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On Cubics and Quartics Through a Canonical Curve

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Abstract. We construct families of quartic and cubic hypersurfaces through a canonical curve, which are parametrized by an open subset in a Grassmannian and a Flag variety respectively. Using G. Kempf's cohomological obstruction theory, we show that these families cut out the canonical curve and that the quartics are birational (via a blowing-up of a linear subspace) to quadric bundles over the projective plane, whose Steinerian curve equals the canonical curve.

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1. – Introduction

Let C be a smooth nonhyperelliptic curve of genus $g \geq 4$ defined over the complex numbers, which we consider as an embedded curve $\iota_\omega : C \hookrightarrow \mathbb{P}^{g-1}$ by its canonical linear series $|\omega|$. Let $I = \bigoplus_{n \geq 2} I(n)$ be the graded ideal of the canonical curve. It was classically known (Noether-Enriques-Petri theorem, see e.g. [ACGH] p. 124) that the ideal I is generated by its elements of degree 2, unless C is trigonal or a plane quintic.

It was also classically known how to construct some distinguished quadrics in $I(2)$. We consider a double point of the theta divisor $\Theta \subset \mathrm{Pic}^{g-1}(C)$, which corresponds by Riemann's singularity theorem to a degree $g-1$ line bundle L satisfying $\dim |L| = \dim |\omega L^{-1}| = 1$ and we observe that the morphism $\iota_L \times \iota_{\omega L^{-1}} : C \longrightarrow C' \subset |L|^* \times |\omega L^{-1}|^* = \mathbb{P}^1 \times \mathbb{P}^1$ (here C' denotes the image curve) followed by the Segre embedding into $\mathbb{P}^3$ factorizes through the canonical space $|\omega|^*$, i.e.,

$$\begin{array}{ccc} C & \hookrightarrow & |\omega|^* \\ \big\downarrow & & \big\downarrow \pi \\ \mathbb{P}^1 \times \mathbb{P}^1 & \hookrightarrow & \mathbb{P}^3, \end{array}$$

where π is projection from a $(g-5)$-dimensional vertex $\mathbb{P}V^\perp$ in $|\omega|^*$. We then

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define the quadric $Q_L := \pi^{-1}(\mathbb{P}^1 \times \mathbb{P}^1)$, which is a rank ≤ 4 quadric in $I(2)$ and coincides with the projectivized tangent cone at the double point $[L] \in \Theta$ under the identification of $H^0(C, \omega)^*$ with the tangent space $T_{[L]} \operatorname{Pic}^{g-1}(C)$. The main result, due to M. Green [Gr], asserts that the set of quadrics $\{Q_L\}$, when L varies over the double points of Θ, linearly spans $I(2)$. From this result one infers a constructive Torelli theorem by intersecting all quadrics Q_L — at least for C general enough.

The geometry of the theta divisor Θ at a double point $[L]$ can also be exploited to produce higher degree elements in the ideal I as follows: we expand in a suitable set of coordinates a local equation θ of Θ near $[L]$ as $\theta = \theta_2 + \theta_3 + \dots$, where θ_i are homogeneous forms of degree i. Having seen that $Q_L = \operatorname{Zeros}(\theta_2)$, we denote by S_L the cubic $\operatorname{Zeros}(\theta_3) \subset |\omega|^*$, the osculating cone of Θ at $[L]$. The cubic S_L has many nice geometric properties: under the blowing-up of the vertex $\mathbb{P}V^\perp \subset S_L$, the cubic S_L is transformed into a quadric bundle $\tilde{S}_L$ over $\mathbb{P}^1 \times \mathbb{P}^1$ and it was shown by G. Kempf and F.-O. Schreyer [KS] that the Hessian and Steinerian curves of $\tilde{S}_L$ are $C' \subset \mathbb{P}^1 \times \mathbb{P}^1$ and $C \subset |\omega|^*$ respectively, which gives another proof of Torelli's theorem.

In this paper we construct and study distinguished cubics and quartics in the ideal I by adapting the methods of [KS] to rank-2 vector bundles over C. Our construction basically goes as follows (Section 2): we consider a general 3-plane $W \subset H^0(C, \omega)$ and define the rank-2 vector bundle E_W as the dual of the kernel of the evaluation map in ω of sections of W. The bundle E_W is stable and admits a theta divisor $D(E_W)$ in the Jacobian JC. Since $D(E_W)$ contains the origin $\mathcal{O} \in JC$ with multiplicity 4, the projectivized tangent cone to $D(E_W)$ at $\mathcal{O}$ is a quartic hypersurface in $\mathbb{P}T_{\mathcal{O}}JC = |\omega|^*$, denoted by F_W and which contains the canonical curve. We therefore obtain a rational map from the Grassmannian $\operatorname{Gr}(3, H^0(\omega))$ to the ideal of quartics $|I(4)|$

$$\mathbf{F}_4 : \operatorname{Gr}(3, H^0(\omega)) \dashrightarrow |I(4)|, \qquad W \mapsto F_W . \tag{1.1}$$

Our main tool to study the tangent cones F_W is G. Kempf's cohomological obstruction theory [K1], [K2], [KS] which in our set-up leads to a simple criterion (Proposition 4.1) for $b \in \mathbb{P}T_{\mathcal{O}}JC = |\omega|^*$ to belong to F_W. We deduce in particular from this criterion that the cubic polar $P_x(F_W)$ of F_W with respect to a point $x \in W^\perp$ also contains the canonical curve. Here $W^\perp$ denotes the annihilator of $W \subset H^0(\omega)$. We therefore obtain a rational map from the flag variety $\operatorname{Fl}(3, g-1, H^0(\omega))$ parametrizing pairs (W, x) to the ideal of cubics $|I(3)|$

$$\mathbf{F}_3 : \operatorname{Fl}(3, g-1, H^0(\omega)) \dashrightarrow |I(3)|, \qquad (W, x) \mapsto P_x(F_W) . \tag{1.2}$$

Our two main results can be stated as follows.

(1) Like the cubic osculating cones S_L, the quartic tangent cones F_W transform under the blowing-up of the vertex $\mathbb{P}W^\perp \subset F_W$ into a quadric bundle $\tilde{F}_W \to \mathbb{P}W^* = \mathbb{P}^2$. Their Hessian and Steinerian curves are the plane curve

Γ, image under the projection with center $\mathbb{P}W^{\perp}$, $\pi : C \to \Gamma \subset \mathbb{P}W^*$, and the canonical curve $C \subset |\omega|^*$ (Theorem 4.8). This surprising analogy with the osculating cones S_L remains however unexplained.

(2) Let us denote by $|F_4| \subset |I(4)|$ and $|F_3| \subset |I(3)|$ the linear subsystems spanned by the quartics F_W and the cubics $P_x(F_W)$ respectively. Then we show (Theorem 6.1) that both base loci of $|F_4|$ and $|F_3|$ coincide with $C \subset |\omega|^*$,i.e., the quartics F_W (resp. the cubics $P_x(F_W)$) cut out the canonical curve.

The starting point of our investigations was the question asked by B. van Geemen and G. van der Geer ([vGvG] page 629) about “these mysterious quartics” which arise as tangent cones to 2θ-divisors in the Jacobian having multiplicity ≥ 4 at the origin. In that paper the authors implicitly conjectured that the base locus of $|F_4|$ equals C, which was subsequently proved by G. Welters [We]. Our proof follows from the fact that $|F_4|$ contains all squares of quadrics in $|I(2)|$.

This paper leaves many questions unanswered (Section 7), like e.g. finding explicit equations of the quartics F_W, their syzygies, the dimensions of $|F_3|$ and $|F_4|$. The techniques used here also apply when replacing $|\omega|^*$ by Prym-canonical space $|\omega\alpha|^*$, and generalizing rank-2 vector bundles to symplectic bundles.

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2. – Some constructions for rank-2 vector bundles with canonical determinant

In this section we briefly recall some known results from [BV], [vGI] and [PP] on rank-2 vector bundles over C.

2.1. – Bundles E with $\dim H^0(C, E) \geq 3$

Let $W \subset H^0(C,\omega)$ be a 3-plane. We denote by $[W] \in \mathrm{Gr}(3, H^0(\omega))$ the corresponding point in the Grassmannian and by $\mathcal{B} \subset \mathrm{Gr}(3, H^0(\omega))$ the codimension 2 subvariety consisting of $[W]$ such that the net $\mathbb{P}W \subset |\omega|$ has a base point. For $[W] \notin \mathcal{B}$ we consider (see [vGI] Section 4) the rank-2 vector bundle E_W defined by the exact sequence

$$0 \longrightarrow E_W^* \longrightarrow \mathcal{O}_C \otimes W \xrightarrow{ev} \omega \longrightarrow 0\,. \tag{2.1}$$

Here E_W^* denotes the dual bundle of E_W. We have $\det E_W = \omega$ and $W^* \subset H^0(C, E_W)$. We denote by $\mathcal{D}$ the effective divisor in $|\mathcal{O}_{\mathrm{Gr}}(g-2)|$ defined by the condition

$$[W] \in \mathcal{D} \iff \dim H^0(C, E_W) \geq 4 .$$

We have the inclusion $\mathcal{B} \subset \mathcal{D}$. If $[W] \notin \mathcal{D}$, then E_W is stable ([vGI] Lemma 4.2).

Let $W^\perp \subset H^0(\omega)^* = H^1(\mathcal{O})$ denote the annihilator of $W \subset H^0(\omega)$. We call the projective subspace $\mathbb{P}W^\perp \subset |\omega|^*$ the *vertex* and denote by

$$\pi : |\omega|^* \dashrightarrow \mathbb{P}W^*, \qquad \pi : C \to \Gamma \subset \mathbb{P}W^* ,$$

the projection with center $\mathbb{P}W^\perp$. Abusing notation we also denote by π a linear lift $\pi : H^0(\omega)^* \to W^*$. If $[W] \notin \mathcal{B}$, then $C \cap \mathbb{P}W^\perp = \emptyset$ and π restricts to a morphism $C \to \mathbb{P}W^*$. Its image is a plane curve Γ of degree $2g-2$. We note that $E_W = \pi^*(T(-1))$, where T is the tangent bundle of $\mathbb{P}W^* = \mathbb{P}^2$.

Conversely any globally generated bundle E with $\det E = \omega$ is of the form E_W.

2.2. – Bundles E with $\dim H^0(C, E) \geq 4$

Following [BV] (see also [PP] Section 5.2) we associate to a bundle E with $\dim H^0(C, E) = 4$ a rank ≤ 6 quadric $Q_E \in |I(2)|$, which is defined as the inverse image of the Klein quadric under the dual μ^* of the exterior product map

$$\mu^* : |\omega|^* \longrightarrow \mathbb{P}(\Lambda^2 H^0(E)^*) \supset \mathrm{Gr}(2, H^0(E)^*), \qquad Q_E := (\mu^*)^{-1}(\mathrm{Gr}) .$$

Composing with the previous construction, we obtain a rational map

$$\alpha : \mathcal{D} \dashrightarrow |I(2)|, \qquad \alpha([W]) = Q_{E_W} .$$

Moreover given a $Q \in |I(2)|$ with $\mathrm{rk} Q \leq 6$ and $\mathrm{Sing}\, Q \cap C = \emptyset$, it is easily shown that

$$\alpha^{-1}(Q) = \{[W] \in \mathcal{D} | \mathbb{P}W^\perp \subset Q\} .$$

If rk $Q = 6$, then $\alpha^{-1}(Q)$ has two connected components, which are isomorphic to $\mathbb{P}^3$.

LEMMA 2.1. *We have $[W] \notin \mathcal{D}$ if and only if the linear map induced by restricting quadrics to the vertex $\mathbb{P}W^\perp$*

$$res : I(2) \longrightarrow H^0(\mathbb{P}W^\perp, \mathcal{O}(2))$$

is an isomorphism.

PROOF. It is enough to observe that the two spaces have the same dimension and that a nonzero element in $\ker res$ corresponds to a $Q \in |I(2)|$ with $\mathrm{rk} Q \leq 6$. □

2.3. – Definition of the quartic F_W

We will now define the main object of this paper. Given $[W] \notin \mathcal{B}$, we consider the 2θ-divisor $D(E_W) \subset JC$ (see e.g. [BV], [vGI], [PP]), whose set-theoretical support equals

$$D(E_W) = \{\xi \in JC \mid \dim H^0(C, \xi \otimes E_W) > 0\}.$$

Since $\text{mult}_{\mathcal{O}} D(E_W) \geq \dim H^0(C, E_W) \geq 3$ and since any 2θ-divisor is symmetric, the first nonzero term of the Taylor expansion of a local equation of $D(E_W)$ at the origin $\mathcal{O}$ is a homogeneous polynomial F_W of degree 4. The hypersurface in $|\omega|^* = \mathbb{P}T_{\mathcal{O}}JC$ associated to F_W is also denoted by F_W. Here we restrict attention to the case $\dim H^0(C, E_W) = 3$ or 4. We have

$$F_W := \text{Cone}_{\mathcal{O}}(D(E_W)) \subset |\omega|^*.$$

The study of the quartics F_W for $[W] \in \text{Gr}(3, H^0(\omega)) \setminus \mathcal{D}$ is the main purpose of this paper. If $[W] \in \mathcal{D}$, the quartics F_W have already been described in [PP] Proposition 5.12.

PROPOSITION 2.2. *If* $\dim H^0(C, E_W) = 4$, *then F_W is a double quadric*

$$F_W = Q^2_{E_W}.$$

Since $|I(2)|$ is linearly spanned by rank ≤ 6 quadrics (see [PP] Section 5), we obtain the following fact, which will be used in Section 6.

PROPOSITION 2.3. *The linear subsystem $|F_4|$ contains all squares of quadrics in $|I(2)|$.*

Although we will not use that fact, we mention that the rational map (1.1) is given by a linear subsystem $\Pi \subset |\mathcal{J}_{\mathcal{B}}(g-1)|$, where $\mathcal{J}_{\mathcal{B}}$ is the ideal sheaf of the subvariety $\mathcal{B}$. If $g = 4$, the inclusion is an equality (see [OPP] Section 6). If $g > 4$, a description of Π is not known.

3. – Kempf's cohomological obstruction theory

In this section we outline Kempf's deformation theory [K1] and apply it to the study of the tangent cones F_W of the divisors $D(E_W)$.

3.1. – Variation of cohomology

Let $\mathcal{E}$ be a vector bundle over the product $C \times S$, where $S = \text{Spec}(A)$ is an affine neighbourhood of the origin of JC. We restrict attention to the case

$$\mathcal{E} = \pi_C^* E_W \otimes \mathcal{L},$$

for some 3-plane W, and recall that Kempf's deformation theory was applied [K1], [K2], [KS] to the case $\mathcal{E} = \pi_C^* M \otimes \mathcal{L}$, for a line bundle M over C. The line bundle $\mathcal{L}$ denotes the restriction of a Poincaré line bundle over $C \times JC$ to the neighbourhood $C \times S$. The fundamental idea to study the variation of cohomology, i.e., the two upper-semicontinuous functions on S

$$s \mapsto h^0(C \times \{s\}, \mathcal{E} \otimes_A \mathbb{C}_s), \qquad s \mapsto h^1(C \times \{s\}, \mathcal{E} \otimes_A \mathbb{C}_s),$$

where $\mathbb{C}_s = A/\mathfrak{m}_s$ and $\mathfrak{m}_s$ is the maximal ideal of $s \in S$, is based on the existence of an approximating homomorphism.

THEOREM 3.1 (Grothendieck, [K1] Section 7). *Given a family $\mathcal{E}$ of vector bundles over $C \times S$, there exist two flat A-modules F and G of finite type and an A-homomorphism $\alpha : F \to G$ such that for all A-modules M, we have isomorphisms*

$$H^0(C \times S, \mathcal{E} \otimes_A M) \cong \ker(\alpha \otimes_A id_M), \quad H^1(C \times S, \mathcal{E} \otimes_A M) \cong \operatorname{coker}(\alpha \otimes_A id_M).$$

By considering a smaller neighbourhood of the origin, we may assume the A-modules F and G to be locally free (Nakayama's lemma). Moreover ([K1] Lemma 10.2) by restricting further the neighbourhood, we may find an approximating homomorphism $\alpha : F \to G$ such that $\alpha \otimes \mathbb{C}_0 : F \otimes_A A/\mathfrak{m}_0 \to G \otimes_A A/\mathfrak{m}_0$ is the zero homomorphism.

We apply this theorem to the family $\mathcal{E} = \pi_C^* E_W \otimes \mathcal{L}$, for $[W] \notin \mathcal{D}$. Since by Riemann-Roch $\chi(\mathcal{E} \otimes \mathbb{C}_s) = \chi(E_W \otimes \mathcal{L}_s) = 0$, $\forall s \in S$, and since $h^0(C, E_W) = 3$, the local equation f of the divisor

$$D(E_W)_{|S} = \{s \in S : | : h^0(C \times \{s\}, E_W \otimes \mathcal{L}_s) > 0\}$$

is given at the origin $\mathcal{O}$ by the determinant of a 3×3 matrix of regular functions f_{ij} on S, with $1 \leq i, j \leq 3$, which vanish at $\mathcal{O}$, i.e., the A-modules F and G are free and of rank 3. Hence

$$f = \det(f_{ij}).$$

The linear part of the regular functions f_{ij} is related to the cup-product as follows ([K1] Lemma 10.3 and Lemma 10.6): let $\mathfrak{m} = \mathfrak{m}_0$ be the maximal ideal of the origin $\mathcal{O} \in S$ and consider the exact sequence of A-modules

$$0 \longrightarrow \mathfrak{m}/\mathfrak{m}^2 \longrightarrow A/\mathfrak{m}^2 \longrightarrow A/\mathfrak{m} \longrightarrow 0.$$

After tensoring with $\mathcal{E}$ over $C \times S$ and taking cohomology, we obtain a coboundary map

$$\begin{aligned} H^0(C, E_W) = H^0(C \times \{s\}, \mathcal{E} \otimes_A A/\mathfrak{m}) &\xrightarrow{\delta} H^1(C \times \{s\}, \mathcal{E} \otimes_A \mathfrak{m}/\mathfrak{m}^2) \\ &= H^1(C, E_W) \otimes \mathfrak{m}/\mathfrak{m}^2, \end{aligned}$$

where $\mathfrak{m}/\mathfrak{m}^2$ is the Zariski cotangent space at $\mathcal{O}$ to JC. Note that we have a canonical isomorphism $(\mathfrak{m}/\mathfrak{m}^2)^* \cong H^1(\mathcal{O})$ and that a tangent vector $b \in H^1(\mathcal{O})$ gives, by composing with the linear form $l_b : \mathfrak{m}/\mathfrak{m}^2 \to \mathbb{C}$, a linear map $\delta_b : H^0(E_W) \to H^1(E_W)$. As in the line bundle case [K1], one proves

LEMMA 3.2. *For any nonzero $b \in H^1(\mathcal{O}) = T_{\mathcal{O}}JC$, we have*

1. *The linear map $\delta_b : H^0(E_W) \to H^1(E_W)$ coincides with the cup-product $(\cup b)$ with the class b, and is* skew-symmetric *after identifying $H^1(E_W)$ with $H^0(E_W)^*$ (Serre duality).*
2. *The coboundary map $\delta : H^0(E_W) \to H^1(E_W) \otimes \mathfrak{m}/\mathfrak{m}^2$ is described by a skew-symmetric 3×3 matrix (x_{ij}), with $x_{ij} \in H^1(\mathcal{O})^*$. Moreover the linear form x_{ij} coincides with the differential $(df_{ij})_0$ of f_{ij} at the origin $\mathcal{O}$.*

The coboundary map δ induces a linear map

$$\Delta : H^1(\mathcal{O}) \longrightarrow \Lambda^2 H^0(E_W)^*, \quad b \longmapsto \delta_b\,,$$

which coincides with the dual of the multiplication map of global sections of E_W. Moreover

$$\ker \Delta = W^\perp = \{x_{12} = x_{13} = x_{23} = 0\}\,.$$

Using a flat structure [K2] we can write the power series expansion of the regular functions f_{ij} around $\mathcal{O}$

$$f_{ij} = x_{ij} + q_{ij} + \dots\,,$$

where x_{ij} and q_{ij} are linear and quadratic polynomials respectively. We easily calculate the expansion of f: by skew-symmetry its cubic term is zero, and its quartic term equals

$$F_W : q_{11}x_{23}^2 + q_{22}x_{13}^2 + q_{33}x_{12}^2 + x_{12}x_{23}(q_{13}+q_{31}) - x_{12}x_{23}(q_{12}+q_{21}) - x_{12}x_{13}(q_{23}+q_{32}).$$

We straightforwardly deduce from this equation the following properties of F_W.

PROPOSITION 3.3.

1. *The quartic F_W is singular along the vertex $\mathbb{P}W^\perp$.*
2. *For any $x \in W^\perp$, the cubic polar $P_x(F_W)$ is singular along the vertex $\mathbb{P}W^\perp$.*

3.2. – Infinitesimal deformations of global sections of E_W

We first recall some elementary facts on principal parts. Let V be an arbitrary vector bundle over C and let $\mathrm{Rat}(V)$ be the space of rational sections of V and p be a point of C. The space of principal parts of V at p is the quotient

$$\mathrm{Prin}_p(V) = \mathrm{Rat}(V)/\mathrm{Rat}_p(V)\,,$$

where $\mathrm{Rat}_p(V)$ denotes the space of rational sections of V which are regular at p. Since a rational section of V has only finitely many poles, we have a natural mapping

(3.1)
$$\mathrm{pp} : \mathrm{Rat}(V) \longrightarrow \mathrm{Prin}(V) := \bigoplus_{p \in C} \mathrm{Prin}_p(V), \quad s \longmapsto (s \mod \mathrm{Rat}_p(V))_{p \in C}\,.$$

Exactly as in the line bundle case ([K1] Lemma 3.3), one proves

LEMMA 3.4. *There are isomorphisms*

$$\ker \mathrm{pp} \cong H^0(C, V), \quad \operatorname{coker} \mathrm{pp} \cong H^1(C, V).$$

In the particular case $V = \mathcal{O}$, we see that a tangent vector $b \in H^1(\mathcal{O}) = T_{\mathcal{O}} JC$ can be represented by a collection $\beta = (\beta_p)_{p \in I}$ of rational functions $\beta_p \in \mathrm{Rat}(\mathcal{O})$, where p varies over a finite set of points $I \subset C$. We then define $\mathrm{pp}(\beta) = (\omega_p)_{p \in I} \in \mathrm{Prin}(\mathcal{O})$, where ω_p is the principal part of β_p at p. We denote by $[\beta] = b$ its cohomology class in $H^1(\mathcal{O})$. Note that we can define powers of β by $\beta^k := (\beta_p^k)_{p \in I}$.

For $i \geq 1$, let D_i be the infinitesimal scheme $\mathrm{Spec}(A_i)$, where A_i is the Artinian ring $\mathbb{C}[\epsilon]/\epsilon^{i+1}$. As explained in [K2] Section 2, a tangent vector $b \in H^1(\mathcal{O})$ determines a morphism

$$\exp_{i,b} : D_i \longrightarrow JC,$$

with $\exp_{i,b}(x_0) = \mathcal{O}$, where x_0 is the closed point of D_i. Let $\mathbb{L}_{i+1}(b)$ denote the pull-back of the Poincaré sheaf $\mathcal{L}$ under the morphism $\exp_{i,b} \times id_C$. Note that we have the following exact sequences

$$D_1 \times C : \quad 0 \longrightarrow \epsilon\mathcal{O} \longrightarrow \mathbb{L}_2(b) \longrightarrow \mathcal{O} \longrightarrow 0, \tag{3.2}$$

$$D_2 \times C : \quad 0 \longrightarrow \epsilon^2\mathcal{O} \longrightarrow \mathbb{L}_3(b) \longrightarrow \mathbb{L}_2(b) \longrightarrow 0. \tag{3.3}$$

The second arrows in each sequence correspond to the restriction to the subschemes $\{x_0\} \times C \subset D_1 \times C$ and $D_1 \times C \subset D_2 \times C$ respectively. As above we choose a representative β of b. Following [K2] Section 2, one shows that the space of global sections $H^0(C \times D_i, \mathbb{L}_{i+1}(b) \otimes E)$, with $E = E_W$ and $[W] \notin \mathcal{D}$, is isomorphic to the A_i-module

$$V_i(\beta) = \{f = f_0 + \ldots + f_i\epsilon^i \in \mathrm{Rat}(E) \otimes A_i \text{ such that } f \exp(\epsilon\beta) \text{ is regular } \forall p \in C\}. \tag{3.4}$$

An element $f \in V_i(\beta)$ is called an i-th order deformation of the global section $f_0 \in H^0(E)$. In the case $i = 2$, the condition $f \in V_i(\beta)$ is equivalent to the following three elements,

$$f_0, \quad f_1 + f_0\beta, \quad f_2 + f_1\beta + f_0\frac{\beta^2}{2}, \tag{3.5}$$

being regular at all points $p \in C$ — for $i = 1$, we consider the first two elements. Alternatively this means that their classes in $\mathrm{Prin}(E)$ are zero. We note that, given two representatives $\beta = (\beta_p)_{p \in I}$ and $\beta' = (\beta'_p)_{p \in I'}$ with $[\beta] = [\beta']$, the two subspaces $V_i(\beta)$ and $V_i(\beta')$ of $\mathrm{Rat}(E) \otimes A_i$ are different and that any rational function $\varphi \in \mathrm{Rat}(\mathcal{O})$ satisfying $\mathrm{pp}(\varphi) = \mathrm{pp}(\beta' - \beta)$ induces an isomorphism $V_i(\beta) \cong V_i(\beta')$.

We consider a class $b \in H^1(\mathcal{O}) \setminus W^\perp$ and a representative β such that $[\beta] = b$. By taking cohomology of (3.2) tensored with E, we observe that a first order deformation of f_0, i.e., a global section $f = f_0 + f_1\epsilon \in V_1(\beta) \cong H^0(C \times D_1, \mathbb{L}_2(b) \otimes E)$ always exists. Since $\mathrm{rk}(\cup b) = 2$, the global section f_0 is uniquely determined up to a scalar

$$f_0 \cdot \mathbb{C} = \ker(\cup b : H^0(E) \longrightarrow H^1(E)) .$$

Moreover any two first order deformations of f_0 differ by an element in $\epsilon H^0(E)$.

We now state a criterion for a tangent vector $b = [\beta]$ to lie on the quartic tangent cone F_W in terms of a second order deformation of $f_0 \in H^0(E)$.

Lemma 3.5. *A cohomology class $b = [\beta] \in H^1(\mathcal{O}) \setminus W^\perp$ is contained in the cone over the quartic F_W if and only if there exists a global section*

$$f = f_0 + f_1\epsilon + f_2\epsilon^2 \in V_2(\beta) \cong H^0(C \times D_2, \mathbb{L}_3(b) \otimes E) .$$

Proof. The proof is similar to [KS] Lemma 4. We work over the Artinian ring A_4, i.e., $\epsilon^5 = 0$. By Theorem 3.1 applied to the family $\mathbb{L}_5(b) \otimes E$ over $C \times D_4$, there exists an approximating homomorphism of A_4-modules

$$A_4^{\oplus 3} \xrightarrow{\varphi} A_4^{\oplus 3} , \tag{3.6}$$

such that $\ker \varphi_{|D_2} \cong H^0(C \times D_2, \mathbb{L}_3(b) \otimes E)$, $\mathrm{coker}\, \varphi_{|D_2} \cong H^1(C \times D_2, \mathbb{L}_3(b) \otimes E)$, and $\varphi \otimes \mathbb{C}_0 = 0$. We denote by $\varphi_{|D_2}$ the homomorphism obtained from (3.6) by projecting to A_2. Note that any A_4-module is free. The matrix φ is equivalent to a matrix

$$M := \begin{pmatrix} \epsilon^u & 0 & 0 \\ 0 & \epsilon^v & 0 \\ 0 & 0 & \epsilon^w \end{pmatrix} .$$

Since $\varphi \otimes \mathbb{C}_0 = 0$, we have $u, v, w \geq 1$. Moreover we can order the exponents so that $1 \leq u \leq v \leq w$. It follows from the definition of $D(E_W)$ as a determinant divisor that the pull-back of $D(E_W)$ by $\exp_4 : D_4 \longrightarrow JC$ is given by the equation (in A_4)

$$\det M = \epsilon^{u+v+w} .$$

We immediately see that $b \in F_W$ if and only if $u + v + w \geq 5$. Let us now restrict φ to D_1,i.e., we project (3.6) to A_1. Since we assume $b \notin W^\perp = \ker \Delta$, the restriction $\varphi_{|D_1}$ is nonzero and by skew-symmetry of rank 2, i.e., $u = v = 1$ and $w \geq 2$. Hence $b \in F_W$ if and only if $w \geq 3$.

On the other hand the A_2-module $\ker \varphi_{|D_2} \cong H^0(C \times D_2, \mathbb{L}_3(b) \otimes E)$ has length $2+w$. Let μ be the multiplication by ϵ^2 on this A_2-module. Then by (3.4) the A_2-module $\ker \mu$ is isomorphic to the A_1-module $H^0(C \times D_1, \mathbb{L}_2(b) \otimes E)$, which is of length 4, provided $b \notin W^\perp$. Hence we obtain that $w \geq 3$ if and only if there exists an $f \in H^0(C \times D_2, \mathbb{L}_3(b) \otimes E)$ such that $\mu(f) = \epsilon^2 f_0$. This proves the lemma. □

4. – Study of the quartic F_W

In this section we prove geometric properties of the quartic F_W.

4.1. – Criteria for $b \in F_W$

We now show that the criterion of Lemma 3.5 simplifies to a criterion involving only a first order deformation $f = f_0 + f_1\epsilon \in V_1(\beta)$ of f_0. As above we assume $b \notin W^\perp$.

First we observe that the rational differential form $f_1 \wedge f_0$ is independent of the choice of the representative β, i.e., $f_1 \wedge f_0$ only depends on the cohomology class $b = [\beta]$: suppose we take $\beta' = \left(\beta_p \cdot \varphi\right)_{p\in I}$, where $\varphi \in \mathrm{Rat}(\omega)$. Then f_0 and f_1 transform into $f_0' = f_0$ and $f_1' = f_1 + \varphi f_0$, from which it is clear that $f_1' \wedge f_0' = f_1 \wedge f_0$.

Secondly one easily sees that $f_0 = \pi(b)$ (Section 2.1) and that, under the canonical identification $\Lambda^2 W^* = \Lambda^2 H^0(E) = W$, the 2-plane $H^0(E) \wedge f_0$ coincides with the intersection $V_b := H_b \cap W$, where H_b denotes the hyperplane determined by $b \in H^1(\mathcal{O})$.

It follows from these two remarks that, given b and W, the form $f_1 \wedge f_0$ is well-defined up to a regular differential form in $V_b \subset W$.

Proposition 4.1. *We have the following equivalence*

$$b \in F_W \quad \Longleftrightarrow \quad f_1 \wedge f_0 \in H_b\,.$$

Proof. Since $f_1 \wedge f_0$ does not depend on β, we may choose a β with simple poles at the points $p \in I$. By Lemma 3.5 and relation (3.5) we see that $b \in F_W$ if and only if the cohomology class $[f_1\beta + f_0\frac{\beta^2}{2}]$ is zero in $H^1(E)/\mathrm{im}(\cup b)$ – we recall that f_1 is defined up to $H^0(E)$.

First we will prove that $[f_0\frac{\beta^2}{2}] \in \mathrm{im}(\cup b)$. The commutativity of the upper right triangle of the diagram (see e.g. [K1])

$$\begin{array}{ccccccc}
 & & & & H^0(E) & & \\
 & & & & \downarrow \cdot\frac{\beta^2}{2} & \searrow & \cup[\frac{\beta^2}{2}] \\
H^0(E) & \longrightarrow & H^0(E(2I)) & \longrightarrow & E(2I)_{|2I} & \longrightarrow & H^1(E) \\
 & & \cap & & \cap & \nearrow & \\
 & & \mathrm{Rat}(E) & \xrightarrow{\mathrm{pp}} & \mathrm{Prin}(E) & &
\end{array}$$

implies that $[f_0\frac{\beta^2}{2}] = f_0 \cup [\frac{\beta^2}{2}]$. Moreover the skew-symmetric cup-product map $\cup b$

$$\cup b = \wedge\overline{b} : H^0(E) = W^* \longrightarrow H^1(E) = W = \Lambda^2 W^*$$

identifies with the exterior product $\wedge\overline{b}$, where $\overline{b} = \pi(b) \in W^*$. It is clear that $\mathrm{im}(\cup b) = \mathrm{im}(\wedge\overline{b}) = \ker(\wedge\overline{b})$, where $\wedge\overline{b}$ also denotes the linear form

$$\wedge\overline{b} : \Lambda^2 W^* \longrightarrow \Lambda^3 W^* \cong \mathbb{C}\,. \tag{4.1}$$

As already observed, we have $f_0 = \overline{b}$. Denoting by $c \in W^*$ the class $\pi([\frac{\beta^2}{2}])$, we see that the relation $(f_0 \wedge c) \wedge \overline{b} = \overline{b} \wedge c \wedge \overline{b} = 0$ implies that $f_0 \cup [\frac{\beta^2}{2}] \in \ker(\wedge\overline{b}) = \mathrm{im}(\cup b)$.

Therefore the previous condition simplifies to $[f_1\beta] \in \mathrm{im}(\cup b)$. We next observe that the linear form $\wedge\overline{b}$ on $H^1(E)$ (4.1) identifies with the exterior product map

$$H^1(E) \xrightarrow{\wedge f_0} H^1(\omega) \cong \mathbb{C}\,.$$

Since we have a commutative diagram

$$\begin{array}{cccccc} f_1 \in & H^0(E(I)) & \xrightarrow{\cdot\beta} & \mathrm{Prin}(E) & \longrightarrow & H^1(E) \\ & & & \downarrow \wedge f_0 & & \downarrow \wedge f_0 \\ f_1 \wedge f_0 \in & H^0(\omega) & \xrightarrow{\cdot\beta} & \mathrm{Prin}(\omega) & \longrightarrow & H^1(\omega), \end{array}$$

and since $f_1 \wedge f_0 \in H^0(\omega) \subset \mathrm{Rat}(\omega)$, we easily see that the condition $[f_1\beta] \in \mathrm{im}(\cup b)$ is equivalent to $f_1 \wedge f_0 \in H_b = \ker(\cup b : H^0(\omega) \longrightarrow H^1(\omega))$. □

In the following proposition we give more details on the element $f_1 \wedge f_0 \in H^0(\omega)$. We additionally assume that $\pi(b) \notin \Gamma$, which implies that the global section $f_0 \in H^0(E)$ does not vanish at any point and hence determines an exact sequence

$$0 \longrightarrow \mathcal{O} \xrightarrow{f_0} E \xrightarrow{\wedge f_0} \omega \longrightarrow 0\,. \tag{4.2}$$

The coboundary map of the associated long exact sequence

$$\cdots \longrightarrow H^0(\omega) \xrightarrow{\cup e} H^1(\mathcal{O}) \longrightarrow \cdots \tag{4.3}$$

is symmetric and coincides (e.g. [K1] Corollary 6.8) with cup-product $\cup e$ with the extension class $e \in \mathbb{P}H^1(\omega^{-1}) = |\omega^2|^*$. Moreover $\cup e$ is the image of e under the dual of the multiplication map

$$H^1(\omega^{-1}) = H^0(\omega^2)^* \hookrightarrow \mathrm{Sym}^2 H^0(\omega)^*, \quad e \longmapsto \cup e\,. \tag{4.4}$$

We note that $\mathrm{corank}(\cup e) = 2$ and that $\ker(\cup e) = V_b$. Hence $(f_1 \wedge f_0) \cup e$ is well-defined.

PROPOSITION 4.2. *If $\pi(b) \notin \Gamma$, then $f_1 \wedge f_0 \notin \ker(\cup e)$ and we have (up to a nonzero scalar)*

$$(f_1 \wedge f_0) \cup e = b \in H^1(\mathcal{O}) .$$

PROOF. We keep the notation of the previous proof. The condition $f_1 \wedge f_0 \in V_b$ implies that f_1 is a regular section and, by (3.5), that f_0 vanishes at the support of b, i.e., $\pi(b) \in \Gamma$. As for the equality of the proposition, we introduce the rank-2 vector bundle $\hat{E}$ which is obtained from E by (positive) elementary transformations at the points $p \in I$ and with respect to the line in E_p spanned by the nonzero vector $f_0(p)$. Then we have $E \subset \hat{E} \subset E(I)$ and $\hat{E}$ fits into the exact sequence

$$0 \longrightarrow E \longrightarrow \hat{E} \longrightarrow \mathcal{O}_I \longrightarrow 0 .$$

Moreover $f_1 \in H^0(\hat{E})$, which follows from condition (3.5). We also have the following exact sequences

$$\begin{array}{ccccccccc} 0 \longrightarrow & \mathcal{O}(I) & \longrightarrow & \hat{E} & \xrightarrow{\wedge f_0} & \omega & \longrightarrow 0 & & (\hat{e}) \\ & \cup & & \cup & & \| & & & \\ 0 \longrightarrow & \mathcal{O} & \xrightarrow{f_0} & E & \xrightarrow{\wedge f_0} & \omega & \longrightarrow 0 & & (e), \end{array}$$

and the extension class $\hat{e} \in H^1(\omega^{-1}(D))$ is obtained from e by the canonical projection $H^1(\omega^{-1}) \to H^1(\omega^{-1}(I))$. Taking the associated long exact sequences, we obtain

$$\begin{array}{ccccc} f_1 \in H^0(\hat{E}) & \xrightarrow{\wedge f_0} & H^0(\omega) & \xrightarrow{\cup \hat{e}} & H^1(\mathcal{O}(I)) \\ \cup & & \| & & \uparrow \pi_I \\ H^0(E) & \xrightarrow{\wedge f_0} & H^0(\omega) & \xrightarrow{\cup e} & H^1(\mathcal{O}) , \end{array}$$

where the two squares commute. This means that

$$\pi_I((f_1 \wedge f_0) \cup e) = (f_1 \wedge f_0) \cup \hat{e} = 0 .$$

Since $f_1 \wedge f_0$ does not depend on β (nor on I), the latter relation holds for any I with $I = \operatorname{supp} \beta$. Hence, denoting by $\langle I \rangle$ the linear span in $|\omega|^*$ of the support I of β, we obtain

$$(f_1 \wedge f_0) \cup e \in \bigcap_{I = \operatorname{supp} \beta} \ker \pi_I = \bigcap_{b \in \langle I \rangle} \langle I \rangle = b . \qquad \square$$

4.2. – Geometric properties of F_W

PROPOSITION 4.3. *For any $[W] \notin \mathcal{D}$ we have the following*

1. *The quartic F_W contains the canonical curve C, i.e., $F_W \in |I(4)|$.*
2. *The quartic F_W contains the secant line $\overline{pq}$, with $p \neq q$, if and only if $\overline{pq} \cap \mathbb{P}W^\perp \neq \emptyset$ or $\dim W \cap H^0(\omega(-2p-2q)) > 0$.*
3. *Let Σ be the set of points p at which the tangent line $\mathbb{T}_p(C)$ intersects the vertex $\mathbb{P}W^\perp$. Then Σ is empty for general $[W]$ and finite for any $[W]$. Moreover any point $p \in C \setminus \Sigma$ is smooth on F_W and the embedded tangent space $\mathbb{T}_p(F_W)$ is the linear span of $\mathbb{T}_p(C)$ and $\mathbb{P}W^\perp$.*

PROOF. All statements are easily deduced from Proposition 4.1. Given a point $p \in C$ we denote by $\mathfrak{p}_p \in \mathrm{Prin}_p(\mathcal{O})$ the principal part supported at p of a rational function with a simple pole at p. Then the class $[\mathfrak{p}_p] \in H^1(\mathcal{O})$ is proportional to $i_\omega(p) \in |\omega|^* = \mathbb{P}H^1(\mathcal{O})$ and the section f_0 vanishes at p. Hence $f_0\mathfrak{p}_p \in \mathrm{Prin}(E)$ is everywhere regular and we may choose $f_1 = 0$. This proves part 1. See also [PP].

As for part 2, we introduce $\beta_{\lambda,\mu} = \lambda\mathfrak{p}_p + \mu\mathfrak{p}_q \in \mathrm{Prin}(\mathcal{O})$ for $\lambda, \mu \in \mathbb{C}$ and denote by s_p and s_q the global sections $\pi([\mathfrak{p}_p])$ and $\pi([\mathfrak{p}_q])$, which vanish at p and q respectively. Then one checks that $f_0 = \lambda s_p + \mu s_q \in \ker(\cup[\beta_{\lambda,\mu}])$ and $\mathrm{pp}(f_1) = \lambda\mu(s_q\mathfrak{p}_p + s_p\mathfrak{p}_q) \in \mathrm{Prin}(E)$. With this notation the condition of Proposition 4.1 transforms into

$$0 = l_{\lambda,\mu}(f_0 \wedge f_1) = \lambda\mu(\lambda^2\gamma_p + \mu^2\gamma_q)\,, \tag{4.5}$$

where $l_{\lambda,\mu}$ is the linear form defined by $[\beta_{\lambda,\mu}] \in H^1(\mathcal{O})$. The scalars γ_p and γ_q are the values of the section $s_p \wedge s_q \in W \cap H^0(\omega(-p-q))$ at p and q respectively. We now conclude noting that $s_p \wedge s_q = 0$ if and only if $\overline{pq} \cap \mathbb{P}W^\perp \neq \emptyset$.

As for part 3, we first observe that the assumption $\Sigma = C$ implies that the restriction $\pi_{|C} : C \to \mathbb{P}W^*$ contracts C to a point, which is impossible. Next we consider the tangent vector t_q at p given by the direction q. By putting $\lambda = 1$ and $\mu = \epsilon$, with $\epsilon^2 = 0$, into equation (4.5) we obtain that $t_q \in \mathbb{T}_p(F_W)$ if and only if $\epsilon\gamma_p = 0$, i.e., $\pi(q) \in \mathbb{T}_{\pi(p)}(\Gamma)$. Hence $\mathbb{T}_p(F_W) = \pi^{-1}(\mathbb{T}_{\pi(p)}(\Gamma))$, which proves part 3. □

4.3. – The cubic polar $P_x(F_W)$

Firstly we deduce from Propositions 4.1 and 4.2 a criterion for $b \in P_x(F_W)$, with $x \in W^\perp$. Let H_x be the hyperplane determined by $x \in H^1(\mathcal{O})$. As above we assume $b \notin W^\perp$ and $\pi(b) \notin \Gamma$, i.e., the pencil $V = V_b$ is base-point-free.

PROPOSITION 4.4. *We have the following equivalence*

$$b \in P_x(F_W) \quad \Longleftrightarrow \quad f_1 \wedge f_0 \in H_x\,.$$

PROOF. We recall from Section 4.1 that $\cup e$ induces a symmetric isomorphism $\cup e : (V^\perp)^* \xrightarrow{\sim} V^\perp$ and we denote by $Q^* \subset \mathbb{P}(V^\perp)^*$ and $Q \subset \mathbb{P}V^\perp$ the two associated smooth quadrics. Note that Q and Q^* are dual to each other. Combining Propositions 4.1, 4.2 and 3.3 (1) we see that the restriction of the quartic F_W to the linear subspace $\mathbb{P}V^\perp \subset |\omega|^*$ splits into a sum of divisors

$$(F_W)_{|\mathbb{P}V^\perp} = 2\mathbb{P}W^\perp + Q\,.$$

We also observe that Q only depends on V (and on W) and not on b. Taking the polar with respect to $x \in W^\perp$, we obtain

$$(P_x(F_W))_{|\mathbb{P}V^\perp} = 2\mathbb{P}W^\perp + P_x(Q)\,.$$

Finally we see that the condition $b \in P_x(Q)$ is equivalent to $f_0 \wedge f_1 = (\cup e)^{-1}(b) \in H_x$. □

We easily deduce from this criterion some properties of $P_x(F_W)$.

PROPOSITION 4.5. *The cubic $P_x(F_W)$ contains the canonical curve C, i.e., $P_x(F_W) \in |I(3)|$.*

PROOF. We first observe that the two closed conditions of Proposition 4.4 are equivalent outside $\pi^{-1}(\Gamma)$. Hence they coincide as well on $\pi^{-1}(\Gamma)$ and we can drop the assumption $\pi(b) \notin \Gamma$. Now, as in the proof of Proposition 4.3 (1), we may choose $f_1 = 0$. □

PROPOSITION 4.6. *We have the following properties*

$$\bigcap_{x \in W^\perp} P_x(F_W) = S_W \cup \mathbb{P}W^\perp \cup \bigcup_{n \geq 2} \Lambda_n ,$$

$$F_W \cap S_W = C \cup \Lambda_1, \quad \textit{and} \quad \Lambda := \bigcup_{n \geq 0} \Lambda_n \subset F_W ,$$

where S_W is an irreducible surface. For $n \geq 0$, we denote by Λ_n the union of $(n+1)$-secant $\mathbb{P}^n$'s to the canonical curve C, which intersect the vertex $\mathbb{P}W^\perp$ along a $\mathbb{P}^{n-1}$. If W is general, then $\Lambda_n = \emptyset$ for $n \geq 2$ and Λ_1 is the union of $2(g-1)(g-3)$ secant lines.

PROOF. We consider b in the intersection of all $P_x(F_W)$ and we first suppose that $\pi(b) \notin \Gamma$. Then by Propositions 4.1 and 4.4 we have

$$f_0 \wedge f_1 \in \bigcap_{x \in W^\perp} H_x = W .$$

Hence we obtain that $\mathbb{P}V^\perp \cap \bigcap_{x \in W^\perp} P_x(F_W)$ is reduced to the point $(\cup e)(W) \in \mathbb{P}V^\perp$. On the other hand a standard computation shows that S_W is the image of $\mathbb{P}^2$ under the linear system of the adjoint curves of Γ. Hence S_W is irreducible.

If $\pi(b) \in \Gamma$, we denote by $p_1, \dots, p_{n+1} \in C$ the points such that $\pi(p_i) = \pi(b)$. Then f_0 vanishes at $p_1, \dots, p_{n+1}$. Since $f_1 \wedge f_0$ does not depend on the support of b, we can choose $\operatorname{supp} b$ such that $p_i \notin \operatorname{supp} b$. Then f_1 is regular at p_i and we deduce that $f_1 \wedge f_0 \in H^0(\omega(-\sum p_i)) \cap W = V_b$. Now any rational f_1 satisfying $f_1 \wedge f_0 \in V_b = \operatorname{im}(\wedge f_0)$ is regular everywhere, which can only happen when f_0 vanishes at the support of b. By uniqueness we have $\operatorname{supp} b \subset \{p_1, \dots, p_{n+1}\}$ and $b \in \Lambda_n$. Note that $\Lambda_0 = C$. This proves the first equality.

If $b \in F_W \cap S_W$, we have $f_1 \wedge f_0 \in W \cap H_b = V_b$ and we conclude as above. Note that Λ_1 is contained in S_W and is mapped by π to the set of ordinary double points of Γ. □

For any $[W] \in \operatorname{Gr}(3, H^0(\omega)) \setminus \mathcal{D}$ we introduce the subspace of $I(3)$

$$L_W = \{R \in I(3) | R \text{ is singular along the vertex } \mathbb{P}W^\perp\} .$$

Then Propositions 4.5 and 3.3 (2) imply that $P_x(F_W) \in L_W$. More precisely, we have

PROPOSITION 4.7. *The restriction of the polar map of the quartic F_W to its vertex $\mathbb{P}W^\perp$*

$$\mathbf{P} : W^\perp \longrightarrow L_W, \quad x \longmapsto P_x(F_W)\,,$$

is an isomorphism.

PROOF. First we show that $\dim L_W = g-3$. We choose a complementary subspace A to $W^\perp$,i.e., $H^0(\omega)^* = W^\perp \oplus A$, and a set of coordinates $x_1, \dots, x_{g-3}$ on $W^\perp$ and a_1, a_2, a_3 on A. This enables us to expand a cubic $F \in S^3H^0(\omega)$

$$F = F_3(x) + F_2(x)G_1(a) + F_1(x)G_2(a) + G_3(a),$$
$$F_i \in \mathbb{C}[x_1, \dots, x_{g-3}],\, G_i \in \mathbb{C}[a_1, a_2, a_3]\,,$$

with $\deg F_i = \deg G_i = i$. Let $\mathcal{S}_A$ denote the subspace of cubics singular along $\mathbb{P}A$,i.e. $G_2 = G_3 = 0$. We consider the linear map

$$\alpha : I(3) \longrightarrow \mathcal{S}_A, \quad F \longmapsto F_3(x) + F_2(x)G_1(a)\,.$$

Since by Lemma 2.1 any monomial $x_ix_j \in H^0(\mathbb{P}W^\perp, \mathcal{O}(2))$ lifts to a quadric $Q_{ij} \in I(2)$, we observe that the monomials $x_ix_jx_k$ and $x_ix_ja_l$, which generate $\mathcal{S}_A$, also lift e.g. to $Q_{ij}x_k$ and $Q_{ij}a_l$ in $I(3)$. Hence α is surjective and $\dim L_W = \dim\ker\alpha$ is easily calculated. One also checks that this computation does not depend on A.

In order to conclude, it will be enough to show that $\mathbf{P}$ is injective. Suppose that the contrary holds, i.e., there exists a point $x \in W^\perp$ with $P_x(F_W) = 0$. Given any base-point-free pencil $V \subset W$ and any $b \in V^\perp$, we obtain by Proposition 4.4 that $f_0 \wedge f_1 \in H_x$. Since $\cup e : (V^\perp)^* \xrightarrow{\sim} V^\perp$ is an isomorphism, we see that for $b \notin (\cup e)^{-1}(H_x)$ the element $f_0 \wedge f_1$ must be zero. This implies that $b \in \Lambda$ and since b varies in an open subset of $|\omega|^*$, we obtain $\Lambda = |\omega|^*$, a contradiction. □

4.4. – The quadric bundle associated to F_W

Let $\tilde{\mathbb{P}}^{g-1}_W \to |\omega|^*$ denote the blowing-up of $|\omega|^*$ along the vertex $\mathbb{P}W^\perp \subset |\omega|^*$. The rational projection $\pi : |\omega|^* \dashrightarrow \mathbb{P}^2 = \mathbb{P}W^*$ resolves into a morphism $\tilde{\pi} : \tilde{\mathbb{P}}^{g-1}_W \to \mathbb{P}^2$. Since F_W is singular along $\mathbb{P}W^\perp$ (Proposition 3.3 (2)), the proper transform $\tilde{F}_W \subset \tilde{\mathbb{P}}^{g-1}_W$ admits a structure of a quadric bundle $\tilde{\pi} : \tilde{F}_W \to \mathbb{P}^2$.

The contents of Propositions 4.3 and 4.5 can be reformulated in a more geometrical way.

THEOREM 4.8. *For any $[W] \in \mathrm{Gr}(3, H^0(\omega)) \setminus \mathcal{D}$, the quadric bundle $\tilde{\pi} : \tilde{F}_W \to \mathbb{P}^2$ has the following properties*

1. *Its* Hessian *curve is $\Gamma \subset \mathbb{P}^2$.*
2. *Its* Steinerian *curve is the (proper transform of the) canonical curve $C \subset |\omega|^*$.*
3. *The rational* Steinerian map $\mathrm{St} : \Gamma \dashrightarrow C$, *which associates to a singular quadric its singular point, coincides with the adjoint map* ad *of the plane curve Γ. Moreover the closure of the image* $\mathrm{ad}(\mathbb{P}^2)$ *equals S_W.*

REMARK 4.9. We note that Theorem 4.8 is analogous to the main result of [KS] (replace $\mathbb{P}^2$ with $\mathbb{P}^1 \times \mathbb{P}^1$). In spite of this striking similarity and the relation between the two parameter spaces Sing and $\mathrm{Gr}(3, H^0(\omega))$ (see [PP]), we were unable to find a common frame for both constructions.

5. – The cubic hypersurface $\Psi_V \subset \mathbb{P}^{g-3}$ associated to a base-point-free pencil $\mathbb{P}V \subset |\omega|$

In this section we show that the symmetric cup-product maps $\cup e \in \mathrm{Sym}^2 H^0(\omega)^*$ (see (4.3)) arise as polar quadrics of a cubic hypersurface Ψ_V, which will be used in the proof of Theorem 6.1.

Let V denote a base-point-free pencil of $H^0(\omega)$. We consider the exact sequence given by evaluation of sections of V

$$0 \longrightarrow \omega^{-1} \longrightarrow \mathcal{O}_C \otimes V \xrightarrow{ev} \omega \longrightarrow 0\,. \tag{5.1}$$

Its extension class $v \in \mathrm{Ext}^1(\omega, \omega^{-1}) \cong H^1(\omega^{-2}) \cong H^0(\omega^3)^*$ corresponds to the hyperplane in $H^0(\omega^3)$, which is the image of the multiplication map

$$\mathrm{im}(V \otimes H^0(\omega^2) \longrightarrow H^0(\omega^3))\,. \tag{5.2}$$

We consider the cubic form Ψ_V defined by

$$\Psi_V : \mathrm{Sym}^3 H^0(\omega) \xrightarrow{\mu} H^0(\omega^3) \xrightarrow{\bar{v}} \mathbb{C}\,,$$

where μ is the multiplication map and $\bar{v}$ the linear form defined by the extension class v. It follows from the description (5.2) that Ψ_V factorizes through the quotient

$$\Psi_V : \mathrm{Sym}^3 \mathcal{V} \longrightarrow \mathbb{C},$$

where $\mathcal{V} := H^0(\omega)/V$. We also denote by $\Psi_V \subset \mathbb{P}\mathcal{V}$ its associated cubic hypersurface.

A 3-plane $W \supset V$ determines a nonzero vector w in the quotient $\mathcal{V} = H^0(\omega)/V$ and a general w determines an extension (4.2) — recall that $W^* \cong H^0(E)$. Hence we obtain an injective linear map $\mathcal{V} \hookrightarrow H^1(\omega^{-1})$, $w \mapsto e$, which we compose with (4.4)

$$\Phi : \mathcal{V} \hookrightarrow H^1(\omega^{-1}) = H^0(\omega^2)^* \hookrightarrow \mathrm{Sym}^2 H^0(\omega)^*, \qquad w \mapsto e \mapsto \cup e.$$

Since $V \subset \ker(\cup e)$, we note that $\mathrm{im}\Phi \subset \mathrm{Sym}^2 \mathcal{V}^*$.

We now can state the main result of this section.

PROPOSITION 5.1. *The linear map* $\Phi : \mathcal{V} \to \mathrm{Sym}^2\mathcal{V}^*$ *coincides with the polar map of the cubic form* Ψ_V*, i.e.,*

$$\forall w \in \mathcal{V}, \qquad \Phi(w) = P_w(\Psi_V).$$

PROOF. This is straightforwardly read from the diagram obtained by relating the exact sequences (5.1) and (2.1) via the inclusion $V \subset W$. We leave the details to the reader. □

We also observe that, by definition of the Hessian hypersurface (see e.g. [DK] Section 3), we have an equality among degree $g-2$ hypersurfaces of $\mathbb{P}\mathcal{V} = \mathbb{P}^{g-3}$

$$\mathrm{Hess}(\Psi_V) = \mathcal{D} \cap \mathbb{P}\mathcal{V}, \tag{5.3}$$

where we use the inclusion $\mathbb{P}\mathcal{V} \subset \mathrm{Gr}(3, H^0(\omega))$.

REMARK 5.2. We recall (see [DK] (5.2.1)) that the Hessian and Steinerian of a cubic hypersurface coincide and that the Steinerian map is a rational involution i. In the case of the cubic Ψ_V, the involution

$$i : \mathrm{Hess}(\Psi_V) \dashrightarrow \mathrm{Hess}(\Psi_V)$$

corresponds to the involution of [BV] Propositions 1.18 and 1.19, i.e., $\forall w \in \mathcal{D} \cap \mathbb{P}\mathcal{V}$, the bundles E_w and $E_{i(w)}$ are related by the exact sequence

$$0 \longrightarrow E^*_{i(w)} \longrightarrow \mathcal{O}_C \otimes H^0(E_w) \xrightarrow{ev} E_w \longrightarrow 0.$$

Since we will not use that result, we leave its proof to the reader.

REMARK 5.3. The construction which associates to a base-point-free pencil $V \subset H^0(\omega)$ the extension class $v \in |\omega^3|^*$ induces a rational map

$$\mathrm{Gr}(2, H^0(\omega)) \dashrightarrow |\omega^3|^*, \qquad V \longmapsto v.$$

It is worthwhile to investigate the possible relations between that map and the Wahl map

$$\mathrm{Gr}(2, H^0(\omega)) \longrightarrow |\omega^3|, \qquad V = \langle s, t\rangle \longmapsto t^{\otimes 2} d\,(s/t)\,.$$

6. – Base loci of $|F_3|$ and $|F_4|$

Let us denote by $|F_3| \subset |I(3)|$ and $|F_4| \subset |I(4)|$ the linear subsystems spanned by the image of the rational maps $\mathbf{F}_3$ and $\mathbf{F}_4$ respectively. Then we have the following

THEOREM 6.1. *The base loci of $|F_3|$ and $|F_4|$ coincide with the canonical curve $C \subset |\omega|^*$.*

PROOF. Let $b \in \mathrm{Bs}\,|F_3|$ and let us suppose that $b \notin C$. We consider a base-point-free pencil $V \subset H_b$. With the notation of section 5, we introduce the rational map

$$r_b : \mathbb{P}\mathcal{V} \dashrightarrow \mathbb{P}\mathcal{V}, \qquad w \mapsto r_b(w) = w', \qquad \text{with } \tilde{\Psi}_V(w, w', \cdot) = b,$$

where $\tilde{\Psi}_V$ is the symmetric trilinear form of Ψ_V. We note (Proposition 4.2) that, for $w \notin \mathbb{P}(H_b/V)$, the element $r_b(w)$ is collinear with the nonzero element $f_0 \wedge f_1 \mod V$ and that r_b is defined away from the hypersurface $\mathrm{Hess}(\Psi_V)$, which we assume to be nonzero. Since $b \in \mathrm{Bs}\,|F_3|$ we obtain by Proposition 4.4 that

$$r_b(w) = \left(\bigcap_{x \in W^\perp} H_x \right) \mod V = W \mod V = w.$$

Hence r_b is the identity map (away from $\mathrm{Hess}(\Psi_V)$). This implies that $\tilde{\Psi}_V(w, w, \cdot) = b$ for any $w \in \mathbb{P}\mathcal{V}$, hence $\Psi_V = x_0^3$, where x_0 is the equation of the hyperplane $\mathbb{P}(H_b/V) \subset \mathbb{P}\mathcal{V}$. This in turn implies that $\mathrm{Hess}(\Psi_V) = 0$, i.e., $\mathbb{P}\mathcal{V} \subset \mathcal{D}$. Since for a general $[W] \in \mathrm{Gr}(3, H^0(\omega))$ the pencil $V = W \cap H_b$ is base-point-free, we obtain that a general $[W]$ lies on the divisor $\mathcal{D}$, which is a contradiction.

As for $|F_4|$, we recall that the fact $\mathrm{Bs}\,|F_4| = C$ follows from [We]. Alternatively, it can also be deduced by noticing (see Proposition 2.3) that $\mathrm{Bs}\,|F_4| \subset \mathrm{Bs}\,|I(2)|$. Hence, if C is not trigonal nor a plane quintic, we are done. In the other cases, the result can be deduced from Proposition 4.3 — we leave the details to the reader. □

7. – Open questions

7.1. – Dimensions

The projective dimensions of the linear systems $|F_3|$ and $|F_4|$ are not known for general g. The known values of $\dim |F_4|$ for a general curve C are given as follows (see [PP]).

g	4	5	6	7
$\dim \|F_4\|$	4	15	40	88

The examples of [PP] section 6 show that $\dim|F_4|$ depends on the gonality of C. Moreover it can be shown that $|F_4| \neq |I(4)|$.

7.2. – Prym-canonical spaces and symplectic bundles

The construction of the quartic hypersurfaces F_W admits various analogues and generalizations, which we briefly outline.

(1) Let $P_\alpha := \mathrm{Prym}(C_\alpha/C)$ denote the Prym variety of the étale double cover $C_\alpha \to C$ associated to the nonzero 2-torsion point $\alpha \in JC$. Given a general 3-plane $Z \subset H^0(C, \omega\alpha)$, we associate the rank-2 vector bundle E_Z defined by

$$0 \longrightarrow E_Z^* \longrightarrow \mathcal{O}_C \otimes Z \xrightarrow{ev} \omega\alpha \longrightarrow 0.$$

By [IP] Proposition 4.1 we can associate to E_Z the divisor $\Delta(E_Z) \in |2\Xi|$, where Ξ is a symmetric principal polarization on P_α. Its projectivized tangent cone at the origin $0 \in P_\alpha$ is a quartic hypersurface F_Z in the Prym-canonical space $\mathbb{P}T_0P_\alpha \cong |\omega\alpha|^*$. Kempf's obstruction theory equally applies to the quartics F_Z. We note that F_Z contains the Prym-canonical curve $i_{\omega\alpha}(C) \subset |\omega\alpha|^*$.

(2) Let W be a vector space of dimension $2n+1$, for $n \geq 1$. We consider a *general* linear map

$$\Phi : \Lambda^2 W^* \longrightarrow H^0(C, \omega).$$

By taking the n-th symmetric power $\mathrm{Sym}^n\Phi$ and using the canonical maps $\mathrm{Sym}^n(\Lambda^2 W^*) \to \Lambda^{2n} W^* \cong W$ and $\mathrm{Sym}^n H^0(\omega) \to H^0(\omega^{\otimes n})$, we obtain a linear map

$$\alpha : W \longrightarrow H^0(\omega^{\otimes n}),$$

which we assume to be injective. We then define the rank $2n$ vector bundle E_Φ by

$$0 \longrightarrow E_\Phi^* \longrightarrow \mathcal{O}_C \otimes W \xrightarrow{ev} \omega^{\otimes n} \longrightarrow 0.$$

The bundle E_Φ carries an ω-valued symplectic form and the projectivized tangent cone at $\mathcal{O} \in JC$ to the divisor $D(E_\Phi)$ is a hypersurface F_Φ in $|\omega|^*$ of degree $2n+2$. Moreover $F_\Phi \in |I(2n+2)|$.

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