersection of Q with a general quadric, \tilde{Q} be the minimal resolution of Q and $\tilde{G} \subset \tilde{Q}$ be the pullback of G . The surface S is the minimal model of the double cover $\pi : V \rightarrow \tilde{Q}$ ramified over $\tilde{G} + \sum_1^{16} A_i$, where $A_i \subset \tilde{Q}, i = 1, \dots, 16$, are the (-2) -curves which contract to the nodes of Q .

It follows from the double cover formulas (cf. [4, Chapter V, Section 22]) that the imposition of a quadruple point to the branch locus decreases K^2 by 2 and the Euler characteristic χ by 1.

We will see that we can impose a quadruple point to the branch locus of the Todorov construction, thus obtaining S with $K_S^2 = 6$. In this case I claim that $p_g(S) = q(S) = 1$. In fact, let W be the surface \tilde{Q} blown-up at the quadruple point, E be the corresponding (-1) -curve, B be the branch locus and L be the line bundle such that $2L \equiv B$. From formula (2.3) in Section 2, one has

$h^0(W, \mathcal{O}_W(2E + L)) = 0$ (thus the bicanonical map of V factors through π), hence also $h^0(W, \mathcal{O}_W(E + L)) = 0$ and then

$$p_g(S) = p_g(W) + h^0(W, \mathcal{O}_W(E + L)) = 1.$$

We will see that $\deg(\phi_2) = 2$, hence $\phi_2(S)$ is a $K3$ surface and so S has no genus 2 fibration.

First we need to obtain an equation of a Kummer surface. The Computational Algebra System *Magma* has a direct way to do this, but I prefer to do it using a beautiful construction that I learned from Miles Reid.

We want a quartic surface $Q \in \mathbb{P}^3$ whose singularities are exactly 16 nodes. Projecting from one of the nodes to \mathbb{P}^2 , one realizes the “Kummer” surface as a double cover

$$\psi : X \longrightarrow \mathbb{P}^2$$

with branch locus the union of 6 lines L_i (see [17, page 774]), each one tangent to a conic C (the image of the projection point) at a point p_i . The surface X contains 15 nodes (from the intersection of the lines) and two (-2) -curves (the pullback $\psi^*(C)$) disjoint from these nodes. To obtain a Kummer surface we have just to contract one of these curves.

Denote also by L_i the defining polynomial of each line L_i . An equation for X is $z^2 = L_1 \cdots L_6$ in the weighted projective space $\mathbb{P}(3, 1, 1, 1)$, with coordinates (z, x_1, x_2, x_3) . We will see that this equation can be written in the form $AB + DE = 0$, where the system $A = B = D = E = 0$ has only the trivial solution and B, E are the defining polynomials of one of the (-2) -curves in $\psi^*(C)$. Now consider the surface X' given by $Bs = D, Es = -A$ in the space $\mathbb{P}(3, 1, 1, 1, 1)$ with coordinates (z, s, x_1, x_2, x_3) . There is a morphism $X \rightarrow X'$ which restricts to an isomorphism

$$X - \{B = E = 0\} \longrightarrow X' - \{[0 : 1 : 0 : 0 : 0]\}$$

and which contracts the curve $\{B = E = 0\}$ to the point $[0 : 1 : 0 : 0 : 0]$. This is an example of *unprojection* (see [21]).

The variable z appears isolated in the equations of X' , therefore eliminating z we obtain the equation of the Kummer Q in \mathbb{P}^3 with variables (s, x_1, x_2, x_3) . All this calculations will be done using *Magma*.

In what follows a line preceded by $>$ is an input line, something preceded by $//$ is a comment. A \backslash at the end of a line means continuation in the next line. The other lines are output ones.

```
> K<e>:=CyclotomicField(6);//e denotes the 6th root of unity.
> //We choose a conic C with equation x1x3-x2^2=0 and fix the
> //p_i's: (1:1:1), (e^2:e:1), (e^4:e^2:1), (e^6:e^3:1),
> //(e^8:e^4:1), (e^10:e^5:1).
> R<z,s,x1,x2,x3>:=PolynomialRing(K,[3,1,1,1,1]);
```

```

> g:={e^(2*i)*x1-2*e^i*x2+x3:i in [0..5]};
> //g is the product of the defining polynomials
> //of the tangent lines L_i to C at p_i.
> X:=z^2-g;
> X eq (z+x1^3-x3^3)*(z-x1^3+x3^3)+4*(x1*x3-4*x2^2)^2*\
> (-x1*x3+x2^2); //The decomposition AB+DE.
true
> i:=Ideal([s*(z-x1^3+x3^3)-4*(x1*x3-4*x2^2)^2,\
> s*(x1*x3-x2^2)-(z+x1^3-x3^3)]);
> j:=EliminationIdeal(i,1);
> j;
Ideal of Graded Polynomial ring of rank 5 over K
Lexicographical Order Variables: z, s, x1, x2, x3
Variable weights: 3 1 1 1 1 Basis:
[-1/2*s^2*x1*x3+1/2*s^2*x2^2+s*x1^3-s*x3^3+2*x1^2*x3^2-
16*x1*x2^2*x3+32*x2^4]
> 2*Basis(j)[1];
-s^2*x1*x3+s^2*x2^2+2*s*x1^3-2*s*x3^3+4*x1^2*x3^2-
32*x1*x2^2*x3+64*x2^4
> //This is the equation of the Kummer Q.

```

We want to find a quadric H such that $H \cap Q$ is a reduced curve $\overline{B'}$ having an ordinary quadruple point pt as only singularity. Since the computer is not fast enough while working with more than 5 or 6 variables, we first need to think what the most probable case is.

Like we have seen in Section 3, the branch locus $B' + \sum_1^{16} A_i$ is contained in 4 fibres F_A^1, \dots, F_A^4 of a fibration f_A of W , where W is the resolution of Q blown-up at pt and the A_i 's are the (-2) -curves which contract to the nodes of Q .

Of course we have a quadric intersecting Q at a curve with a quadruple point pt : the tangent space T to Q at pt counted twice. But this one is double, so we need to find an irreducible one (and these two induce f_A), the curve $\overline{B'}$. These curves $2T$ and $\overline{B'}$ are good candidates for $\overline{F_A^1}$ and $\overline{F_A^2}$ (in the notation of Sections 3 and 4). If this configuration exists, then the 16 nodes must be contained in the other two fibres, $\overline{F_A^3}$ and $\overline{F_A^4}$. These fibres are divisible by 2, because $\overline{F_A^1} = 2T$, and are double outside the nodes. Since in a $K3$ surface only 0, 8 or 16 nodes can have sum divisible by 2, it is reasonable to try the following configuration: each of $\overline{F_A^3}$ and $\overline{F_A^4}$ contain 8 nodes with sum divisible by 2 and is double outside the nodes.

It is well known (see e. g. [17]) that the Kummer surface Q has 16 double hyperplane sections T_i such that each one contains 6 nodes of Q and that any two of them intersect in 2 nodes. The sum of the 8 nodes contained in

$$N := (T_1 \cup T_2) \setminus (T_1 \cap T_2)$$

is divisible by 2. Magma will give 3 generators h_1, h_2, h_3 for the linear system of quadrics through these nodes.

```

> K<e>:=CyclotomicField(6);
> P3<s,x1,x2,x3>:=ProjectiveSpace(K,3);
> F:=-s^2*x1*x3+s^2*x2^2+2*s*x1^3-2*s*x3^3+4*x1^2*x3^2-\
> 32*x1*x2^2*x3+64*x2^4;
> Q:=Scheme(P3,F);/*The Kummer*/; SQ:=SingularSubscheme(Q);
> T1:=Scheme(P3,x1-2*x2+x3); T2:=Scheme(P3,s);
> N:=Difference((T1 join T2) meet SQ, T1 meet T2);
> s:=SetToSequence(RationalPoints(N));
> //s is the sequence of the 8 nodes.
> L:=LinearSystem(P3,2);
> //This will give the h_i's:
> LinearSystem(L,[P3!s[i] : i in [1..8]]);
Linear system on Projective Space of dimension 3
Variables: s, x1, x2, x3 with 3 sections:

s*x1-2*x1^2-4*x1*x2-2*x1*x3+8*x2^2+4*x2*x3+2*x3^2
s*x2-2*x1^2-4*x1*x2+4*x2*x3+2*x3^2
s*x3-2*x1^2-4*x1*x2+2*x1*x3-8*x2^2+4*x2*x3+2*x3^2

```

Now we want to find a quadric H in the form $h_1 + bh_2 + ch_3$, for some b, c (or, less probably, in the form $bh_2 + ch_3$) such that the projection of $H \cap Q$ to \mathbb{P}^2 (by elimination) is a curve with a quadruple point. To find a quadruple point we just have to impose the annulation of the derivatives up to order 3 and ask Magma to do the rest.

```

> R<s,b,c,x1,x2,x3>:=PolynomialRing(Rationals(),6);
> F:=-s^2*x1*x3+s^2*x2^2+2*s*x1^3-2*s*x3^3+4*x1^2*x3^2-\
> 32*x1*x2^2*x3+64*x2^4;
> h1:=s*x1-2*x1^2-4*x1*x2-2*x1*x3+8*x2^2+4*x2*x3+2*x3^2;
> h2:=s*x2-2*x1^2-4*x1*x2+4*x2*x3+2*x3^2;
> h3:=s*x3-2*x1^2-4*x1*x2+2*x1*x3-8*x2^2+4*x2*x3+2*x3^2;
> H:=h1+b*h2+c*h3;
> I:=ideal<R|[F,H]>;
> I1:=EliminationIdeal(I,1);
> q0:=Evaluate(Basis(I1)[1],x3,1);//We work in the affine plane.
> R4<B,C,X1,X2>:=PolynomialRing(Rationals(),4);
> h:=hom<R->R4|[0,B,C,X1,X2,0]>;
> q:=h(q0);q1:=Derivative(q,X1);q2:=Derivative(q,X2);
> q3:=Derivative(q1,X1);q4:=Derivative(q1,X2);q5:=Derivative\
> (q2,X2);q6:=Derivative(q3,X1);q7:=Derivative(q3,X2);
> q8:=Derivative(q4,X2);q9:=Derivative(q5,X2);
> A4:=AffineSpace(R4);
> S:=Scheme(A4,[q,q1,q2,q3,q4,q5,q6,q7,q8,q9]);
> Dimension(S);
0
> PointsOverSplittingField(S);

```

This last command gives the points of S , as well as the necessary equations to define the field extensions where they belong. There are various solutions. One of them gives the desired quadruple point. The confirmation is as follows:


```

> R<x>:=PolynomialRing(Rationals());
> K<r13>:=ext<Rationals()|x^4 + x^3 + 1/4*x^2 + 3/32>;
> P3<s,x1,x2,x3>:=ProjectiveSpace(K,3);
> F:=-s^2*x1*x3+s^2*x2^2+2*s*x1^3-2*s*x3^3+4*x1^2*x3^2-\
> 32*x1*x2^2*x3+64*x2^4;
> b:=64/55*r13^3-272/55*r13^2-96/55*r13-46/55;
> c:=-2176/605*r13^3+448/605*r13^2+624/605*r13-361/605;
> H:=(s*x1-2*x1^2-4*x1*x2-2*x1*x3+8*x2^2+4*x2*x3+2*x3^2)+\
> b*(s*x2-2*x1^2-4*x1*x2+4*x2*x3+2*x3^2)+\
> c*(s*x3-2*x1^2-4*x1*x2+2*x1*x3-8*x2^2+4*x2*x3+2*x3^2);
> Q:=Scheme(P3,F);
> C:=Scheme(Q,H);
> IsReduced(C);
false
> RC:=ReducedSubscheme(C);
> #SingularPoints(RC);//# means ``number of``.
1
> HasSingularPointsOverExtension(RC);
false
> pt:=Representative(SingularPoints(RC));
> pt in SingularSubscheme(Q);//pt is not a node of Q.
false
> T:=DefiningPolynomial(TangentSpace(Q,pt));
> T2:=Scheme(Q,T^2);
> #RationalPoints(T2 meet C);
1
> pt in RationalPoints(T2 meet C);
true
> HasPointsOverExtension(T2 meet C);
false

```

This way $T2$ and C generate a pencil with a quadruple base point and the curve $\overline{B'}$ is a general element of this pencil.

Finally, it remains to be shown that the degree of the bicanonical map ϕ_2 is 2. As $(2K_S)^2 = 24$, it suffices to show that $\phi_2(S)$ is of degree 12. Since, in the notation of diagram (3.1), $h^*|2K_S| = \pi^*|2K_W + B'|$ then $\phi_2(S)$ is the image of W via the map $\tau : W \rightarrow \phi_2(S)$ given by $|2K_W + B'|$. The projection of this linear system on Q is the linear system of the quadrics whose intersection with Q has a double point at pt . In order to easily write this linear system, we will translate the point pt to the origin (in affine coordinates).

```

> QA:=AffinePatch(Scheme(P3,F),4);
> p:=Representative(RationalPoints(AffinePatch(Cluster(pt),4)));
> A3<x,y,z>:=Ambient(QA);
> psi:=map<A3->A3|[x-p[1],y-p[2],z-p[3]]>;Q0:=psi(QA);
> FA:=DefiningPolynomial(Q0);
> j:=[Evaluate(Derivative(FA,A3.i),Origin(A3)):i in [1,2,3]];
> J:=LinearSystem(A3,[j[1]*x+j[2]*y+j[3]*z,x^2,x*y,x*z,y^2,y*z,\

```

```
> z^2]);
> P6:=ProjectiveSpace(K,6);
> tau:=map<A3->P6|Sections(J)>;
> Degree(tau(Q0));
12
```

Example 5.2. Here we will construct a surface of general type S , with $p_g = q = 1$ and $K^2 = 4$, as the minimal model of a double cover of a surface W such that $\text{Kod}(W) = p_g(W) = 1$ and $q(W) = 0$.

Step 1. Construction of W .

Consider five distinct lines $L_1, \dots, L_5 \subset \mathbb{P}^2$ meeting in one point p_0 . Let $p_1 \in L_4, p_2, p_3 \in L_5$ be points distinct from p_0 . Choose three distinct non-degenerate conics, C_1, C_2, C_3 , tangent to L_4 at p_1 and passing through p_2, p_3 . Define

$$D := L_1 + \dots + L_4 + C_1 + C_2.$$

Denote by p_4, \dots, p_{15} the 12 nodes of D contained in $L_1 + L_2 + L_3$. To resolve the $(3, 3)$ point of D at p_1 we must do two blow-ups: one at p_1 and other at an infinitely near point p'_1 . Let $\mu : X \rightarrow \mathbb{P}^2$ be the blow-up with centers $p_0, p_1, p'_1, p_2, \dots, p_{15}$ and $E_0, E_1, E'_1, E_2, \dots, E_{15}$ be the corresponding exceptional divisors (with self-intersection -1). Consider

$$D' := \mu^*(D) - 4E_0 - 2E_1 - 4E'_1 - 2 \sum_2^{15} E_i.$$

Let $\psi : \tilde{X} \rightarrow X$ be the double cover of X with branch locus D' . The surface \tilde{X} is the canonical resolution of the double cover of \mathbb{P}^2 ramified over D . Let W be the minimal model of \tilde{X} and ν be the corresponding morphism.

$$\begin{array}{ccc} \tilde{X} & \xrightarrow{\nu} & W \\ \psi \downarrow & & \\ X & \xrightarrow{\mu} & \mathbb{P}^2. \end{array}$$

Notice that ν contracts two (-1) -curves contained in $(\mu \circ \psi)^*(L_4)$.

We have $K_X \equiv -\mu^*(3L) + E'_1 + \sum_0^{15} E_i$, where L denotes a general line of \mathbb{P}^2 . Hence, using the double cover formulas (cf. (2.1)),

$$K_{\tilde{X}} \equiv \psi^* \left(K_X + \frac{1}{2} D' \right) \equiv \psi^*(\mu^*(L) - E_0 - E'_1) \equiv \psi^*(\widehat{L}_4 + (E_1 - E'_1) + E'_1),$$

where $\widehat{L}_4 \subset X$ is the strict transform of L_4 . Since \widehat{L}_4 and $E_1 - E'_1$ are (-2) -curves contained in the branch locus D' , then $\frac{1}{2}\psi^*(\widehat{L}_4)$ and $\frac{1}{2}\psi^*(E_1 - E'_1)$ are (-1) -curves in \tilde{X} , thus

$$K_W \equiv \nu(\psi^*(E'_1)).$$

The divisor $2\nu(\psi^*(E'_1)) \equiv 2K_W$ is a (double) fibre of the elliptic fibration of W induced by the pencil of lines through p_0 . So $p_g(W) = 1$ and W has Kodaira dimension 1.

From (2.1) one has

$$\chi(\mathcal{O}_W) = 2 + \frac{1}{8}D'(2K_X + D') = 2 + \frac{1}{8}(28 - 28) = 2.$$

Step 2. The branch locus in W .

Since the strict transforms $\widehat{L}_1, \dots, \widehat{L}_4 \subset X$ are in the branch locus D' , then there are curves $l_1, \dots, l_4 \subset \widetilde{X}$ such that

$$\begin{aligned} (\mu \circ \psi)^*(L_1 + \dots + L_4) &= 2l_1 + \dots + 2l_4 + 4\psi^*(E_0) + \psi^*(E_1 - E'_1) \\ &\quad + 2\psi^*(E'_1) + \sum_4^{15} A_i, \end{aligned}$$

where each $A_i := \psi^*(E_i)$ is a (-2) -curve. But also $E_1 - E'_1$ is in the branch locus, thus $\psi^*(E_1 - E'_1) \equiv 0 \pmod{2}$ and then

$$\sum_4^{15} A_i \equiv 0 \pmod{2}.$$

The strict transform \widehat{L}_5 is a (-2) -curve which do not intersect D' thus

$$\psi^*(\widehat{L}_5) = A_{16} + A_{17},$$

with A_{16}, A_{17} disjoint (-2) -curves.

Denote by $\widehat{C}_3 \subset X$ the strict transform of the conic C_3 . We have

$$\begin{aligned} (\mu \circ \psi)^*(C_3 + L_4 + L_5) &= \psi^*(\widehat{C}_3) + 2l_4 + A_{16} + A_{17} \\ &\quad + 2\psi^*(E_0 + \dots + E_3) + 2\psi^*(E'_1) \equiv 0 \pmod{2}. \end{aligned}$$

With this we conclude that

$$\psi^*(\widehat{C}_3) + \sum_4^{17} A_i \equiv 0 \pmod{2}.$$

Notice that $F \cdot \nu(\psi^*(\widehat{C}_3)) = 4$ for a fibre F of the elliptic fibration of W , thus $K_W \cdot \nu(\psi^*(\widehat{C}_3)) = 2$.

Step 3. Construction of S .

Let $\pi : V \rightarrow W$ be the double cover with branch locus

$$B := \nu \left(\psi^*(\widehat{C}_3) + \sum_4^{17} A_i \right)$$

and S be the minimal model of V . From the double cover formulas (2.1) we obtain

$$2K_V^2 = (2K_W + B)^2 = 4K_W^2 + 4K_W B + B^2 = 4 \cdot 0 + 4 \cdot 2 + (-28) = -20$$

and, by contraction of the (-1) -curves $\frac{1}{2}\pi^*(\nu(A_i))$,

$$K_S^2 = K_V^2 + 14 = -10 + 14 = 4.$$

Let $L := \frac{1}{2}B$. Formulas (2.1) give

$$\chi(\mathcal{O}_S) = 2\chi(\mathcal{O}_W) + \frac{1}{2}L(K_W + L) = 4 - 3 = 1.$$

Using now formula (2.3) we obtain $h^0(W, \mathcal{O}_W(2K_W + L)) = 0$, which means that the bicanonical map of V factors through π .

Because K_W is effective then also $h^0(W, \mathcal{O}_W(K_W + L)) = 0$ and

$$p_g(S) = p_g(W) + h^0(W, \mathcal{O}_W(K_W + L)) = 1.$$

Hence $q(S) = 1$ and then, as we noticed in the beginning of Section 4, the curve $\nu(\psi^*(\widehat{C}_3))$ is contained in the fibration of W which induces the Albanese fibration of S . As $\nu(\psi^*(\widehat{C}_3))^2 = 0$, we conclude that the Albanese fibration of S is the one induced by the pencil $|\widehat{C}_3|$. It is of genus 2 because $\widehat{C}_3 D' = \widehat{C}_3(\widehat{L}_1 + \widehat{L}_2 + \widehat{L}_3) = 6$.

Example 5.3. Now we will obtain a surface of general type S , with $p_g = q = 1$ and $K^2 = 2$, as the minimal model of a double cover of a surface of general type W such that $K_W^2 = p_g(W) = 1$ and $q(W) = 0$.

Step 1. Construction of W .

Let $p_0, \dots, p_3 \in \mathbb{P}^2$ be distinct points and L_i be the line through p_0 and p_i , $i = 1, 2, 3$. For each $j \in \{1, 2, 3\}$ let C_j be the conic through p_1, p_2, p_3 tangent to the L_i 's except for L_j . Denote by D a general element of the linear system generated by $3C_1 + 2L_1$, $3C_2 + 2L_2$ and $3C_3 + 2L_3$. The singularities of D are a $(3, 3)$ -point at p_i , tangent to L_i , $i = 1, 2, 3$, and a double point at p_0 . Let L_4 be a line through p_0 transverse to D .

Denote by W' the canonical resolution of the double cover of \mathbb{P}^2 with branch locus

$$D + L_1 + \dots + L_4$$

and by W the minimal model of W' . The formulas of [4, Chapter V, Section 22] give $\chi(W) = 2$ and $K_W^2 = 1$ (notice that the map $W' \rightarrow W$ contracts three (-1) -curves contained in the pullback of $L_1 + L_2 + L_3$). Since $K^2 \geq 2p_g$ for an irregular surface ([14]), W is regular and then $p_g(W) = \chi(W) - 1 = 1$.

Step 2. The branch locus in W .

The pencil of lines through p_0 induces a (genus 2) fibration of W . Let F_i be the fibre induced by L_i , $i = 1, \dots, 4$. The fibre F_4 is the union of six disjoint (-2) -curves (corresponding to the nodes of $D - p_0$) with a double component (the strict transform of L_4). Each F_i , $i = 1, 2, 3$, is the union of two (-2) -curves with a double component (cf. [24, Section 1]). Thus $F_1 + \dots + F_4$ contain disjoint (-2) -curves A_1, \dots, A_{12} such that

$$\sum_1^{12} A_i \equiv 0 \pmod{2}.$$

Step 3. Construction of S .

Let V be the double cover of W with branch locus $\sum_1^{12} A_i$ and S be the minimal model of V . From (2.1) we obtain $\chi(\mathcal{O}_S) = 1$ and $K_V^2 = -10$. The A_i 's lift to (-1) -curves in V , thus $K_S^2 = -10 + 12 = 2$. We have $1 = p_g(W) \leq p_g(S)$, hence $q(S) \neq 0$ and then $2 = K_S^2 \geq 2p_g(S)$. So $p_g(S) = q(S) = 1$.

The genus 2 fibration of W induces the Albanese fibration of S .

References

- [1] A. BEAUVILLE, "Surfaces Algébriques Complexes", Astérisque 54, 1978.
- [2] E. BOMBIERI, *Canonical models of surfaces of general type*, Publ. Math. Inst. Hautes Étud. Sci. **42** (1972), 171–219.
- [3] G. BORRELLI, *On the classification of surfaces of general type with non-birational bicanonical map and Du Val double planes*, preprint, math.AG/0312351.
- [4] W. BARTH, C. PETERS and A. VAN DE VEN, "Compact Complex Surfaces", *Ergebn. der Math.* 3, Folge, Band 4, Springer-Verlag, Berlin, 1984.
- [5] F. CATANESE, *On a class of surfaces of general type*, In: "Algebraic Surfaces", CIME, Liguori, 1981, 269–284.
- [6] F. CATANESE, *Singular bidouble covers and the construction of interesting algebraic surfaces*, Amer. Math. Soc. Contemp. Math. **241** (1999), 97–120.
- [7] F. CATANESE and C. CILIBERTO, *Surfaces with $p_g = q = 1$* , In: Sympos. Math., Vol. 32, Academic Press, 1991, 49–79.
- [8] F. CATANESE, C. CILIBERTO and M. MENDES LOPES, *On the classification of irregular surfaces of general type with nonbirational bicanonical map*, Trans. Amer. Math. Soc. **350** (1988), 275–308.
- [9] A. CALABRI, C. CILIBERTO and M. MENDES LOPES *Numerical Godeaux surfaces with an involution*, Trans. Amer. Math. Soc., to appear.
- [10] C. CILIBERTO, P. FRANCA and M. MENDES LOPES, *Remarks on the bicanonical map for surfaces of general type*, Math. Z. **224** (1997), 137–166.

- [11] C. CILIBERTO, *The bicanonical map for surfaces of general type*, Proc. Sympos. Pure Math. **62** (1997), 57–84.
- [12] C. CILIBERTO and M. MENDES LOPES, *On surfaces with $p_g = q = 2$ and non-birational bicanonical map*, Adv. Geom. **2** (2002), 281–300.
- [13] C. CILIBERTO and M. MENDES LOPES, *On surfaces with $p_g = 2$, $q = 1$ and non-birational bicanonical map*, In: “Algebraic Geometry”, Beltrametti, Mauro C. *et al.* (eds.), a volume in memory of Paolo Francia, de Gruyter, Berlin, 2002, 117–126.
- [14] O. DEBARRE, *Inégalités numériques pour les surfaces de type général*, Bull. Soc. Math. Fr. **110** (1982), 319–346.
- [15] B. SAINT-DONAT, *Projective models of $K-3$ surfaces*, Amer. J. Math. **96** (1974), 602–639.
- [16] P. DU VAL, *On surfaces whose canonical system is hyperelliptic*, Canadian J. Math. **4** (1952), 204–221.
- [17] P. GRIFFITHS and J. HARRIS, “Principles of Algebraic Geometry”, Wiley Classics Library, New York, 1994.
- [18] K. KONNO, *Non-hyperelliptic fibrations of small genus and certain irregular canonical surfaces*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) **20** (1993), 575–595.
- [19] F. POLIZZI, *On surfaces of general type with $p_g = q = 1$, $K^2 = 3$* , Collect. Math. **56** (2005), 181–234.
- [20] F. POLIZZI, *Surfaces of general type with $p_g = q = 1$, $K^2 = 8$ and bicanonical map of degree 2*, Trans. Amer. Math. Soc. **358** (2006), 759–798.
- [21] M. REID, *Graded rings and birational geometry*, In: “Proceedings of Algebraic Symposium” (Kinosaki, Oct 2000), K. Ohno (ed.) 1–72, available from www.maths.warwick.ac.uk/miles/3folds
- [22] I. REIDER, *Vector bundles of rank 2 and linear systems on algebraic surfaces*, Ann. of Math. (2) **127** (1988), 309–316.
- [23] A. TODOROV, *A construction of surfaces with $p_g = 1$, $q = 0$ and $2 \leq K^2 \leq 8$. Counter examples of the global Torelli theorem*, Invent. Math. **63** (1981), 287–304.
- [24] G. XIAO, “Surfaces fibrées en courbes de genre deux”, Lecture Notes in Mathematics, Vol. 1137, Springer-Verlag, Berlin, 1985.
- [25] G. XIAO, *Degree of the bicanonical map of a surface of general type*, Amer. J. Math. **112** (1990), 713–736.

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