

Second order parabolic equations and weak uniqueness of diffusions with discontinuous coefficients

DOYOON KIM

Abstract. We prove the unique solvability of parabolic equations with discontinuous leading coefficients in $W_p^{1,2}((0, T) \times \mathbb{R}^d)$. Using this result, we establish the uniqueness of diffusion processes with time-dependent discontinuous coefficients.

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1. Introduction

The unique solvability of second order parabolic equations in non-divergence form is well established if the leading coefficients are uniformly continuous (see [8] and [9]). More precisely, the parabolic equation

$$u_t = a_{jk}(t, x) u_{x_j x_k} + b_j(t, x) u_{x_j} + c(t, x) u + f(t, x)$$

with appropriate initial and boundary conditions has a unique solution $u \in W_p^{1,2}((0, T) \times \Omega)$, $1 < p < \infty$, if $a_{jk}(t, x)$ are uniformly continuous with respect to x uniformly in t (a_{jk} satisfy the uniform ellipticity condition). However, even if $a_{jk}(t, x)$ are discontinuous, there are some cases where one can expect the unique solvability of the above equation. The most intensively investigated case is when the coefficients a_{jk} are in the space of VMO. As noted in [13] (also see references therein to find a list of many papers which deal with elliptic and parabolic equations with VMO coefficients), VMO contains the space of bounded uniformly continuous functions as a proper subset. However, for example, if a_{jk} are piecewise constant, then they are not in the space of VMO. This means that we may try to find another class of discontinuous coefficients a_{jk} (a class that includes piecewise constant coefficients) so that equations with leading coefficients from the class have strong unique solutions.

In this paper we prove the existence and uniqueness of a solution u in $W_p^{1,2}((0, T) \times \mathbb{R}^d)$, $1 < p < \infty$, to the above parabolic equation with coefficients $a_{jk}(t, x)$ being discontinuous at the hyper-plane $\{(t, 0, x') : t \in \mathbb{R}, x' \in \mathbb{R}^{d-1}\}$. Specifically, we assume that

$$a_{jk}(t, x) = a_{jk}^+(t, x) \text{ for } x_1 > 0 \text{ and } a_{jk}(t, x) = a_{jk}^-(t, x) \text{ for } x_1 < 0,$$

where $a_{jk}^+(t, x)$ and $a_{jk}^-(t, x)$ are defined on $\overline{\mathbb{R} \times \mathbb{R}_+^d}$ and $\overline{\mathbb{R} \times \mathbb{R}_-^d}$ respectively and uniformly continuous with respect to spatial variables uniformly in t . In addition, $a_{jk}^\pm(t, 0, x')$ are uniformly continuous as functions of (t, x') . We see that a_{jk}^+ do not have any relation with a_{jk}^- , so the coefficients a_{jk} can be discontinuous at the hyper-plane $\{(t, 0, x') : t \in \mathbb{R}, x' \in \mathbb{R}^{d-1}\}$. Clearly, this contains piecewise constant a_{jk} with discontinuity at the hyper-plane. An a priori estimate, as usual, is the main step we achieve, and to do that, we make use of trace operators and multipliers.

When a_{jk} , b_j , and c are piecewise constant with discontinuity at the above hyper-plane, Salsa [12] solved in $W_2^{1,2}((0, \infty) \times \mathbb{R}^d)$ the equation $u_t = a_{jk}u_{x_j x_k} + b_j u_{x_j} + cu$ with a non-zero initial condition. For elliptic equations with similar leading coefficients, see [10], [11], and [5].

In this paper we also discuss the well-posedness of the martingale problem for L_t , where

$$L_t = \frac{1}{2} a_{jk}(t, \cdot) \frac{\partial^2}{\partial x_j \partial x_k} + b_j(t, \cdot) \frac{\partial}{\partial x_j}.$$

Here the coefficients $a_{jk}(t, x)$ are allow to be discontinuous at infinitely many parallel hyper-planes. For more details, see assumptions in Section 3. In fact, there are papers which investigate the well-posedness of martingale problems (or well-posedness of diffusion processes) when the diffusion coefficients $a_{jk}(x)$ are discontinuous in x . Some of them can be [2], [3], and [7]. However, they dealt with coefficients $a_{jk}(x)$ which are independent of t . We here consider time-dependent coefficients which are discontinuous at hyper-planes $\{(t, \gamma_j, x') : t \in \mathbb{R}, x' \in \mathbb{R}^{d-1}\}$, where $\{\gamma_j\}$ is a subset in \mathbb{R} with no limit points. As is shown in [15], the well-posedness of the martingale problem for L_t follows from the unique solvability of the corresponding parabolic equation (if the equation is uniquely solvable). Hence our result on parabolic equations in Section 2 plays a major role in the discussion of the martingale problem. For complete details about martingale problems, see the monograph [15].

This paper consists of two sections excluding this introduction. We discuss parabolic equations and martingale problems in Section 2 and 3 respectively.

A few words about notation. We denote by (t, x) a point in \mathbb{R}^{d+1} . That is, $(t, x) = (t, x_1, x') \in \mathbb{R} \times \mathbb{R}^d = \mathbb{R}^{d+1}$, where $t \in \mathbb{R}$, $x_1 \in \mathbb{R}$, $x' \in \mathbb{R}^{d-1}$, and $x \in \mathbb{R}^d$. We write \mathbb{R}_+^d and \mathbb{R}_-^d for half-spaces

$$\{(x_1, x') : x_1 > 0, x' \in \mathbb{R}^{d-1}\} \quad \text{and} \quad \{(x_1, x') : x_1 < 0, x' \in \mathbb{R}^{d-1}\}$$

respectively. Throughout the paper we set $L_p = L_p(\mathbb{R}^{d+1})$.

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2. Second order parabolic equations with discontinuous coefficients

We consider parabolic equations of the form

$$\begin{cases} u_t(t, x) = Lu(t, x) + f(t, x) & \text{in } (0, T) \times \mathbb{R}^d \\ u(0, x) = 0 \end{cases},$$

where

$$Lu(t, x) = a_{jk}(t, x)u_{x_j x_k}(t, x) + b_j(t, x)u_{x_j}(t, x) + c(t, x)u(t, x).$$

We prove the existence and uniqueness of solutions to the parabolic equations as above in the Sobolev space $W_p^{1,2}((0, T) \times \mathbb{R}^d)$, $1 < p < \infty$. The definition of $W_p^{1,2}((T_1, T_2) \times \Omega)$ can be found in [8] or [9], where $-\infty \leq T_1 < T_2 \leq \infty$ and Ω is an open set in \mathbb{R}^d .

Let us first state the assumptions on coefficients $a_{jk}(t, x)$, $b_j(t, x)$, and $c(t, x)$. Note that in the below the coefficients $a_{jk}(t, x)$ are not uniformly continuous in $x \in \mathbb{R}^d$. The function ω below is an increasing function defined on $[0, \infty)$ such that $\omega(\varepsilon) \rightarrow 0$ as $\varepsilon \searrow 0$.

Assumption 2.1. The coefficients a_{jk} , $j, k = 1, \dots, d$, are defined by

$$a_{jk}(t, x) = \begin{cases} a_{jk}^+(t, x) & \text{if } x_1 > 0 \\ a_{jk}^-(t, x) & \text{if } x_1 < 0 \end{cases},$$

where a_{jk}^\pm satisfy the following.

1. $a_{jk}^+(t, x)$ are defined on $\overline{\mathbb{R} \times \mathbb{R}_+^d}$ and $a_{jk}^-(t, x)$ are defined on $\overline{\mathbb{R} \times \mathbb{R}_-^d}$.
2. $a_{jk}^\pm(t, x) = a_{kj}^\pm(t, x)$, and there exists a constant $\kappa \in (0, 1)$ such that, for any $t \in \mathbb{R}$, $x \in \mathbb{R}^d$, and $\vartheta \in \mathbb{R}^d$,

$$\kappa |\vartheta|^2 \leq \sum_{j,k=1}^d a_{jk}^\pm(t, x) \vartheta_j \vartheta_k \leq \kappa^{-1} |\vartheta|^2.$$

3. For each $t \in \mathbb{R}$,

$$|a_{jk}^+(t, x) - a_{jk}^+(t, y)| \leq \omega(|x - y|) \quad \text{for } x, y \in \overline{\mathbb{R}_+^d},$$

$$|a_{jk}^-(t, x) - a_{jk}^-(t, y)| \leq \omega(|x - y|) \quad \text{for } x, y \in \overline{\mathbb{R}_-^d}.$$

4. For $t, s \in \mathbb{R}$ and $x', y' \in \mathbb{R}^{d-1}$,

$$|a_{jk}^+(t, 0, x') - a_{jk}^+(s, 0, y')| \leq \omega(|t - s| + |x' - y'|),$$

$$|a_{jk}^-(t, 0, x') - a_{jk}^-(s, 0, y')| \leq \omega(|t - s| + |x' - y'|).$$

Assumption 2.2. The coefficients $b_j(t, x)$ and $c(t, x)$ are measurable functions defined on $\mathbb{R} \times \mathbb{R}^d$ satisfying

$$|b_j(t, x)| + |c(t, x)| \leq K$$

for some positive constant K .

The first case we consider is

$$u_t(t, x) = L_0 u(t, x) - \lambda u(t, x) + f(t, x) \quad \text{in } \mathbb{R} \times \mathbb{R}^d$$

with

$$L_0 = \begin{cases} L_0^+ = a_{jk}^+ D_{jk} & \text{in } \mathbb{R} \times \mathbb{R}_+^d \\ L_0^- = a_{jk}^- D_{jk} & \text{in } \mathbb{R} \times \mathbb{R}_-^d \end{cases},$$

where a_{jk}^+ and a_{jk}^- are constant. Recall that $L_p = L_p(\mathbb{R}^{d+1})$.

Theorem 2.3. For any $\lambda \geq 0$ and $u \in C_0^\infty(\mathbb{R}^{d+1})$,

$$\lambda \|u\|_{L_p} + \|u_t\|_{L_p} + \|u_{xx}\|_{L_p} \leq N \|L_0 u - \lambda u - u_t\|_{L_p}, \quad (2.1)$$

where N depends only on d , p , and κ .

To prove this theorem we need the following function spaces and trace theorem which can be found in [8, 17]. For a non-negative real number l , let

$$W_p^{l/2, l}(\mathbb{R} \times \mathbb{R}^{d-1}) = L_p(\mathbb{R}, W_p^l(\mathbb{R}^{d-1})) \cap W_p^{l/2}(\mathbb{R}, L_p(\mathbb{R}^{d-1})),$$

where $W_p^s(\mathbb{R}^{d-1})$, $s = 0, 1, 2, \dots$, are Sobolev spaces and $W_p^s(\mathbb{R}^{d-1})$, $s \neq \text{integer}$, $s > 0$, are Slobodeckij spaces.

Theorem 2.4. Set $l = 2 - 1/p$. There exists the trace operator T from

$$W_p^{1,2}(\mathbb{R} \times \mathbb{R}_+^d) \quad \text{onto} \quad W_p^{l/2, l}(\mathbb{R} \times \mathbb{R}^{d-1}) \times W_p^{(l-1)/2, l-1}(\mathbb{R} \times \mathbb{R}^{d-1})$$

such that

$$Tu = (u(t, 0, x'), u_{x_1}(t, 0, x'))$$

and

$$\|u(\cdot, 0, \cdot)\|_{W_p^{l/2, l}(\Gamma)} + \|u_{x_1}(\cdot, 0, \cdot)\|_{W_p^{(l-1)/2, l-1}(\Gamma)} \leq N \|u\|_{W_p^{1,2}(\mathbb{R} \times \mathbb{R}_+^d)} \quad (2.2)$$

for $u \in W_p^{1,2}(\mathbb{R} \times \mathbb{R}_+^d)$, where $\Gamma = \mathbb{R} \times \mathbb{R}^{d-1}$ and N depends only on d and p . Moreover, T has a bounded linear right inverse (extension operator).

We also need the following notations, similar to those in [10, 12]. Set

$$\alpha^\pm(\xi') = i \sum_{j=2}^d a_{1j}^\pm \xi_j, \quad \xi' = (\xi_2, \dots, \xi_d) \in \mathbb{R}^{d-1}, \quad i = \sqrt{-1},$$

$$\beta^\pm(\xi') = \sum_{j,k=2}^d a_{jk}^\pm \xi_j \xi_k + 1,$$

and

$$\begin{aligned} H^\pm(\xi') &= (\alpha^\pm(\xi'))^2 + a_{11}^\pm \beta^\pm(\xi') \\ &= - \left(\sum_{j=2}^d a_{1j}^\pm \xi_j \right)^2 + a_{11}^\pm \left(\sum_{j,k=2}^d a_{jk}^\pm \xi_j \xi_k + 1 \right). \end{aligned}$$

Let $s \in \mathbb{R}$. We denote

$$\begin{aligned} z^+(s, \xi') &= \frac{-\alpha^+(\xi') - \sqrt{H^+(\xi') + i a_{11}^+ s}}{a_{11}^+}, \\ z^-(s, \xi') &= \frac{-\alpha^-(\xi') + \sqrt{H^-(\xi') + i a_{11}^- s}}{a_{11}^-}, \\ z(s, \xi') &= z^+(s, \xi') - z^-(s, \xi'), \end{aligned}$$

where \sqrt{w} means, from now on, that branch of the analytic function $w^{1/2}$ that has non-negative real part. Note that

$$\Re[z^+(s, \xi')] = -\frac{1}{a_{11}^+} \Re \left[\sqrt{H^+(\xi') + i a_{11}^+ s} \right]$$

and

$$\Re[z^-(s, \xi')] = \frac{1}{a_{11}^-} \Re \left[\sqrt{H^-(\xi') + i a_{11}^- s} \right].$$

The following estimates hold true for $\Re[z^\pm(s, \xi')]$. One can find the same estimates in [12].

Lemma 2.5.

$$-N_2 \sqrt{1 + |\xi'|^2 + |s|} \leq \Re[z^+(s, \xi')] \leq -N_1 \sqrt{1 + |\xi'|^2 + |s|}$$

and

$$N_1 \sqrt{1 + |\xi'|^2 + |s|} \leq \Re[z^-(s, \xi')] \leq N_2 \sqrt{1 + |\xi'|^2 + |s|},$$

where N_1 and N_2 depend only on κ .

Proof. The lemma follows from the fact that $\kappa \leq a_{11}^{\pm} \leq \kappa^{-1}$ and

$$N_1 |\xi'|^2 \leq H^{\pm}(\xi') - a_{11}^{\pm} \leq N_2 |\xi'|^2, \quad (2.3)$$

which is proved in Lemma 3 in [10]. Here the constants N_1 and N_2 depend only on κ . \square

The main step in the proof of Theorem 2.3 is to prove that an operator \mathcal{T} defined by

$$\widetilde{\mathcal{T}}h = \widetilde{h}/z(s, \xi'), \quad h \in C_0^{\infty}(\mathbb{R} \times \mathbb{R}^{d-1}),$$

can be extended to a bounded operator from

$$W_p^{(l-1)/2, l-1}(\mathbb{R} \times \mathbb{R}^{d-1}) \quad \text{into} \quad W_p^{l/2, l}(\mathbb{R} \times \mathbb{R}^{d-1}),$$

where $l = 2 - 1/p$ and $\mathbb{R} \times \mathbb{R}^{d-1} = \{(t, 0, x') : t \in \mathbb{R}, x' \in \mathbb{R}^{d-1}\}$. Here \widetilde{f} is the Fourier transform of f in $\mathbb{R} \times \mathbb{R}^{d-1}$. To prove this, we start with a lemma showing that $(1 + s^2)^{1/4}/z(s, \xi')$ and $i\xi_j/z(s, \xi')$ are multipliers. Note that we sometimes mean by \mathbb{R}^d , especially in Lemma 2.6 and 2.7, the space $\{(t, 0, x') : t \in \mathbb{R}, x' \in \mathbb{R}^{d-1}\}$.

Lemma 2.6.

$$\frac{(1 + s^2)^{1/4}}{z(s, \xi')} \quad \text{and} \quad \frac{i\xi_j}{z(s, \xi')}, \quad j = 2, \dots, d,$$

are multipliers for $L_p(\mathbb{R}^d)$. That is, if we set $\widetilde{\mathcal{T}}f(s, \xi') = m(s, \xi')\widetilde{f}$, where $f \in L_2(\mathbb{R}^d) \cap L_p(\mathbb{R}^d)$ and $m(s, \xi')$ is either of the above multipliers, then we have

$$\|\mathcal{T}f\|_{L_p(\mathbb{R}^d)} \leq N\|f\|_{L_p(\mathbb{R}^d)},$$

where N depends only on d , p , and κ .

Proof. Set $s = \xi_1$ (recall that $\xi' = (\xi_2, \dots, \xi_d)$). We prove

$$\left| \frac{\partial^m}{\partial \xi_{j_1} \cdots \partial \xi_{j_m}} \left(\frac{(1 + \xi_1^2)^{1/4}}{z(\xi_1, \xi')} \right) \right| \leq \frac{N}{|\xi_{j_1} \cdots \xi_{j_m}|} \quad (2.4)$$

and

$$\left| \frac{\partial^m}{\partial \xi_{j_1} \cdots \partial \xi_{j_m}} \left(\frac{i\xi_j}{z(\xi_1, \xi')} \right) \right| \leq \frac{N}{|\xi_{j_1} \cdots \xi_{j_m}|}, \quad j = 2, \dots, d, \quad (2.5)$$

where $m = 1, \dots, d$, $j_k \in \{1, \dots, d\}$, and $j_k \neq j_l$ if $k \neq l$. Especially, the constant N depends only on d and κ . Once these inequalities are proved, the lemma follows from Mikhlin's theorem (Theorem 6' on page 109 [14] or see page 289 [8]).

First, we prove that, for a multi-index α , $\alpha = (0, \alpha_2, \dots, \alpha_d)$,

$$\left| D^\alpha \sqrt{H^\pm(\xi') + i a_{11}^\pm \xi_1} \right| \leq N(1 + |\xi'|^2 + |\xi_1|)^{-\frac{|\alpha|-1}{2}}, \quad (2.6)$$

where N depends only on d , α , and κ . Using the matrix $\sqrt{[a_{jk}^+]}$ and an appropriate orthogonal change of variables, we can find a matrix $A = [\sigma_{jk}]$ satisfying the following:

- (i) The mapping $x \rightarrow Ax$ is a diffeomorphism from \mathbb{R}_+^d onto itself.
- (ii) For a twice differentiable function $v(x)$ defined on \mathbb{R}_+^d , set $w(x) = v(Ax)$. Then $L_0^+ v(Ax) = \Delta w(x)$ in \mathbb{R}_+^d , where, as we recall, $L_0^+ v = a_{jk}^+ v_{x_j x_k}$.

Notice that from the above two properties of $A = [\sigma_{jk}]$ we have $a_{jk}^+ = \sum_{l=1}^d \sigma_{jl} \sigma_{kl}$, $k = 1, \dots, d$, and $\sigma_{1k} = 0$, $k = 2, \dots, d$. Denote by B the sub-matrix formed by removing the first row and column of A . Then B is a non-singular matrix. Observe that

$$\begin{aligned} |B^* \xi'|^2 &= \sum_{j,k=2}^d (a_{jk}^+ - \sigma_{j1} \sigma_{k1}) \xi_j \xi_k = - \sum_{j,k=2}^d \sigma_{j1} \sigma_{k1} \xi_j \xi_k + \sum_{j,k=2}^d a_{jk}^+ \xi_j \xi_k \\ &= - \left(\sum_{j=2}^d \sigma_{j1} \xi_j \right)^2 + \sum_{j,k=2}^d a_{jk}^+ \xi_j \xi_k = - \frac{1}{\sigma_{11}^2} \left(\sum_{j=2}^d \sigma_{11} \sigma_{j1} \xi_j \right)^2 + \sum_{j,k=2}^d a_{jk}^+ \xi_j \xi_k \\ &= - \frac{1}{a_{11}^+} \left(\sum_{j=2}^d a_{1j}^+ \xi_j \right)^2 + \sum_{j,k=2}^d a_{jk}^+ \xi_j \xi_k = \frac{1}{a_{11}^+} H^+(\xi') - 1. \end{aligned}$$

Thus

$$\sqrt{H^+(\xi') + i a_{11}^+ \xi_1} = \sqrt{a_{11}^+ (|B^* \xi'|^2 + 1 + i \xi_1)}.$$

From this and (2.3) we see that

$$\left| D^\alpha \sqrt{H^+(\xi') + i a_{11}^+ \xi_1} \right| \leq N(1 + |\xi'|^2 + |\xi_1|)^{-\frac{|\alpha|-1}{2}},$$

where N depends only on d , α , and κ . By the same calculation, we also have the above inequality with $H^-(\xi')$ and a_{11}^- in place of $H^+(\xi')$ and a_{11}^+ . Therefore, (2.6) is proved. Moreover, it follows easily that

$$\left| D^\alpha \left[(H^\pm(\xi') + i a_{11}^\pm \xi_1)^{-1/2} \right] \right| \leq N(1 + |\xi'|^2 + |\xi_1|)^{-\frac{|\alpha|+1}{2}}. \quad (2.7)$$

Next, we prove that, for $\alpha = (0, \alpha_2, \dots, \alpha_d)$,

$$\left| D^\alpha \left(\frac{1}{z(\xi_1, \xi')} \right) \right| \leq N(1 + |\xi'|^2 + |\xi_1|)^{-\frac{|\alpha|+1}{2}}, \quad (2.8)$$

where N depends only on d, α , and κ . If $\alpha = 0$, then by Lemma 2.5

$$\left| \frac{1}{z(\xi_1, \xi')} \right| \leq \frac{1}{-\Re[z^+(\xi_1, \xi')] + \Re[z^-(\xi_1, \xi')]} \leq N(1 + |\xi'|^2 + |\xi_1|)^{-\frac{1}{2}},$$

where N depends only on κ . For a multi-index $|\alpha| \geq 1$, the inequality (2.8) is justified using (2.6) and induction.

Finally, we prove that, for $\alpha = (0, \alpha_2, \dots, \alpha_d)$,

$$\left| D^\alpha \left[\frac{\partial}{\partial \xi_1} \left(\frac{1}{z(\xi_1, \xi')} \right) \right] \right| \leq N(1 + |\xi'|^2 + |\xi_1|)^{-\frac{|\alpha|+3}{2}}, \quad (2.9)$$

where N depends only on d, α , and κ . This follows from (2.7), (2.8), and

$$\frac{\partial}{\partial \xi_1} \left(\frac{1}{z(\xi_1, \xi')} \right) = \frac{i}{2z(\xi_1, \xi')^2} \left(\frac{a_{11}^+}{\sqrt{H^+(\xi')} + i a_{11}^+ \xi_1} + \frac{a_{11}^-}{\sqrt{H^-(\xi')} + i a_{11}^- \xi_1} \right).$$

Now we use (2.8) and (2.9) to prove (2.4) and (2.5). For example, if $\xi_{j_1} = \xi_1$ and $D^\alpha = \frac{\partial^{m-1}}{\partial \xi_{j_2} \cdots \partial \xi_{j_m}}$, then

$$\begin{aligned} & \left| \frac{\partial^m}{\partial \xi_{j_1} \cdots \partial \xi_{j_m}} \left(\frac{(1 + \xi_1^2)^{1/4}}{z(\xi_1, \xi')} \right) \right| \\ &= \left| \frac{\xi_1}{2(1 + \xi_1^2)^{3/4}} D^\alpha \left(\frac{1}{z(\xi_1, \xi')} \right) + (1 + \xi_1^2)^{1/4} D^\alpha \left[\frac{\partial}{\partial \xi_1} \left(\frac{1}{z(\xi_1, \xi')} \right) \right] \right| \\ &\leq N \left(\frac{|\xi_1|}{(1 + \xi_1^2)^{3/4} (1 + |\xi'|^2 + |\xi_1|)^{(|\alpha|+1)/2}} + \frac{(1 + \xi_1^2)^{1/4}}{(1 + |\xi'|^2 + |\xi_1|)^{(|\alpha|+3)/2}} \right) \\ &\leq \frac{N}{|\xi_1| (1 + |\xi'|^2 + |\xi_1|)^{\frac{|\alpha|}{2}}} \leq \frac{N}{|\xi_{j_1} \cdots \xi_{j_m}|}. \end{aligned}$$

The other cases are proved in a similar way. The lemma is proved. \square

Using the above result and interpolation, we prove the following lemma.

Lemma 2.7. *Let \mathcal{T} be an operator such that*

$$\widetilde{\mathcal{T}h} = \frac{1}{z(s, \xi')} \widetilde{h}, \quad h \in C_0^\infty(\mathbb{R}^d),$$

where $\widetilde{\mathcal{T}h}$ and \widetilde{h} are Fourier transforms of $\mathcal{T}h$ and h in \mathbb{R}^d respectively. Set $l = 2 - 1/p$, then

$$\|\mathcal{T}h\|_{W_p^{l/2, l}(\mathbb{R} \times \mathbb{R}^{d-1})} \leq N \|h\|_{W_p^{(l-1)/2, l-1}(\mathbb{R} \times \mathbb{R}^{d-1})}, \quad (2.10)$$

where N depends only on d , p , and κ .

Proof. Define an operator \mathcal{S} by

$$\widetilde{\mathcal{S}h} = \frac{(1 + s^2)^{1/4}}{z(s, \xi')} \widetilde{h},$$

where $h \in C_0^\infty(\mathbb{R}^d)$. By Lemma 2.6 the operator \mathcal{S} can be extended to a bounded operator from $L_p(\mathbb{R}, L_p(\mathbb{R}^{d-1}))$ to $L_p(\mathbb{R}, L_p(\mathbb{R}^{d-1}))$ and from $W_p^1(\mathbb{R}, L_p(\mathbb{R}^{d-1}))$ to $W_p^1(\mathbb{R}, L_p(\mathbb{R}^{d-1}))$. Here we used the fact that $\mathcal{S}(D^\alpha h) = D^\alpha \mathcal{S}(h)$. By real interpolation (see [1]) we have

$$(L_p(\mathbb{R}, L_p(\mathbb{R}^{d-1})), W_p^1(\mathbb{R}, L_p(\mathbb{R}^{d-1})))_{(l-1)/2, p} = W_p^{(l-1)/2}(\mathbb{R}, L_p(\mathbb{R}^{d-1})).$$

Thus

$$\|\mathcal{S}h\|_{W_p^{(l-1)/2}(\mathbb{R}, L_p(\mathbb{R}^{d-1}))} \leq N \|h\|_{W_p^{(l-1)/2}(\mathbb{R}, L_p(\mathbb{R}^{d-1}))}, \quad (2.11)$$

where N depends only on d , p , and κ . Since $(1 + s^2)^{1/4}$ is an isometric multiplier from $W_p^{l/2}(\mathbb{R}) (=B_{p,p}^{l/2}(\mathbb{R}))$ to $W_p^{(l-1)/2}(\mathbb{R}) (=B_{p,p}^{(l-1)/2}(\mathbb{R}))$,

$$\begin{aligned} \|\mathcal{T}h\|_{W_p^{l/2}(\mathbb{R}, L_p(\mathbb{R}^{d-1}))}^p &= \int_{\mathbb{R}^{d-1}} \|\mathcal{T}h(\cdot, x')\|_{W_p^{l/2}(\mathbb{R})}^p dx' \\ &= \int_{\mathbb{R}^{d-1}} \|\mathcal{F}_1^{-1}(1 + s^2)^{1/4} \mathcal{F}_1(\mathcal{T}h)(\cdot, x')\|_{W_p^{(l-1)/2}(\mathbb{R})}^p dx' \\ &= \|\mathcal{S}h\|_{W_p^{(l-1)/2}(\mathbb{R}, L_p(\mathbb{R}^{d-1}))}^p, \end{aligned}$$

where \mathcal{F}_1 and \mathcal{F}_1^{-1} are the Fourier transform and its inverse on \mathbb{R} . This and the inequality (2.11) prove

$$\|\mathcal{T}h\|_{W_p^{l/2}(\mathbb{R}, L_p(\mathbb{R}^{d-1}))} \leq N \|h\|_{W_p^{(l-1)/2}(\mathbb{R}, L_p(\mathbb{R}^{d-1}))}, \quad (2.12)$$

where N depends only on d , p , and κ .

On the other hand, from Lemma 2.6, we know that $i\xi_j/z(s, \xi')$, $j = 2, \dots, d$, are multipliers for $L_p(\mathbb{R}^d)$. This implies that

$$\int_{-\infty}^{\infty} \|\mathcal{T}h(t, \cdot)\|_{W_p^{k+1}(\mathbb{R}^{d-1})}^p dt \leq N \int_{-\infty}^{\infty} \|h(t, \cdot)\|_{W_p^k(\mathbb{R}^{d-1})}^p dt,$$

where $k = 0, 1$ and N depends only on d, p , and κ . Hence \mathcal{T} can be extended to a bounded operator from

$$L_p(\mathbb{R}, W_p^k(\mathbb{R}^{d-1})) \quad \text{to} \quad L_p(\mathbb{R}, W_p^{k+1}(\mathbb{R}^{d-1})),$$

where $k = 0, 1$. Note that by real interpolation (see [1] or Theorem 1.18.4 in [16]), we have

$$\left(L_p(\mathbb{R}, L_p(\mathbb{R}^{d-1})), L_p(\mathbb{R}, W_p^1(\mathbb{R}^{d-1})) \right)_{l-1, p} = L_p(\mathbb{R}, W_p^{l-1}(\mathbb{R}^{d-1}))$$

and

$$\left(L_p(\mathbb{R}, W_p^1(\mathbb{R}^{d-1})), L_p(\mathbb{R}, W_p^2(\mathbb{R}^{d-1})) \right)_{l-1, p} = L_p(\mathbb{R}, W_p^l(\mathbb{R}^{d-1})).$$

Therefore,

$$\|\mathcal{T}h\|_{L_p(\mathbb{R}, W_p^l(\mathbb{R}^{d-1}))} \leq N \|h\|_{L_p(\mathbb{R}, W_p^{l-1}(\mathbb{R}^{d-1}))}, \quad (2.13)$$

where N depends only on d, p , and κ .

Finally, recall that

$$W_p^{s/2, s}(\mathbb{R} \times \mathbb{R}^{d-1}) = L_p(\mathbb{R}, W_p^s(\mathbb{R}^{d-1})) \cap W_p^{s/2}(\mathbb{R}, L_p(\mathbb{R}^{d-1})),$$

where s is either l or $l - 1$. From this, (2.12), and (2.13), the inequality (2.10) follows. \square

We need the Fourier transform of a solution to a parabolic equation defined on a parabolic half-space (e.g. $\mathbb{R} \times \mathbb{R}_+^d$).

Lemma 2.8. For $g(t, x') \in C_0^\infty(\mathbb{R} \times \mathbb{R}^{d-1}) (= C_0^\infty(\mathbb{R}^d))$, let $v^+ \in W_p^{1,2}(\mathbb{R} \times \mathbb{R}_+^d)$ and $v^- \in W_p^{1,2}(\mathbb{R} \times \mathbb{R}_-^d)$ be the solutions of the equations

$$\begin{cases} v_t^+ = L_0^+ v^+ - v^+ & \text{in } \mathbb{R} \times \mathbb{R}_+^d \\ v^+(t, 0, x') = g(t, x') \end{cases}, \quad \begin{cases} v_t^- = L_0^- v^- - v^- & \text{in } \mathbb{R} \times \mathbb{R}_-^d \\ v^-(t, 0, x') = g(t, x'). \end{cases}$$

Then

$$\widetilde{v_{x_1}^+}(s, 0, \xi') = z^+(s, \xi') \widetilde{g}(s, \xi') \quad \text{and} \quad \widetilde{v_{x_1}^-}(s, 0, \xi') = z^-(s, \xi') \widetilde{g}(s, \xi'),$$

where Fourier transforms are taken with respect to $(t, x') \in \mathbb{R} \times \mathbb{R}^{d-1}$.

Proof. First consider

$$\begin{cases} v_t = \Delta v - v & \text{in } \mathbb{R} \times \mathbb{R}_+^d, \\ v(t, 0, x') = g(t, x') \end{cases}, \quad (2.14)$$

where $g \in C_0^\infty(\mathbb{R} \times \mathbb{R}^{d-1})$. Let

$$K(t, x_1, x') = \begin{cases} \frac{1}{(4\pi t)^{d/2}} \frac{x_1}{t} e^{-\frac{1}{4t}|x|^2-t} & \text{for } t > 0, \quad x_1 > 0 \\ 0 & \text{for } t \leq 0, \quad x_1 > 0. \end{cases}$$

Then the solution to (2.14) is

$$v(t, x_1, x') = [K(\cdot, x_1, \cdot) * g(\cdot, \cdot)](t, x').$$

We see that

$$\int_{\mathbb{R}^d} e^{-i(st+\xi' \cdot x')} K(t, x_1, x') dt dx' = e^{-x_1 \sqrt{1+|\xi'|^2+is}}.$$

Thus we have

$$\tilde{v}(s, x_1, \xi') = \tilde{g}(s, \xi') e^{-x_1 \sqrt{1+|\xi'|^2+is}}$$

and

$$\tilde{v}_{x_1}(s, 0, \xi') = -\tilde{g}(s, \xi') \sqrt{1+|\xi'|^2+is}, \quad (2.15)$$

where the second equality can be justified as follows. Note that

$$v(t, x_1, x') = c_d \int_{\mathbb{R}^d} e^{i(ts+x' \cdot \xi')} \tilde{g}(s, \xi') e^{-x_1 \sqrt{1+|\xi'|^2+is}} ds d\xi',$$

where $c_d = (2\pi)^{-d/2}$. Thus

$$\begin{aligned} v_{x_1}(t, 0, x') &= \lim_{h \rightarrow 0} c_d \int_{\mathbb{R}^d} e^{i(ts+x' \cdot \xi')} \tilde{g}(s, \xi') \frac{e^{-h\sqrt{1+|\xi'|^2+is}} - 1}{h} ds d\xi' \\ &= -c_d \int_{\mathbb{R}^d} e^{i(ts+x' \cdot \xi')} \tilde{g}(s, \xi') \sqrt{1+|\xi'|^2+is} ds d\xi'. \end{aligned}$$

By taking Fourier transforms, we arrive at (2.15). This, together with the definition of $z^+(s, \xi')$, proves the case $L_0^+ = \Delta$. The case $L_0^- = \Delta$ is proved similarly. For general L_0^+ and L_0^- , we use the change of variables in the proof of Lemma 2.6. \square

Proof of Theorem 2.3. We prove the inequality (2.1) only for $\lambda > 0$ because the inequality with $\lambda = 0$ is obtained by continuity (with respect to λ). In the case $\lambda > 0$, since we can use a dilation argument, it is enough to prove (i.e., the inequality (2.1) with $\lambda = 1$)

$$\|u\|_{W_p^{1,2}(\mathbb{R} \times \mathbb{R}^d)} \leq N \|L_0 u - u - u_t\|_{L_p}, \quad (2.16)$$

where N depends only on d , p , and κ .

Let $f = u_t + u - L_0 u$ and $g(t, x') = u(t, 0, x')$. Consider

$$\begin{cases} v_t^+ = L_0^+ v^+ - v^+ & \text{in } \mathbb{R} \times \mathbb{R}_+^d \\ v^+(t, 0, x') = g(t, x'), \end{cases} \quad \begin{cases} w_t^+ = L_0^+ w^+ - w^+ + f & \text{in } \mathbb{R} \times \mathbb{R}_+^d \\ w^+(t, 0, x') = 0, \end{cases}$$

and

$$\begin{cases} v_t^- = L_0^- v^- - v^- & \text{in } \mathbb{R} \times \mathbb{R}_-^d \\ v^-(t, 0, x') = g(t, x'), \end{cases} \quad \begin{cases} w_t^- = L_0^- w^- - w^- + f & \text{in } \mathbb{R} \times \mathbb{R}_-^d \\ w^-(0, t, x') = 0. \end{cases}$$

As is well known, the above equations have unique solutions

$$v^+, w^+ \in W_p^{1,2}(\mathbb{R} \times \mathbb{R}_+^d), \quad v^-, w^- \in W_p^{1,2}(\mathbb{R} \times \mathbb{R}_-^d)$$

satisfying (we set $l = 2 - 1/p$.)

$$\|v^+\|_{W_p^{1,2}(\mathbb{R} \times \mathbb{R}_+^d)} \leq N \|g\|_{W_p^{l/2,l}(\mathbb{R} \times \mathbb{R}^{d-1})}, \quad \|w^+\|_{W_p^{1,2}(\mathbb{R} \times \mathbb{R}_+^d)} \leq N \|f\|_{L_p(\mathbb{R} \times \mathbb{R}_+^d)}, \quad (2.17)$$

and

$$\|v^-\|_{W_p^{1,2}(\mathbb{R} \times \mathbb{R}_-^d)} \leq N \|g\|_{W_p^{l/2,l}(\mathbb{R} \times \mathbb{R}^{d-1})}, \quad \|w^-\|_{W_p^{1,2}(\mathbb{R} \times \mathbb{R}_-^d)} \leq N \|f\|_{L_p(\mathbb{R} \times \mathbb{R}_-^d)}, \quad (2.18)$$

where N depends only on d , p , and κ . We see that

$$u = \begin{cases} v^+ + w^+ & \text{in } \mathbb{R} \times \mathbb{R}_+^d \\ v^- + w^- & \text{in } \mathbb{R} \times \mathbb{R}_-^d. \end{cases}$$

It then follows that

$$\|u\|_{W_p^{1,2}(\mathbb{R} \times \mathbb{R}^d)} \leq N \left(\|f\|_{L_p} + \|g\|_{W_p^{l/2,l}(\mathbb{R} \times \mathbb{R}^{d-1})} \right),$$

where N depends only on d , p , and κ . Therefore, to prove (2.16) we need only prove

$$\|g\|_{W_p^{l/2,l}(\mathbb{R} \times \mathbb{R}^{d-1})} \leq N \|f\|_{L_p}, \quad (2.19)$$

where N depends only on d , p , and κ . We prove this inequality as follows.

Observe that (in the trace sense)

$$u_{x_1}(t, 0, x') = v_{x_1}^+(t, 0, x') + w_{x_1}^+(t, 0, x') = v_{x_1}^-(t, 0, x') + w_{x_1}^-(t, 0, x').$$

Thus

$$v_{x_1}^+(t, 0, x') - v_{x_1}^-(t, 0, x') = w_{x_1}^-(t, 0, x') - w_{x_1}^+(t, 0, x'). \quad (2.20)$$

Set

$$h(t, x') = w_{x_1}^-(t, 0, x') - w_{x_1}^+(t, 0, x').$$

Then by Theorem 2.4 we have $h \in W_p^{(l-1)/2, l-1}(\mathbb{R} \times \mathbb{R}^{d-1})$. In addition, from the inequality (2.2) and the estimates for w^\pm in (2.17) and (2.18), we have

$$\|h\|_{W_p^{(l-1)/2, l-1}(\mathbb{R} \times \mathbb{R}^{d-1})} \leq N \|f\|_{L_p},$$

where N depends only on d , p , and κ . From this inequality, we notice that the inequality (2.19) follows if we prove

$$\|g\|_{W_p^{l/2, l}(\mathbb{R} \times \mathbb{R}^{d-1})} \leq N \|h\|_{W_p^{(l-1)/2, l-1}(\mathbb{R} \times \mathbb{R}^{d-1})}. \quad (2.21)$$

Notice that by Lemma 2.8, (2.20), and the definition of $z(s, \xi')$,

$$z(s, \xi') \tilde{g}(s, \xi') = \tilde{h}(s, \xi'), \quad \text{i.e.,} \quad \tilde{g}(s, \xi') = \frac{1}{z(s, \xi')} \tilde{h}(s, \xi').$$

Then Lemma 2.7 proves the inequality (2.21). The theorem is proved. \square

Now we solve the first case. Specifically, for a given $f \in L_p = L_p(\mathbb{R}^{d+1})$ and $\lambda > 0$, there exists a unique $u \in W_p^{1,2}(\mathbb{R} \times \mathbb{R}^d)$ such that

$$u_t = L_0 u - \lambda u + f.$$

This can be done, as is well known, using the estimate in Theorem 2.3 and the method of continuity. In addition, we can find a unique solution $u \in W_p^{1,2}((0, T) \times \mathbb{R}^d)$ of the equation

$$\begin{cases} u_t = L_0 u - \lambda u + f & \text{in } (0, T) \times \mathbb{R}^d \\ u(0, x) = 0 \end{cases},$$

where $0 < T \leq \infty$.

We now consider the general case. That is, we deal with

$$u_t(t, x) = Lu(t, x) - \lambda u(t, x) + f(t, x) \quad \text{in } \mathbb{R} \times \mathbb{R}^d, \quad (2.22)$$

where the coefficients of L satisfy Assumption 2.1 and 2.2. The problem can be solved by freezing the coefficients a_{jk} and using the estimate in Theorem 2.3. Indeed, we can proceed as follows.

Lemma 2.9. *There exist constants $\varepsilon > 0$, λ_0 , and N , depending only on d , K , p , κ , and ω , such that if $u \in C_0^\infty(\mathbb{R}^{d+1})$ satisfies*

$$\text{supp } u \subset \{(t, x_1, x') : |t - t_0| < \varepsilon, |x' - x'_0| < \varepsilon\}$$

for some $(t_0, x'_0) \in \mathbb{R} \times \mathbb{R}^{d-1}$, then

$$\lambda \|u\|_{L_p} + \|u_t\|_{L_p} + \|u_{xx}\|_{L_p} \leq N \|Lu - \lambda u - u_t\|_{L_p} \quad (2.23)$$

for any $\lambda > \lambda_0$.

Proof. We can assume that $t_0 = 0$ and $x'_0 = 0$. Choose an infinitely differentiable function $\eta(x_1)$ defined on \mathbb{R} satisfying $\eta \geq 0$ and

$$\eta(x_1) = \begin{cases} 1 & -\varepsilon/2 \leq x_1 \leq \varepsilon/2 \\ 0 & x_1 \geq \varepsilon \text{ or } x_1 \leq -\varepsilon \end{cases},$$

where ε will be specified later. Set $\mu = 1 - \eta$.

First notice that $\mu u \in W_p^{1,2}(\mathbb{R} \times \mathbb{R}_+^d)$, $\mu u \in W_p^{1,2}(\mathbb{R} \times \mathbb{R}_-^d)$, and $(\mu u)(t, 0, x') = 0$. Thus by the well-known result (see, e.g., Chapter VII and Exercise 7.5 in [9]), there exist λ' and N , depending only on d , K , p , κ , and ω , such that, for any $\lambda > \lambda'$,

$$\begin{aligned} \lambda \|\mu u\|_{L_p(\mathbb{R} \times \mathbb{R}_+^d)} + \|(\mu u)_t\|_{L_p(\mathbb{R} \times \mathbb{R}_+^d)} + \|(\mu u)_{xx}\|_{L_p(\mathbb{R} \times \mathbb{R}_+^d)} \\ \leq N \|L(\mu u) - \lambda(\mu u) - (\mu u)_t\|_{L_p(\mathbb{R} \times \mathbb{R}_+^d)}. \end{aligned}$$

The same estimate holds true with \mathbb{R}_-^d in place of \mathbb{R}_+^d . Hence we have

$$\lambda \|\mu u\|_{L_p} + \|(\mu u)_t\|_{L_p} + \|(\mu u)_{xx}\|_{L_p} \leq N \|L(\mu u) - \lambda(\mu u) - (\mu u)_t\|_{L_p} \quad (2.24)$$

for any $\lambda > \lambda'$. Note that the above N is independent of ε .

Now consider ηu . We see that

$$\text{supp}(\eta u) \subset \{(t, x_1, x') : |t| \leq \varepsilon, |x_1| \leq \varepsilon, |x'| \leq \varepsilon\}.$$

Set

$$L_0 = \begin{cases} a_{jk}^+(0) D_{jk} & \text{if } x_1 > 0 \\ a_{jk}^-(0) D_{jk} & \text{if } x_1 < 0, \end{cases}$$

where $0 \in \mathbb{R}^{d+1}$. Then by Theorem 2.3, we have

$$\lambda \|\eta u\|_{L_p} + \|(\eta u)_t\|_{L_p} + \|(\eta u)_{xx}\|_{L_p} \leq N \|L_0(\eta u) - \lambda(\eta u) - (\eta u)_t\|_{L_p} \quad (2.25)$$

for $\lambda > 0$, where N is the same as in Theorem 2.3. Observe that

$$\begin{aligned} \|L_0(\eta u) - \lambda(\eta u) - (\eta u)_t\|_{L_p} \\ \leq \|L(\eta u) - \lambda(\eta u) - (\eta u)_t\|_{L_p} + \|(L_0 - L)(\eta u)\|_{L_p} \end{aligned}$$

and

$$\|(L_0 - L)(\eta u)\|_{L_p} \leq N\|(\eta u)_x\|_{L_p} + N\|\eta u\|_{L_p} + J_1 + J_2,$$

where N depends only on d and K , and

$$J_1 = \sup_{\substack{(t,x) \in \text{supp}(\eta u) \\ x_1 \geq 0}} \left(|a_{jk}^+(0) - a_{jk}^+(t, x)| \right) \|(\eta u)_{xx}\|_{L_p(\mathbb{R} \times \mathbb{R}_+^d)},$$

$$J_2 = \sup_{\substack{(t,x) \in \text{supp}(\eta u) \\ x_1 \leq 0}} \left(|a_{jk}^-(0) - a_{jk}^-(t, x)| \right) \|(\eta u)_{xx}\|_{L_p(\mathbb{R} \times \mathbb{R}_-^d)}.$$

Also observe that

$$\begin{aligned} |a_{jk}^+(0) - a_{jk}^+(t, x_1, x')| &\leq |a_{jk}^+(0) - a_{jk}^+(t, 0, x')| + |a_{jk}^+(t, 0, x') - a_{jk}^+(t, x_1, x')| \\ &\leq \omega(|t| + |x'|) + \omega(|x_1|). \end{aligned}$$

Similarly,

$$|a_{jk}^-(0) - a_{jk}^-(t, x_1, x)| \leq \omega(|t| + |x'|) + \omega(|x_1|).$$

Hence we can find a sufficiently small $\varepsilon > 0$ depending only on d, K, p, κ , and ω , such that the following inequality holds.

$$\begin{aligned} &\lambda\|\eta u\|_{L_p} + \|(\eta u)_t\|_{L_p} + \|(\eta u)_{xx}\|_{L_p} \\ &\leq N\|L(\eta u) - \lambda(\eta u) - (\eta u)_t\|_{L_p} + \frac{1}{4}\|(\eta u)_{xx}\|_{L_p} + N\|(\eta u)_x\|_{L_p} + N\|\eta u\|_{L_p}. \end{aligned}$$

Using the interpolation inequality, we find that the right hand side of the above inequality is not greater than

$$N\|L(\eta u) - \lambda(\eta u) - (\eta u)_t\|_{L_p} + \frac{1}{2}\|(\eta u)_{xx}\|_{L_p} + N'\|\eta u\|_{L_p},$$

where N is the same as in (2.25) and N' depends only on d, K, p , and κ . This implies that there exists $\lambda'' > 0$ (specifically, $\lambda'' \geq 2N'$) depending only on d, K, p , and κ , such that

$$\lambda\|\eta u\|_{L_p} + \|(\eta u)_t\|_{L_p} + \|(\eta u)_{xx}\|_{L_p} \leq 2N\|L(\eta u) - \lambda(\eta u) - (\eta u)_t\|_{L_p}$$

for any $\lambda > \lambda''$, where N is the one in (2.25). From this inequality and the inequality (2.24) we can find λ_0 and N depending only on d, K, p, κ , and ω , such that the inequality (2.23) holds true. The lemma is proved. \square

Lemma 2.10. *There exist λ_1 and N , depending only on d, K, p, κ , and ω , such that, for any $\lambda > \lambda_1$ and $u \in C_0^\infty(\mathbb{R}^{d+1})$, the estimate (2.23) holds true.*

Proof. First note that (recall again that $L_p = L_p(\mathbb{R}^{d+1})$)

$$\begin{aligned} \|u_t\|_{L_p} &= \|Lu - \lambda u - (Lu - \lambda u - u_t)\|_{L_p} \\ &\leq \|Lu - \lambda u - u_t\|_{L_p} + N\|u_{xx}\|_{L_p} + N\|u_x\|_{L_p} + (\lambda + N)\|u\|_{L_p}. \end{aligned}$$

Thus we need only to prove that there exist λ_1 and N such that, for $\lambda > \lambda_1$,

$$\lambda\|u\|_{L_p} + \|u_{xx}\|_{L_p} \leq N\|Lu - \lambda u - u_t\|_{L_p}.$$

Choose an infinitely differentiable function $\zeta(t, x')$ defined on \mathbb{R}^d (recall that $t \in \mathbb{R}$, $x' \in \mathbb{R}^{d-1}$) such that $\|\zeta\|_{L_p(\mathbb{R}^d)} = 1$ and

$$\text{supp } \zeta \subset \{(t, x') \in \mathbb{R} \times \mathbb{R}^{d-1} : |t| < \varepsilon, |x'| < \varepsilon\},$$

where ε is as in Lemma 2.9. We denote

$$\zeta^{s,y'}(t, x') = \zeta(t - s, x' - y').$$

Then

$$|u_{xx}(t, x)|^p = \int_{\mathbb{R}^d} |u_{xx}(t, x)\zeta^{s,y'}(t, x')|^p ds dy'. \quad (2.26)$$

Using the fact that $u_{x_j x_k} \zeta = (u\zeta)_{x_j x_k} - u_{x_j} \zeta_{x_k} - u_{x_k} \zeta_{x_j} - u \zeta_{x_j x_k}$, we obtain

$$\begin{aligned} |u_{x_j x_k}(t, x)\zeta^{s,y'}(t, x')|^p &\leq 2^p |(u(t, x)\zeta^{s,y'}(t, x'))_{x_j x_k}|^p \\ &\quad + N(|u(t, x)|^p + |u_x(t, x)|^p)\varphi(t - s, x' - y'), \end{aligned}$$

where $\varphi = |\zeta_x|^p + |\zeta_{xx}|^p \in L_1(\mathbb{R}^d)$. Combining this and (2.26), we have

$$\|u_{xx}\|_{L_p}^p \leq N \int_{\mathbb{R}^d} \|(u\zeta^{s,y'})_{xx}\|_{L_p}^p ds dy' + N \left(\|u\|_{L_p}^p + \|u_x\|_{L_p}^p \right).$$

For each $(s, y') \in \mathbb{R}^d$, we see that $u(t, x)\zeta^{s,y'}(t, x')$ satisfies the assumptions in Lemma 2.9. Thus

$$\|(u\zeta^{s,y'})_{xx}\|_{L_p} \leq N \left\| \left(L - \lambda - \frac{\partial}{\partial t} \right) (u\zeta^{s,y'}) \right\|_{L_p}$$

for $\lambda > \lambda_0$, where λ_0 and N are as in Lemma 2.9. We then have

$$\|u_{xx}\|_{L_p}^p \leq N \int_{\mathbb{R}^d} \left\| \left(L - \lambda - \frac{\partial}{\partial t} \right) (u\zeta^{s,y'}) \right\|_{L_p}^p ds dy' + N \left(\|u\|_{L_p}^p + \|u_x\|_{L_p}^p \right). \quad (2.27)$$

Note that

$$\begin{aligned} \left(L - \lambda - \frac{\partial}{\partial t} \right) (u \zeta^{s,y'}) &= \zeta^{s,y'} (Lu - \lambda u - u_t) \\ &\quad + 2a_{jk} u_{x_j} \zeta^{s,y'}_{x_k} + u \left[a_{jk} \zeta^{s,y'}_{x_j x_k} + b_j \zeta^{s,y'}_{x_j} - \zeta_t^{s,y'} \right], \end{aligned}$$

where, for instance, $\zeta_{x_j x_k}^{s,y'}(t, x') = \zeta_{x_j x_k}(t - s, x' - y')$. Thus

$$\begin{aligned} \int_{\mathbb{R}^d} \left\| \left(L - \lambda - \frac{\partial}{\partial t} \right) (u \zeta^{s,y'}) \right\|_{L_p}^p ds dy' \\ \leq N \left(\|Lu - \lambda u - u_t\|_{L_p}^p + \|u\|_{L_p}^p + \|u_x\|_{L_p}^p \right). \end{aligned}$$

From this and (2.27) we have

$$\|u_{xx}\|_{L_p}^p \leq N \left(\|Lu - \lambda u - u_t\|_{L_p}^p + \|u\|_{L_p}^p + \|u_x\|_{L_p}^p \right). \quad (2.28)$$

Similarly,

$$\begin{aligned} \lambda^p \|u\|_{L_p}^p &= \lambda^p \int_{\mathbb{R}^d} \|u \zeta^{s,y'}\|_{L_p}^p ds dy' \\ &\leq N \left(\|Lu - \lambda u - u_t\|_{L_p}^p + \|u\|_{L_p}^p + \|u_x\|_{L_p}^p \right). \quad (2.29) \end{aligned}$$

Therefore, from (2.28) and (2.29) we have

$$\lambda \|u\|_{L_p} + \|u_{xx}\|_{L_p} \leq N_1 \|Lu - \lambda u - u_t\|_{L_p} + N_2 (\|u_x\|_{L_p} + \|u\|_{L_p}),$$

where N_1 and N_2 depend only on d, K, p, κ , and ω . Now we make use of the argument in the proof of Lemma 2.9 to finish the proof. \square

Using the estimate proved in the above lemma and the method of continuity, we arrive at the following theorem.

Theorem 2.11. *Let $\lambda > \lambda_1$, where λ_1 is as in Lemma 2.10. For $f \in L_p(\mathbb{R} \times \mathbb{R}^d)$, there exists a unique solution $u \in W_p^{1,2}(\mathbb{R} \times \mathbb{R}^d)$ satisfying the parabolic equation (2.22). Moreover, the solution u satisfies the estimate (2.23).*

Remark 2.12. It follows from Theorem 2.11 that there exists a unique solution $u \in W_p^{1,2}((0, T) \times \mathbb{R}^d)$ to the equation

$$\begin{cases} u_t(t, x) = Lu(t, x) + f(t, x) & \text{in } (0, T) \times \mathbb{R}^d \\ u(0, x) = 0 \end{cases}.$$

Moreover, the solution u satisfies

$$\|u\|_{W_p^{1,2}((0,T) \times \mathbb{R}^d)} \leq N \|f\|_{L_p((0,T) \times \mathbb{R}^d)},$$

where N depends on d, K, T, p, κ , and ω .

3. Uniqueness of diffusions with discontinuous coefficients

In this section we investigate the well-posedness of the martingale problem for L_t

$$L_t = \frac{1}{2} a_{jk}(t, \cdot) \frac{\partial^2}{\partial x_j \partial x_k} + b_j(t, \cdot) \frac{\partial}{\partial x_j}, \quad (3.1)$$

where the coefficient a_{jk} and b_j satisfy the following assumptions.

Assumption 3.1. $a_{jk}(t, x)$ and $b_j(t, x)$ are Borel measurable functions defined on $[0, \infty) \times \mathbb{R}^d$ such that, for all $t \in [0, \infty)$, $x, \vartheta \in \mathbb{R}^d$,

$$a_{jk} = a_{kj}, \quad a_{jk}(t, x) \vartheta_j \vartheta_k \geq \kappa |\vartheta|^2,$$

$$|a_{jk}(t, x)| + |b_j(t, x)|^2 \leq K(1 + |x|^2).$$

Assumption 3.2. Let I be the set of all integers. There exists a sequence $\{\gamma_j, j \in I\} \subset \mathbb{R}$, $\gamma_j < \gamma_{j+1}$, with no limit points such that

(i) for $x = (x_1, x')$, $x_1 \notin \{\gamma_j, j \in I\}$,

$$\lim_{y \rightarrow x} \sup_{0 \leq t \leq T} |a_{jk}(t, y) - a_{jk}(t, x)| = 0$$

for all $T > 0$,

(ii) for $x = (\gamma_j, x')$, $j \in I$, there exist $a_{jk}^+(t, \gamma_j, x')$ and $a_{jk}^-(t, \gamma_j, x')$ defined on $[0, \infty) \times \mathbb{R}^{d-1}$ such that they are continuous (in (t, x')), and they satisfy

$$\lim_{\substack{(y_1, y') \rightarrow (\gamma_j, x') \\ y_1 > \gamma_j}} \sup_{0 \leq t \leq T} |a_{jk}^+(t, \gamma_j, x') - a_{jk}(t, y_1, y')| = 0$$

and

$$\lim_{\substack{(y_1, y') \rightarrow (\gamma_j, x') \\ y_1 < \gamma_j}} \sup_{0 \leq t \leq T} |a_{jk}^-(t, \gamma_j, x') - a_{jk}(t, y_1, y')| = 0$$

for all $T > 0$.

Let $\Omega = C([0, \infty), \mathbb{R}^d)$ and $\mathcal{F} = \mathcal{B}(\Omega)$, where $C([0, \infty), \mathbb{R}^d)$ is the Polish space of continuous \mathbb{R}^d -valued functions given on $[0, \infty)$ with metric

$$d(\omega_1, \omega_2) = \sum_{n=1}^{\infty} \frac{1}{2^n} \max_{0 \leq t \leq n} (|\omega_1(t) - \omega_2(t)| \wedge 1).$$

$\mathcal{B}(\Omega)$ is, as usual, the Borel σ -field of subsets of Ω . Define a random variable X on Ω by $X_t(\omega) = \omega(t)$, $\omega \in \Omega$.

We say that the martingale problem for L_t (or martingale problem for a_{jk} and b_j) is well-posed (see [15], [4]) if, for each $(s, x) \in [0, \infty) \times \mathbb{R}^d$, there exists a unique probability measure $P_{s,x}$ on (Ω, \mathcal{F}) satisfying

$$P_{s,x}(X_t = x, \quad 0 \leq t \leq s) = 1$$

such that

$$f(X_t) - \int_s^t L_r f(X_r) dr$$

is a $P_{s,x}$ -martingale after time s for all $f \in C_0^\infty(\mathbb{R}^d)$.

Remark 3.3. The well-posedness of the martingale problem for the operator L_t in (3.1) is equivalent to that of the stochastic integral equation

$$\xi_t = x + \int_0^t \sigma(r, \xi_r) dW_r + \int_0^t b(r, \xi_r) dr,$$

where W_t is a Brownian motion, σ is the positive square root of the matrix $[a_{jk}]_{j,k=1}^d$, and $b = (b_j)_{j=1}^d$. The stochastic integral equation is said to be well-posed if, for every initial condition $x \in \mathbb{R}^d$, it admits a weak solution which is unique in the sense of probability law. For details, see [4].

We first note that, for each $(s, x) \in [0, \infty) \times \mathbb{R}^d$, there exists a solution to the martingale problem for L_t . This follows from Remark 2.1 in [7] and references therein.

To prove uniqueness, we start with coefficients a_{jk} and b_j which satisfy assumptions in Section 2.

Lemma 3.4. *Let $a_{jk}(t, x)$ and $b_j(t, x)$ be Borel measurable functions defined on $[0, \infty) \times \mathbb{R}^d$ satisfying Assumption 2.1 and 2.2 in Section 2. Then the martingale problem for L_t is well-posed.*

Proof. Clearly, we need to prove only the uniqueness of a solution to martingale problem for a_{jk} and b_j starting from $(s, x) \in [0, \infty) \times \mathbb{R}^d$. By Corollary 6.2.6 in [15], this can be done if we prove that, for each $0 \leq s < T$ and $f \in C_0^\infty([s, T] \times \mathbb{R}^d)$, there is a bounded measurable function u defined on $[0, T) \times \mathbb{R}^d$ such that

$$u(s, x) = -E^P \left[\int_s^T f(t, X_t) dt \right] \tag{3.2}$$

whenever P solves the martingale problem for L_t starting from $(s, x) \in [0, \infty) \times \mathbb{R}^d$.

Let $P = P_{s,x}$ is a solution to the martingale problem for L_t starting from $(s, x) \in [0, \infty) \times \mathbb{R}^d$. For a given $f \in C_0^\infty([s, T] \times \mathbb{R}^d)$, we set

$$\tilde{f} = \begin{cases} f(t, x) & \text{if } s \leq t \leq T \\ 0 & \text{otherwise.} \end{cases}$$

We also fix a real number p such that $d + 1 < p < \infty$. Then by remark 2.12 we can find a unique solution $u \in W_p^{1,2}((0, T) \times \mathbb{R}^d)$ of the equation

$$\begin{cases} u_t + L_t u(t, x) = \tilde{f} & \text{in } (0, T) \times \mathbb{R}^d \\ u(T, x) = 0 \end{cases}.$$

Now we find a sequence $u_n \in C^{1,2}([0, T] \times \mathbb{R}^d)$ such that each u_n is bounded (derivatives are bounded as well), $u_n \rightarrow u$ in $W_p^{1,2}((0, T) \times \mathbb{R}^d)$, and $u_n(T, x) = 0$. Note that we have (see Lemma 6.3.1 in [15])

$$u_n(s, x) = -E^P \left[\int_s^T (u_{nt} + L_t u_n)(t, X_t) dt \right]$$

for each n , where, as we recall,

$$\begin{aligned} (u_{nt} + L_t u_n)(t, X_t) \\ = u_{nt}(t, X_t) + \frac{1}{2} a_{jk}(t, X_t) u_{nx_j x_k}(t, X_t) + b_j(t, X_t) u_{nx_j}(t, X_t). \end{aligned}$$

Since $u_n(s, x) \rightarrow u(s, x)$ pointwise by the Sobolev embedding theorem, to prove (3.2), it only remains to prove that

$$E^P \left[\int_s^T (u_{nt} + L_t u_n)(t, X_t) dt \right] \rightarrow E^P \left[\int_s^T (u_t + L_t u)(t, X_t) dt \right].$$

This follows easily from

$$\begin{aligned} E^P \int_s^T |(u_t^n + L_t u^n)(t, X_t) - (u_t + L_t u)(t, X_t)| dt \\ \leq N \|(u_{nt} + L_t u_n) - (u_t + L_t u)\|_{L_p((0, T) \times \mathbb{R}^d)} \\ \leq N \|u_n - u\|_{W_p^{1,2}((0, T) \times \mathbb{R}^d)} \rightarrow 0, \end{aligned}$$

where the first inequality is due to Theorem 2.3.4 in [6]. The lemma is proved. \square

Now we can say that the martingale problem for L_t , where coefficients a_{jk} and b_j satisfying Assumption 3.1 and 3.1, is well-posed. The justification of this claim follows from Corollary 10.1.2 and Theorem 6.6.1 in [15]. Indeed, we may prove this as follows.

As noted earlier, we have the existence for the martingale problem for L_t . Thus by Corollary 10.1.2 in [15] we need only check the well-posedness of the martingale problem for L_t with bounded a_{jk} and b_j satisfying Assumption 3.1 and 3.2. In that case by Theorem 6.6.1 (and Exercise 6.7.4) in [15] it is enough to show that, for each $(s, x) \in [0, \infty) \times \mathbb{R}^d$, there exist coefficients \tilde{a}_{jk} and \tilde{b}_j defined on $[0, \infty) \times \mathbb{R}^d$

such that

- (i) $\tilde{a}_{jk} = a_{jk}$ and $\tilde{b}_j = b_j$ in a neighborhood of (s, x) ,
- (ii) the martingale for \tilde{L}_t is well-posed, where

$$\tilde{L}_t = \frac{1}{2} \tilde{a}_{jk}(t, \cdot) \frac{\partial^2}{\partial x_j \partial x_k} + \tilde{b}_j(t, \cdot) \frac{\partial}{\partial x_j}.$$

Note that $\{\gamma_j, j \in I\}$ has no limit points. Thus, for points $(s, x) \in [0, \infty) \times \mathbb{R}^d$, where $x_1 \notin \{\gamma_j, j \in I\}$, we can find coefficients \tilde{a}_{jk} and \tilde{b}_j satisfying the above two conditions. Especially, (ii) is satisfied by results in Ch. 7 in [15].

For points $(s, x) \in [0, \infty) \times \mathbb{R}^d$, where $x_1 \in \{\gamma_j, j \in I\}$ (say, for example, $x_1 = 0$ and $0 \in \{\gamma_j, j \in I\}$), we find coefficients \tilde{a}_{jk} and \tilde{b}_j defined on $[0, \infty) \times \mathbb{R}^d$ such that they satisfy Assumption 2.1 and 2.2 in Section 2 as well as (i) in the above. In fact, we may set

$$\tilde{a}_{jk} = \varphi a_{jk} + (1 - \varphi) \delta_{jk}, \quad \tilde{b}_j = b_j,$$

where φ is a non-negative infinitely differentiable function defined on \mathbb{R}^{d+1} such that $\varphi = 1$ in a sufficiently small neighborhood of $(s, 0, x')$ and $\text{supp } \varphi$ does not intersect $\{(t, \gamma_k, x') : (t, x') \in [0, \infty) \times \mathbb{R}^{d-1}\}$ for all $\gamma_k \in \{\gamma_j, j \in I\} \setminus \{0\}$. Now we prove (ii), i.e., the well-posedness of the martingale problem for \tilde{a}_{jk} and \tilde{b}_j , using Lemma 3.4. Therefore, we have proved

Theorem 3.5. *Under Assumption 3.1 and 3.2, the martingale problem for L_t , where L_t is as in (3.1), is well-posed.*

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School of Mathematics
University of Minnesota
127 Vincent Hall
206 Church St. SE
Minneapolis, MN 55455, U.S.A.
dykim@math.umn.edu