

## Riemann Maps in Almost Complex Manifolds

BERNARD COUPET – HERVÉ GAUSSIÉ – ALEXANDRE SUKHOV

**Abstract.** We prove the existence of stationary discs in the ball for small almost complex deformations of the standard structure. We define a local analogue of the Riemann map and establish its main properties. These constructions are applied to study the local geometry of almost complex manifolds and their morphisms.

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### Introduction

The notion of stationary disc was introduced by L. Lempert in [13]. A holomorphic disc in a domain  $\Omega$  in  $\mathbb{C}^n$  is a holomorphic map from the unit disc  $\Delta$  to  $\Omega$ . A proper holomorphic disc  $f : \Delta \rightarrow \Omega$ , continuous up to  $\partial\Delta$ , is called *stationary* if there is a meromorphic lift  $\hat{f}$  of  $f$  to the cotangent bundle  $T^*(\mathbb{C}^n)$  with an only possible pole of order at most one at origin, such that the image  $\hat{f}(\partial\Delta)$  is contained in the conormal bundle  $\Sigma(\partial\Omega)$  of  $\partial\Omega$ . L. Lempert proved in [13] that for a strictly convex domain stationary discs coincide with extremal discs for the Kobayashi metric and studied their basic properties. Using these discs he introduced a multi dimensional analogue of the Riemann map (see also [20] by S. Semmes and [1] by Z. Balogh and Ch. Leuenberger). The importance of this object comes from its links with the complex potential theory [13], [14], moduli spaces of Cauchy-Riemann structures and contact geometry [3], [4], [5], [16], [20], and the mapping problem [15], [17], [23].

Stationary discs are natural global biholomorphic invariants of complex manifolds with boundary. In view of the recent progress in symplectic geometry due to the application of almost complex structures and pseudo holomorphic curves, it seems relevant to find an analogue of Lempert's theory in the almost complex case. In this paper we restrict ourselves to small almost complex deformations of the standard integrable structure in  $\mathbb{C}^n$ . Since every almost complex structure may be represented locally in such a form, our results can be viewed

as a local analogue of Lempert's theory in almost complex manifolds. We prove the existence of stationary discs in the unit ball for small deformations of the structure and show that they form a foliation of the unit ball (singular at the origin) in Section 4. Then we define an analogue of the Riemann map and establish its main properties: regularity, holomorphy along the leaves and commutativity with (pseudo) biholomorphic maps (Theorem 1). As an application we prove the following criterion for the boundary regularity of diffeomorphisms in almost complex manifolds, which is a partial generalization of Fefferman's theorem [9]: let  $M$  and  $M'$  be two  $\mathcal{C}^\infty$  smooth real  $2n$ -dimensional manifolds,  $D \subset M$  and  $D' \subset M'$  be relatively compact domains. Suppose that there exists an almost complex structure  $J$  of class  $\mathcal{C}^\infty$  on  $\bar{D}$  such that  $(D, J)$  is strictly pseudoconvex. Then a  $\mathcal{C}^1$  diffeomorphism  $\phi$  between  $\bar{D}$  and  $\bar{D}'$  is of class  $\mathcal{C}^\infty(\bar{D})$  if and only if the direct image  $J' := \phi^*(J)$  is of class  $\mathcal{C}^\infty(\bar{D}')$  and  $(D', J')$  is strictly pseudoconvex. (See Theorem 4.) This result is partially motivated by Eliashberg's question on the existence of a symplectic analogue of Fefferman's theorem, see [2]. Finally we point out that the Riemann map contains an essential information on the local geometry of an almost complex manifold. It can be used to solve the local equivalence problem for almost complex structures (Theorem 5).

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## 1. – Preliminaries

Everywhere in this paper  $\Delta$  denotes the unit disc in  $\mathbb{C}$  and  $\mathbb{B}_n$  the unit ball in  $\mathbb{C}^n$ . Let  $(M, J)$  be an almost complex manifold ( $J$  is a smooth  $\mathcal{C}^\infty$ -field on the tangent bundle  $TM$  of  $M$ , satisfying  $J^2 = -I$ ). By an abuse of notation  $J_0$  is the standard structure on  $\mathbb{C}^k$  for every integer  $k$ . A  $J$ -holomorphic disc in  $M$  is a smooth map from  $\Delta$  to  $M$ , satisfying the quasilinear elliptic equation  $df \circ J_0 = J \circ df$  on  $\Delta$ .

An important special case of an almost complex manifold is a bounded domain  $D$  in  $\mathbb{C}^n$  equipped with an almost complex structure  $J$ , defined in a neighborhood of  $\bar{D}$ , and sufficiently close to the standard structure  $J_0$  in the  $\mathcal{C}^2$  norm on  $\bar{D}$ . Every almost complex manifold may be represented locally in such a form. More precisely, we have the following lemma.

LEMMA 1. *Let  $(M, J)$  be an almost complex manifold. Then for every point  $p \in M$  and every  $\lambda_0 > 0$  there exist a neighborhood  $U$  of  $p$  and a coordinate diffeomorphism  $z : U \rightarrow \mathbb{B}$  such that  $z(p) = 0$ ,  $dz(p) \circ J(p) \circ dz^{-1}(0) = J_0$  and the direct image  $J' = z^*(J)$  satisfies  $\|J' - J_0\|_{\mathcal{C}^2(\bar{\mathbb{B}})} \leq \lambda_0$ .*

PROOF. There exists a diffeomorphism  $z$  from a neighborhood  $U'$  of  $p \in M$  onto  $\mathbb{B}$  satisfying  $z(p) = 0$  and  $dz(p) \circ J(p) \circ dz^{-1}(0) = J_0$ . For  $\lambda > 0$  consider the dilation  $d_\lambda : t \mapsto \lambda^{-1}t$  in  $\mathbb{C}^n$  and the composition  $z_\lambda = d_\lambda \circ z$ . Then  $\lim_{\lambda \rightarrow 0} \|z_\lambda^*(J) - J_0\|_{C^2(\bar{\mathbb{B}})} = 0$ . Setting  $U = z_\lambda^{-1}(\mathbb{B})$  for  $\lambda > 0$  small enough, we obtain the desired statement.  $\square$

In particular, every almost complex structure  $J$  sufficiently close to the standard structure  $J_0$  will be written locally  $J = J_0 + \mathcal{O}(\|z\|)$ . Finally by a small perturbation (or deformation) of the standard structure  $J_0$  defined in a neighborhood of  $\bar{D}$ , where  $D$  is a domain in  $\mathbb{C}^n$ , we will mean a smooth one parameter family  $(J_\lambda)_\lambda$  of almost complex structures defined in a neighborhood of  $\bar{D}$ , the real parameter  $\lambda$  belonging to a neighborhood of the origin, and satisfying:  $\lim_{\lambda \rightarrow 0} \|J_\lambda - J_0\|_{C^2(\bar{D})} = 0$ .

Let  $(M, J)$  be an almost complex manifold. We denote by  $T(M)$  the real tangent bundle of  $M$  and by  $T_{\mathbb{C}}(M)$  its complexification. Then:

$$T_{(1,0)}(M) := \{X \in T_{\mathbb{C}}(M) : JX = iX\} = \{\zeta - iJ\zeta, \zeta \in T(M)\},$$

$$T_{(0,1)}(M) := \{X \in T_{\mathbb{C}}(M) : JX = -iX\} = \{\zeta + iJ\zeta, \zeta \in T(M)\}.$$

Let  $T^*(M)$  denote the cotangent bundle of  $M$ . Identifying  $\mathbb{C} \otimes T^*M$  with  $T_{\mathbb{C}}^*(M) := \text{Hom}(T_{\mathbb{C}}M, \mathbb{C})$  we may consider a complex one form on  $M$  as an element in  $T_{\mathbb{C}}^*(M)$ . We define the set of complex forms of type  $(1, 0)$  on  $M$  by:

$$T_{(1,0)}^*(M) = \{w \in T_{\mathbb{C}}^*(M) : w(X) = 0, \forall X \in T_{(0,1)}(M)\}$$

and the set of complex forms of type  $(0, 1)$  on  $M$  by:

$$T_{(0,1)}^*(M) = \{w \in T_{\mathbb{C}}^*(M) : w(X) = 0, \forall X \in T_{(1,0)}(M)\}.$$

Let  $\Gamma$  be a real smooth submanifold in  $M$  and let  $p \in \Gamma$ . We denote by  $H^J(\Gamma)$  the  $J$ -holomorphic tangent bundle  $T\Gamma \cap JT\Gamma$ .

DEFINITION 1. A real submanifold  $\Gamma$  in  $M$  is *totally real* if  $H_p^J(\Gamma) = \{0\}$  for every  $p \in \Gamma$ .

In what follows we will need the notion of the Levi form of a hypersurface.

DEFINITION 2. Let  $\Gamma = \{r = 0\}$  be a smooth real hypersurface in  $M$  ( $r$  is any smooth defining function of  $\Gamma$ ) and let  $p \in \Gamma$ .

- (i) The *Levi form* of  $\Gamma$  at  $p$  is the map defined on  $H_p^J(\Gamma)$  by  $\mathcal{L}_\Gamma^J(X_p) = (J^*dr)[X, JX]_p$ , where a vector field  $X$  is a section of the  $J$ -holomorphic tangent bundle  $H^J\Gamma$  such that  $X(p) = X_p$ .
- (ii) A real smooth hypersurface  $\Gamma = \{r = 0\}$  in  $M$  is *strictly  $J$ -pseudoconvex* if its Levi form  $\mathcal{L}_\Gamma^J$  is positive definite on  $H^J(\Gamma)$ .

We recall the notion of conormal bundle of a real submanifold in  $\mathbb{C}^n$  ([23]). Let  $\pi : T^*(\mathbb{C}^n) \rightarrow \mathbb{C}^n$  be the natural projection. Then  $T_{(1,0)}^*(\mathbb{C}^n)$  can be canonically identified with the cotangent bundle of  $\mathbb{C}^n$ . In the canonical complex

coordinates  $(z, t)$  on  $T_{(1,0)}^*(\mathbb{C}^n)$  an element of the fiber at  $z$  is a holomorphic form  $w = \sum_j t_j dz^j$ .

Let  $N$  be a real smooth generic submanifold in  $\mathbb{C}^n$ . The conormal bundle  $\Sigma(N)$  of  $N$  is a real subbundle of  $T_{(1,0)}^*(\mathbb{C}^n)|_N$  whose fiber at  $z \in N$  is defined by  $\Sigma_z(N) = \{\phi \in T_{(1,0)}^*(\mathbb{C}^n) : \operatorname{Re} \phi|_{T_{(1,0)}(N)} = 0\}$ .

Let  $\rho_1, \dots, \rho_d$  be local defining functions of  $N$ . Then the forms  $\partial\rho_1, \dots, \partial\rho_d$  form a basis in  $\Sigma_z(N)$  and every section  $\phi$  of the bundle  $\Sigma(N)$  has the form  $\phi = \sum_{j=1}^d c_j \partial\rho_j$ ,  $c_1, \dots, c_d \in \mathbb{R}$ . We will use the following (see [23]):

**LEMMA 2.** *Let  $\Gamma$  be a real  $C^2$  hypersurface in  $\mathbb{C}^n$ . The conormal bundle  $\Sigma(\Gamma)$  is a totally real submanifold of dimension  $2n$  in  $T_{(1,0)}^*(\mathbb{C}^n)$  if and only if the Levi form of  $\Gamma$  is nondegenerate.*

In Section 5 we will introduce an analogue of this notion in the almost complex case.

## 2. – Existence of discs attached to a real submanifold of an almost complex manifold

### 2.1. – Partial indices and the Riemann-Hilbert problem

In this section we introduce basic tools of the linear Riemann-Hilbert problem.

Let  $V \subset \mathbb{C}^N$  be an open set. We denote by  $\mathcal{C}^k(V)$  the Banach space of (real or complex valued) functions of class  $\mathcal{C}^k$  on  $V$  such that

$$\|r\|_k = \sum_{|\nu| \leq k} \sup\{|D^\nu r(w)| : w \in V\}$$

is finite.

For a positive real number  $\alpha < 1$  and a Banach space  $X$ , we denote by  $\mathcal{C}^\alpha(\partial\Delta, X)$  the Banach space of all functions  $f : \partial\Delta \rightarrow X$  such that

$$\|f\|_\alpha := \sup_{\zeta \in \partial\Delta} \|f(\zeta)\| + \sup_{\theta, \eta \in \partial\Delta, \theta \neq \eta} \frac{\|f(\theta) - f(\eta)\|}{|\theta - \eta|^\alpha} < \infty.$$

If  $\alpha = m + \beta$  with an integer  $m \geq 0$  and  $\beta \in ]0, 1[$ , then we consider the Banach space

$$\mathcal{C}^\alpha(V) := \{r \in \mathcal{C}^m(V, \mathbb{R}) : D^\nu r \in \mathcal{C}^\beta(V), \nu : |\nu| \leq m\}$$

and we set  $\|r\|_\alpha = \sum_{|\nu| \leq m} \|D^\nu r\|_\beta$ .

Then a map  $f$  is in  $\mathcal{C}^\alpha(V, \mathbb{C}^k)$  if and only if its components belong to  $\mathcal{C}^\alpha(V)$  and we say that  $f$  is of class  $\mathcal{C}^\alpha$ .

Consider the following situation:

- $B$  is an open ball centered at the origin in  $\mathbb{C}^N$  and  $r^1, \dots, r^N$  are smooth  $C^\infty$  functions defined in a neighborhood of  $B \times \partial\Delta$  in  $\mathbb{C}^N \times \mathbb{C}$
- $f$  is a map of class  $C^\alpha$  from  $\partial\Delta$  to  $B$ , where  $\alpha > 1$  is a noninteger real number
- for every  $\zeta \in \partial\Delta$ 
  - (i)  $E(\zeta) = \{z \in B : r^j(z, \zeta) = 0, 1 \leq j \leq N\}$  is a maximal totally real submanifold in  $\mathbb{C}^N$ ,
  - (ii)  $f(\zeta) \in E(\zeta)$ ,
  - (iii)  $\partial_z r^1(z, \zeta) \wedge \dots \wedge \partial_z r^N(z, \zeta) \neq 0$  on  $B \times \partial\Delta$ .

Such a family  $E = \{E(\zeta)\}$  of manifolds with a fixed disc  $f$  is called a *totally real fibration* over the unit circle. A disc attached to a fixed totally real manifold ( $E$  is independent of  $\zeta$ ) is a special case of a totally real fibration.

Assume that the defining function  $r := (r^1, \dots, r^N)$  of  $E$  depends smoothly on a small real parameter  $\varepsilon$ , namely  $r = r(z, \zeta, \varepsilon)$ , and that the fibration  $E_0 := E(\zeta, 0)$  corresponding to  $\varepsilon = 0$  coincides with the above fibration  $E$ . Then for every sufficiently small  $\varepsilon$  and for every  $\zeta \in \partial\Delta$  the manifold  $E_\varepsilon := E(\zeta, \varepsilon) := \{z \in B : r(z, \zeta, \varepsilon) = 0\}$  is totally real. We call  $E_\varepsilon$  a *smooth totally real deformation* of the totally real fibration  $E$ . By a holomorphic disc  $\tilde{f}$  attached to  $E_\varepsilon$  we mean a holomorphic map  $\tilde{f} : \Delta \rightarrow B$ , continuous on  $\bar{\Delta}$ , satisfying  $r(f(\zeta), \zeta, \varepsilon) = 0$  on  $\partial\Delta$ .

For every positive real noninteger  $\alpha$  we denote by  $(\mathcal{A}^\alpha)^N$  the space of maps defined on  $\bar{\Delta}$ ,  $J_0$ -holomorphic on  $\Delta$ , and belonging to  $(C^\alpha(\bar{\Delta}))^N$ .

**2.2. – Almost complex perturbation of discs**

We recall that for  $\lambda$  small enough the  $(J_0, J_\lambda)$ -holomorphy condition for a map  $f : \Delta \rightarrow \mathbb{C}^N$  may be written in the form

$$(1) \quad \bar{\partial}_{J_\lambda} f = \bar{\partial} f + q(\lambda, f)\partial f = 0$$

where  $q$  is a smooth function satisfying  $q(0, \cdot) \equiv 0$ , uniquely determined by  $J_\lambda$  ([21]).

Let  $E_\varepsilon = \{r_j(z, \zeta, \varepsilon) = 0, 1 \leq j \leq N\}$  be a smooth totally real deformation of a totally real fibration  $E$ . A disc  $f \in (C^\alpha(\bar{\Delta}))^N$  is attached to  $E_\varepsilon$  and is  $J_\lambda$ -holomorphic if and only if it satisfies the following nonlinear boundary Riemann-Hilbert type problem:

$$\begin{cases} r(f(\zeta), \zeta, \varepsilon) = 0, & \zeta \in \partial\Delta \\ \bar{\partial}_{J_\lambda} f(\zeta) = 0, & \zeta \in \Delta. \end{cases}$$

Let  $f^0 \in (\mathcal{A}^\alpha)^N$  be a disc attached to  $E$  and let  $\mathcal{U}$  be a neighborhood of  $(f^0, 0, 0)$  in the space  $(C^\alpha(\bar{\Delta}))^N \times \mathbb{R} \times \mathbb{R}$ . Given  $(f, \varepsilon, \lambda)$  in  $\mathcal{U}$  define the maps  $v_{f,\varepsilon,\lambda} : \zeta \in \partial\Delta \mapsto r(f(\zeta), \zeta, \varepsilon)$  and

$$\begin{aligned} u : \mathcal{U} &\rightarrow (C^\alpha(\partial\Delta))^N \times C^{\alpha-1}(\Delta) \\ (f, \varepsilon, \lambda) &\mapsto (v_{f,\varepsilon,\lambda}, \bar{\partial}_{J_\lambda} f). \end{aligned}$$

Denote by  $X$  the Banach space  $(C^\alpha(\bar{\Delta}))^N$ . Since  $r$  is of class  $C^\infty$ , the map  $u$  is smooth and the tangent map  $D_X u(f^0, 0, 0)$  (we consider the derivative with respect to the space  $X$ ) is a linear map from  $X$  to  $(C^\alpha(\partial\Delta))^N \times C^{\alpha-1}(\Delta)$ , defined for every  $h \in X$  by

$$D_X u(f^0, 0, 0)(h) = \left( \begin{matrix} 2\operatorname{Re}[Gh] \\ \bar{\partial}_{J_0} h \end{matrix} \right),$$

where for  $\zeta \in \partial\Delta$

$$G(\zeta) = \begin{pmatrix} \frac{\partial r_1}{\partial z^1}(f^0(\zeta), 0) & \cdots & \frac{\partial r_1}{\partial z^N}(f^0(\zeta), 0) \\ \cdots & \cdots & \cdots \\ \frac{\partial r_N}{\partial z^1}(f^0(\zeta), 0) & \cdots & \frac{\partial r_N}{\partial z^N}(f^0(\zeta), 0) \end{pmatrix}$$

(see [10]).

PROPOSITION 1. *Let  $f^0 : \bar{\Delta} \rightarrow \mathbb{C}^N$  be a  $J_0$ -holomorphic disc attached to a totally real fibration  $E$  in  $\mathbb{C}^N$ . Let  $E_\varepsilon$  be a smooth totally real deformation of  $E$  and  $J_\lambda$  be a smooth almost complex deformation of  $J_0$  in a neighborhood of  $f(\bar{\Delta})$ . Assume that for some  $\alpha > 1$  the linear map from  $(\mathcal{A}^\alpha)^N$  to  $(C^{\alpha-1}(\Delta))^N$  given by  $h \mapsto 2\operatorname{Re}[Gh]$  is surjective and has a  $k$  dimensional kernel. Then there exist  $\delta_0, \varepsilon_0, \lambda_0 > 0$  such that for every  $0 \leq \varepsilon \leq \varepsilon_0$  and for every  $0 \leq \lambda \leq \lambda_0$ , the set of  $J_\lambda$ -holomorphic discs  $f$  attached to  $E_\varepsilon$  and such that  $\|f - f^0\|_\alpha \leq \delta_0$  forms a smooth  $(N + k)$ -dimensional submanifold  $\mathcal{A}_{\varepsilon, \lambda}$  in the Banach space  $(C^\alpha(\bar{\Delta}))^N$ .*

PROOF. According to the implicit function theorem, the proof of Proposition 1 reduces to the proof of the surjectivity of  $D_X u$ . It follows by classical one-variable results on the resolution of the  $\bar{\partial}$ -problem in the unit disc that the linear map from  $X$  to  $C^{\alpha-1}(\Delta)$  given by  $h \mapsto \bar{\partial}h$  is surjective. More precisely, given  $g \in C^{\alpha-1}(\Delta)$  consider the Cauchy transform

$$T_\Delta(g) : \tau \in \partial\Delta \mapsto \frac{1}{2i\pi} \int \int_\Delta \frac{g(\zeta)}{\zeta - \tau} d\zeta \wedge d\bar{\zeta}.$$

For every function  $g \in C^{\alpha-1}(\Delta)$  the solutions  $h \in X$  of the equation  $\bar{\partial}h = g$  have the form  $h = h_0 + T_\Delta(g)$  where  $h_0$  is an arbitrary function in  $(\mathcal{A}^\alpha)^N$ . Consider the equation

$$(2) \quad D_X u(f^0, 0, 0)(h) = \begin{pmatrix} g_1 \\ g_2 \end{pmatrix},$$

where  $(g_1, g_2)$  is a vector-valued function with components  $g_1 \in C^{\alpha-1}(\partial\Delta)$  and  $g_2 \in C^{\alpha-1}(\Delta)$ . Solving the  $\bar{\partial}$ -equation for the second component, we reduce (2) to

$$2\operatorname{Re}[G(\zeta)h_0(\zeta)] = g_1 - 2\operatorname{Re}[G(\zeta)T_\Delta(g_2)(\zeta)]$$

with respect to  $h_0 \in (\mathcal{A}^\alpha)^N$ . The surjectivity of the map  $h_0 \mapsto 2\operatorname{Re}[Gh_0]$  gives the result. □

**2.3. – Riemann-Hilbert problem on the (co)tangent bundle of an almost complex manifold**

Let  $(J_\lambda)_\lambda$  be an almost complex deformation of the standard structure  $J_0$  on  $\mathbb{B}_n$ , satisfying  $J_\lambda(0) = J_0$  and consider a (1,1) tensor field defined on the bundle  $\mathbb{B}_n \times \mathbb{R}^{2n}$ , represented in the  $(x, y)$  coordinates by a  $(4n \times 4n)$ -matrix

$$T_\lambda = \begin{pmatrix} J_\lambda(x) & 0 \\ \sum y_k A_\lambda^k(x) & B_\lambda(x) \end{pmatrix},$$

where  $A_\lambda^k(x) = A^k(\lambda, x)$ ,  $B_\lambda(x) = B^k(\lambda, x)$  are smooth  $(2n \times 2n)$ -matrix functions ( $B_\lambda$  will be either  $J_\lambda$  or  ${}^t J_\lambda$ ). We stress that we do not assume  $T_\lambda$  to be an almost complex structure, namely we do not require the identity  $T_\lambda^2 = -\text{Id}$ . In what follows we always assume that for every  $k$ :

$$(3) \quad A_0^k(x) \equiv 0, \quad \text{for every } k$$

and that one of the following two conditions holds:

$$(4) \quad B_0(x) = B_\lambda(0) = J_0, \quad \text{for every } \lambda, x,$$

$$(5) \quad B_0(x) = B_\lambda(0) = {}^t J_0, \quad \text{for every } \lambda, x.$$

In what follows the trivial bundle  $\mathbb{B} \times \mathbb{R}^{2n}$  over the unit ball will be a local coordinate representation of the tangent or cotangent bundle of an almost complex manifold. We denote by  $x = (x^1, \dots, x^{2n}) \in \mathbb{B}_n$  and  $y = (y_1, \dots, y_{2n}) \in \mathbb{R}^{2n}$  the coordinates on the base and fibers respectively. We identify the base space  $(\mathbb{R}^{2n}, x)$  with  $(\mathbb{C}^n, z)$ . Since  ${}^t J_0$  is orthogonally equivalent to  $J_0$  we may identify  $(\mathbb{R}^{2n}, {}^t J_0)$  with  $(\mathbb{C}^n, J_0)$ . After this identification the  ${}^t J_0$ -holomorphy is expressed by the  $\bar{\partial}$ -equation in the usual  $t$  coordinates in  $\mathbb{C}^n$ .

By analogy with the almost complex case we say that a smooth map  $\hat{f} = (f, g) : \Delta \rightarrow \mathbb{B} \times \mathbb{R}^{2n}$  is  $(J_0, T_\lambda)$ -holomorphic if it satisfies

$$T_\lambda(f, g) \circ d\hat{f} = d\hat{f} \circ J_0$$

on  $\Delta$ .

For  $\lambda$  small enough this can be rewritten as the following Beltrami type quasilinear elliptic equation:

$$(\mathcal{E}) \begin{cases} \bar{\partial} f + q_1(\lambda, f)\partial f = 0 \\ \bar{\partial} g + q_2(\lambda, f)\partial g + q_3(\lambda, f)g = 0, \end{cases}$$

where the first equation coincides with the  $(J_0, J_\lambda)$ -holomorphy condition for  $f$  that is  $\bar{\partial}_{J_\lambda} f = \bar{\partial} f + q_1(\lambda, f)\partial$ . The coefficient  $q_1$  is uniquely determined by  $J_\lambda$  and, in view of (3), (4), (5) the coefficient  $q_k$  satisfies, for  $k = 2, 3$ :

$$(6) \quad q_k(0, \cdot) \equiv 0, \quad q_k(\cdot, 0) \equiv 0.$$

We point out that in  $(\mathcal{E})$  the equations for the fiber component  $g$  are obtained as a small perturbation of the  $\bar{\partial}$ -operator. An important feature of this system is that the second equation is *linear* with respect to the fiber component  $g$ .

DEFINITION 3. We call the above tensor field  $T_\lambda$  a prolongation of the structure  $J_\lambda$  to the bundle  $\mathbb{B}_n \times \mathbb{R}^{2n}$ . The operator

$$\bar{\partial}_{T_\lambda} : \begin{pmatrix} f \\ g \end{pmatrix} \mapsto \begin{pmatrix} \bar{\partial} f + q_1(\lambda, f)\partial f \\ \bar{\partial} g + q_2(\lambda, f)\partial g + q_3(\lambda, f)g \end{pmatrix}$$

is called an elliptic prolongation of the operator  $\bar{\partial}_{J_\lambda}$  to the bundle  $\mathbb{B}_n \times \mathbb{R}^{2n}$  associated with  $T_\lambda$ .

Let  $r^j(z, t, \lambda)$ ,  $j = 1, \dots, 4n$  be  $C^\infty$ -smooth real functions on  $\mathbb{B} \times \mathbb{B} \times [0, \lambda_0]$  and let  $r := (r^1, \dots, r^{4n})$ . In what follows we consider the following nonlinear boundary Riemann-Hilbert type problem for the operator  $\bar{\partial}_{T_\lambda}$ :

$$(\mathcal{BP}_\lambda) \begin{cases} r(f(\zeta), \zeta^{-1}g(\zeta), \lambda) = 0 \text{ on } \partial\Delta, \\ \bar{\partial}_{T_\lambda}(f, g) = 0, \end{cases}$$

on the space  $C^\alpha(\bar{\Delta}, \mathbb{B}_n \times \mathbb{B}_n)$ .

The boundary problem  $(\mathcal{BP}_\lambda)$  has the following geometric sense. Consider the disc  $(\hat{f}, \hat{g}) := (f, \zeta^{-1}g)$  on  $\Delta \setminus \{0\}$  and the set  $E_\lambda := \{(z, t) : r(z, t, \lambda) = 0\}$ . The boundary condition in  $(\mathcal{BP}_\lambda)$  means that

$$(\hat{f}, \hat{g})(\partial\Delta) \subset E_\lambda.$$

This boundary problem has the following *invariance property*. Let  $(f, g)$  be a solution of  $(\mathcal{BP}_\lambda)$  and let  $\phi$  be a automorphism of  $\Delta$ . Then  $(f \circ \phi, cg \circ \phi)$  also satisfies the  $\bar{\partial}_{T_\lambda}$  equation for every complex constant  $c$ . In particular, if  $\theta \in [0, 2\pi]$  is fixed, then the disc  $(f(e^{i\theta}\zeta), e^{-i\theta}g(e^{i\theta}\zeta))$  satisfies the  $\bar{\partial}_{T_\lambda}$ -equation on  $\Delta \setminus \{0\}$  and the boundary of the disc  $(f(e^{i\theta}\zeta), e^{-i\theta}\zeta^{-1}g(e^{i\theta}\zeta))$  is attached to  $E_\lambda$ . This implies the following

LEMMA 3. *If  $(f, g)$  is a solution of  $(\mathcal{BP}_\lambda)$ , then  $\zeta \mapsto (f(e^{i\theta}\zeta), e^{-i\theta}g(e^{i\theta}\zeta))$  is also a solution of  $(\mathcal{BP}_\lambda)$ .*

Suppose that this problem has a solution  $(f^0, g^0)$  for  $\lambda = 0$  (in view of the above assumptions this solution is holomorphic on  $\Delta$  with respect to the standard structure on  $\mathbb{C}^n \times \mathbb{C}^n$ ). Using the implicit function theorem we study, for sufficiently small  $\lambda$ , the solutions of  $(\mathcal{BP}_\lambda)$  close to  $(f^0, g^0)$ . Similarly to Section 2.2 consider the map  $u$  defined in a neighborhood of  $(f^0, g^0, 0)$  in  $(C^\alpha(\bar{\Delta}))^{4n} \times \mathbb{R}$  by:

$$u : (f, g, \lambda) \mapsto \begin{pmatrix} \zeta \in \partial\Delta \mapsto r(f(\zeta), \zeta^{-1}g(\zeta), \lambda) \\ \bar{\partial} f + q_1(\lambda, f)\partial f \\ \bar{\partial} g + q_2(\lambda, f)\partial g + q_3(\lambda, f)g \end{pmatrix}.$$



If  $X := (\mathcal{C}^\alpha(\bar{\Delta}))^{4n}$  then its tangent map at  $(f^0, g^0, 0)$  has the form

$$D_X u(f^0, g^0, 0) : h = (h_1, h_2) \mapsto \begin{pmatrix} \zeta \in \partial\Delta \mapsto 2\text{Re}[G(f^0(\zeta), \zeta^{-1}g^0(\zeta), 0)h] \\ \bar{\partial}h_1 \\ \bar{\partial}h_2 \end{pmatrix}$$

where for  $\zeta \in \partial\Delta$  one has

$$G(\zeta) = \begin{pmatrix} \frac{\partial r_1}{\partial w_1}(f^0(\zeta), \zeta^{-1}g^0(\zeta), 0) & \cdots & \frac{\partial r_1}{\partial w_N}(f^0(\zeta), \zeta^{-1}g^0(\zeta), 0) \\ \cdots & \cdots & \cdots \\ \frac{\partial r_N}{\partial w_1}(f^0(\zeta), \zeta^{-1}g^0(\zeta), 0) & \cdots & \frac{\partial r_N}{\partial w_N}(f^0(\zeta), \zeta^{-1}g^0(\zeta), 0) \end{pmatrix}$$

with  $N = 4n$  and  $w = (z, t)$ .

If the tangent map  $D_X u(f^0, g^0, 0) : (\mathcal{A}^\alpha)^N \rightarrow (\mathcal{C}^{\alpha-1}(\Delta))^N$  is surjective and has a finite-dimensional kernel, we may apply the implicit function theorem as in Section 2.2 (see Proposition 1) and conclude to the existence of a finite-dimensional variety of nearby discs. In particular, consider the fibration  $E$  over the disc  $(f^0, g^0)$  with fibers  $E(\zeta) = \{(z, t) : r^j(z, t, \zeta) = 0\}$ . Suppose that this fibration is totally real. Then we have:

**PROPOSITION 2.** *Suppose that the fibration  $E$  is totally real. If the tangent map  $D_X u(f^0, g^0, 0) : (\mathcal{A}^\alpha)^{4n} \rightarrow (\mathcal{C}^{\alpha-1}(\Delta))^{4n}$  is surjective and has a finite-dimensional kernel, then for every sufficiently small  $\lambda$  the solutions of the boundary problem  $(\mathcal{B}P_\lambda)$  form a smooth submanifold in the space  $(\mathcal{C}^\alpha(\Delta))^{4n}$ .*

In the next section we present a sufficient condition for the surjectivity of the map  $D_X u(f^0, g^0, 0)$ . This is due to J. Globevnik [10], [11] for the integrable case and relies on the partial indices of the totally real fibration along  $(f^0, g^0)$ .

### 3. – Generation of stationary discs

Let  $D$  be a smoothly bounded domain in  $\mathbb{C}^n$  with the boundary  $\Gamma$ . According to [13] a continuous map  $f : \bar{\Delta} \setminus \{0\} \rightarrow \bar{D}$ , holomorphic on  $\Delta \setminus \{0\}$ , is called a *stationary disc* for  $D$  (or for  $\Gamma$ ) if there exists a holomorphic map  $\hat{f} : \Delta \setminus \{0\} \rightarrow T_{(1,0)}^*(\mathbb{C}^n)$ ,  $\hat{f} \neq 0$ , continuous on  $\bar{\Delta} \setminus \{0\}$  and such that

- (i)  $\pi \circ \hat{f} = f$
- (ii)  $\zeta \mapsto \zeta \hat{f}(\zeta)$  is in  $\mathcal{O}(\Delta)$
- (iii)  $\hat{f}(\zeta) \in \Sigma_{f(\zeta)}(\Gamma)$  for every  $\zeta$  in  $\partial\Delta$ .

We call  $\hat{f}$  a *lift* of  $f$  to the conormal bundle of  $\Gamma$  (this is a meromorphic map from  $\Delta$  into  $T_{(1,0)}^*(\mathbb{C}^n)$  whose values on the unit circle lie on  $\Sigma(\Gamma)$ ).

We point out that originally Lempert gave this definition in a different form, using the natural coordinates on the cotangent bundle of  $\mathbb{C}^n$ . The present more geometric version in terms of the conormal bundle is due to Tumanov [23]. This form is particularly useful for our goals since it can be transferred to the almost complex case.

Let  $f$  be a stationary disc for  $\Gamma$ . It follows from Lemma 2 that if  $\Gamma$  is a Levi nondegenerate hypersurface, the conormal bundle  $\Sigma(\Gamma)$  is a totally real fibration along  $f^*$ . Conditions (i), (ii), (iii) may be viewed as a nonlinear boundary problem considered in Section 2. If the associated tangent map is surjective, Proposition 2 gives a description of all stationary discs  $\tilde{f}$  close to  $f$ , for a small deformation of  $\Gamma$ . When dealing with the standard complex structure on  $\mathbb{C}^n$ , the bundle  $T_{(1,0)}^*(\mathbb{C}^n)$  is a holomorphic vector bundle which can be identified, after projectivization of the fibers, with the holomorphic bundle of complex hyperplanes that is with  $\mathbb{C}^n \times \mathbb{P}^{n-1}$ . The conormal bundle  $\Sigma(\Gamma)$  of a real hypersurface  $\Gamma$  may be naturally identified, after this projectivization, with the bundle of holomorphic tangent spaces  $H(\Gamma)$  over  $\Gamma$ . According to S. Webster [25] this is a totally real submanifold in  $\mathbb{C}^n \times \mathbb{P}^{n-1}$ . When dealing with the standard structure, we may therefore work with projectivizations of lifts of stationary discs attached to the holomorphic tangent bundle  $H(\Gamma)$ . The technical advantage is that after such a projectivization lifts of stationary discs become holomorphic, since the lifts have at most one pole of order 1 at the origin. This idea was first used by L. Lempert and then applied by several authors [1], [6], [22].

When we consider almost complex deformations of the standard structure (and not just deformations of  $\Gamma$ ) the situation is more complicated. The main geometric difficulty is to prolong an almost complex structure  $J$  from  $\mathbb{R}^{2n}$  to the cotangent bundle  $T^*(\mathbb{R}^{2n})$  in a certain “natural” way. As we will see later, such a prolongation is *not* unique in contrast with the case of the integrable structure. Moreover, if the cotangent bundle  $T^*(\mathbb{R}^{2n})$  is equipped with an almost complex structure, there is no natural possibility to transfer this structure to the space obtained by the projectivization of the fibers. Consequently we do not work with projectivization of the cotangent bundle but we will deal with meromorphic lifts of stationary discs. Representing such lifts  $(\hat{f}, \hat{g})$  in the form  $(\hat{f}, \hat{g}) = (f, \xi^{-1}g)$ , we will consider  $(J_0, T_\lambda)$ -holomorphic discs close to the  $(J_0, T_0)$ -holomorphic disc  $(f, g)$ . The disc  $(f, g)$  satisfies a nonlinear boundary problem of Riemann-Hilbert type  $(\mathcal{B}P_\lambda)$ . When an almost complex structure on the cotangent bundle is fixed, we may view it as an elliptic prolongation of an initial almost complex structure on the base and apply the implicit function theorem as in previous section. This avoids difficulties coming from the projectivization of almost complex fibre spaces.

### 3.1. – Maslov index and Globevnik’s condition

We denote by  $GL(N, \mathbb{C})$  the group of invertible  $(N \times N)$  complex matrices and by  $GL(N, \mathbb{R})$  the group of all such matrices with real entries. Let  $0 < \alpha < 1$

and let  $B : \partial\Delta \rightarrow GL(N, \mathbb{C})$  be of class  $C^\alpha$ . According to [24] (see also [8])  $B$  admits the factorization  $B(\tau) = F^+(\tau)\Lambda(\tau)F^-(\tau)$ ,  $\tau \in \partial\Delta$ , where:

- $\Lambda$  is a diagonal matrix of the form  $\Lambda(\tau) = \text{diag}(\tau^{k_1}, \dots, \tau^{k_N})$ ,
- $F^+ : \bar{\Delta} \rightarrow GL(N, \mathbb{C})$  is of class  $C^\alpha$  on  $\bar{\Delta}$  and holomorphic in  $\Delta$ ,
- $F^- : [\mathbb{C} \cup \{\infty\}] \setminus \Delta \rightarrow GL(N, \mathbb{C})$  is of class  $C^\alpha$  on  $[\mathbb{C} \cup \{\infty\}] \setminus \Delta$  and holomorphic on  $[\mathbb{C} \cup \{\infty\}] \setminus \bar{\Delta}$ .

The integers  $k_1 \geq \dots \geq k_n$  are called the partial indices of  $B$ .

Let  $E$  be a totally real fibration over the unit circle. For every  $\zeta \in \partial\Delta$  consider the “normal” vectors  $v_j(\zeta) = (r_{\bar{z}_1}^j(f(\zeta), \zeta), \dots, r_{\bar{z}_N}^j(f(\zeta), \zeta))$ ,  $j = 1, \dots, N$ . We denote by  $K(\zeta) \in GL(N, \mathbb{C})$  the matrix with rows  $v_1(\zeta), \dots, v_N(\zeta)$  and we set  $B(\zeta) := -\overline{K(\zeta)}^{-1}K(\zeta)$ ,  $\zeta \in \partial\Delta$ . The partial indices of the map  $B : \partial\Delta \rightarrow GL(N, \mathbb{C})$  are called the *partial indices* of the fibration  $E$  along the disc  $f$  and their sum is called the *total index* or the *Maslov index of  $E$  along  $f$* . The following result is due to J. Globevnik [11]:

**THEOREM.** *Suppose that all the partial indices of the totally real fibration  $E$  along  $f$  are  $\geq -1$  and denote by  $k$  the Maslov index of  $E$  along  $f$ . Then the linear map from  $(A^\alpha)^N$  to  $(C^{\alpha-1}(\Delta))^N$  given by  $h \mapsto 2\text{Re}[Gh]$  is surjective and has a  $k$  dimensional kernel.*

Proposition 1 may be restated in terms of partial indices as follows:

**PROPOSITION 3.** *Let  $f^0 : \bar{\Delta} \rightarrow \mathbb{C}^N$  be a  $J_0$ -holomorphic disc attached to a totally real fibration  $E$  in  $\mathbb{C}^N$ . Suppose that all the partial indices of  $E$  along  $f^0$  are  $\geq -1$ . Denote by  $k$  the Maslov index of  $E$  along  $f^0$ . Let also  $E_\varepsilon$  be a smooth totally real deformation of  $E$  and  $J_\lambda$  be a smooth almost complex deformation of  $J_0$  in a neighborhood of  $f(\bar{\Delta})$ . Then there exists  $\delta_0, \varepsilon_0, \lambda_0 > 0$  such that for every  $0 \leq \varepsilon \leq \varepsilon_0$  and for every  $0 \leq \lambda \leq \lambda_0$  the set of  $J_\lambda$ -holomorphic discs attached to  $E_\varepsilon$  and such that  $\|f - f^0\|_\alpha \leq \delta_0$  forms a smooth  $(N + k)$ -dimensional submanifold  $A_{\varepsilon, \lambda}$  in the Banach space  $(C^\alpha(\bar{\Delta}))^N$ .*

Globevnik’s result was applied to the study of stationary discs in some classes of domains in  $\mathbb{C}^n$  by M. Cerne [6] and A. Spiro-S. Trapani [22]. Since they worked with the projectivization of the conormal bundle, we explicitly compute, for reader’s convenience and completeness of exposition, partial indices of meromorphic lifts of stationary discs for the unit sphere in  $\mathbb{C}^n$ .

Consider the unit sphere  $\Gamma := \{z \in \mathbb{C}^n : z^1\bar{z}^1 + \dots + z^n\bar{z}^n - 1 = 0\}$  in  $\mathbb{C}^n$ . The conormal bundle  $\Sigma(\Gamma)$  is given in the  $(z, t)$  coordinates by the equations

$$(S) \begin{cases} z^1\bar{z}^1 + \dots + z^n\bar{z}^n - 1 = 0, \\ t_1 = c\bar{z}^1, \dots, t_n = c\bar{z}^n, \quad c \in \mathbb{R}. \end{cases}$$

According to [13], every stationary disc for  $\Gamma$  is extremal for the Kobayashi metric. Therefore, such a stationary disc  $f^0$  centered at the origin is linear by the Schwarz lemma. So, up to a unitary transformation, we have  $f^0(\zeta) = (\zeta, 0, \dots, 0)$  with lift  $(\widehat{f^0}, \widehat{g^0})(\zeta) = (\zeta, 0, \dots, 0, \zeta^{-1}, 0, \dots, 0) = (f^0, \zeta^{-1}g^0)$  to the conormal bundle. Representing nearby meromorphic discs in the form

$(z, \zeta^{-1}w)$  and eliminating the parameter  $c$  in system (S) we obtain that holomorphic discs  $(z, w)$  close to  $(f^0, g^0)$  satisfy for  $\zeta \in \partial\Delta$ :

$$\begin{aligned} r^1(z, w) &= z^1 \bar{z}^1 + \dots + z^n \bar{z}^n - 1 = 0, \\ r^2(z, w) &= iz^1 w_1 \zeta^{-1} - i\bar{z}^1 \bar{w}_1 \zeta = 0, \\ r^3(z, w) &= \bar{z}^1 w_2 \zeta^{-1} - \bar{z}^2 w_1 \zeta^{-1} + z^1 \bar{w}_2 \zeta - z^2 \bar{w}_1 \zeta = 0, \\ r^4(z, w) &= i\bar{z}^1 w_2 \zeta^{-1} - i\bar{z}^2 w_1 \zeta^{-1} - iz^1 \bar{w}_2 \zeta + iz^2 \bar{w}_1 \zeta = 0, \\ r^5(z, w) &= \bar{z}^1 w_3 \zeta^{-1} - \bar{z}^3 w_1 \zeta^{-1} + z^1 \bar{w}_3 \zeta - z^3 \bar{w}_1 \zeta = 0, \\ r^6(z, w) &= i\bar{z}^1 w_3 \zeta^{-1} - i\bar{z}^3 w_1 \zeta^{-1} - iz^1 \bar{w}_3 \zeta + iz^3 \bar{w}_1 \zeta = 0, \\ &\dots \\ r^{2n-1}(z, w) &= \bar{z}^1 w_n \zeta^{-1} - \bar{z}^n w_1 \zeta^{-1} + z^1 \bar{w}_n \zeta - z^n \bar{w}_1 \zeta = 0, \\ r^{2n}(z, w) &= i\bar{z}^1 w_n \zeta^{-1} - i\bar{z}^n w_1 \zeta^{-1} - iz^1 \bar{w}_n \zeta + iz^n \bar{w}_1 \zeta = 0. \end{aligned}$$

Hence the  $(2n \times 2n)$ -matrix  $K(\zeta)$  has the following expression:

$$\begin{pmatrix} \zeta & 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 \\ -i\zeta & 0 & 0 & \dots & 0 & -i & 0 & 0 & \dots & 0 \\ 0 & -\zeta^{-1} & 0 & \dots & 0 & 0 & \zeta^2 & 0 & \dots & 0 \\ 0 & -i\zeta^{-1} & 0 & \dots & 0 & 0 & -i\zeta^2 & 0 & \dots & 0 \\ 0 & 0 & -\zeta^{-1} & \dots & 0 & 0 & 0 & \zeta^2 & \dots & 0 \\ 0 & 0 & -i\zeta^{-1} & \dots & 0 & 0 & 0 & -i\zeta^2 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & -\zeta^{-1} & 0 & 0 & 0 & \dots & \zeta^2 \\ 0 & 0 & 0 & \dots & -i\zeta^{-1} & 0 & 0 & 0 & \dots & -i\zeta^2 \end{pmatrix}$$

and a direct computation shows that  $-B = \bar{K}^{-1}K$  has the form

$$\begin{pmatrix} C_1 & C_2 \\ C_3 & C_4 \end{pmatrix},$$

where the  $(n \times n)$  matrices  $C_1, \dots, C_4$  are given by

$$\begin{aligned} C_1 &= \begin{pmatrix} \zeta^2 & 0 & \cdot & 0 \\ 0 & 0 & \cdot & 0 \\ 0 & 0 & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdot & 0 \end{pmatrix}, \quad C_2 = \begin{pmatrix} 0 & 0 & \cdot & 0 \\ 0 & -\zeta & \cdot & 0 \\ 0 & 0 & -\zeta & 0 \\ \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & -\zeta \end{pmatrix}, \\ C_3 &= \begin{pmatrix} -2\zeta & 0 & \cdot & 0 \\ 0 & -\zeta & \cdot & 0 \\ 0 & 0 & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdot & -\zeta \end{pmatrix}, \quad C_4 = \begin{pmatrix} -1 & 0 & \cdot & 0 \\ 0 & 0 & \cdot & 0 \\ 0 & 0 & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdot & 0 \end{pmatrix}. \end{aligned}$$

We point out that the matrix

$$\begin{pmatrix} \zeta^2 & 0 \\ \zeta & 1 \end{pmatrix}$$

admits the following factorization:

$$\begin{pmatrix} 1 & \zeta \\ 0 & 1 \end{pmatrix} \times \begin{pmatrix} -\zeta & 0 \\ 0 & \zeta \end{pmatrix} \times \begin{pmatrix} 0 & 1 \\ 1 & \zeta^{-1} \end{pmatrix}.$$

Permutating the lines (that is multiplying  $B$  by some nondegenerate matrices with constant coefficients) and using the above factorization of  $(2 \times 2)$  matrices, we obtain the following

PROPOSITION 4. *All the partial indices of the conormal bundle of the unit sphere along a meromorphic disc of a stationary disc are equal to one and the Maslov index is equal to  $2n$ .*

Proposition 4 enables to apply Proposition 1 to construct the family of stationary discs attached to the unit sphere after a small deformation of the complex structure. Indeed denote by  $r^j(z, w, \zeta, \lambda)$   $C^\infty$ -smooth functions coinciding for  $\lambda = 0$  with the above functions  $r^1, \dots, r^{2n}$ .

In the end of this subsection we make the two following assumptions:

- (i)  $r^1(z, w, \zeta, \lambda) = z^1 \bar{z}^1 + \dots + z^n \bar{z}^n - 1$ , meaning that the sphere is not deformed
- (ii)  $r^j(z, tw, \zeta, \lambda) = tr^j(z, w, \zeta, \lambda)$  for every  $j \geq 2, t \in \mathbb{R}$ .

Geometrically this means that given  $\lambda$ , the set  $\{(z, w) : r^j(z, w, \lambda) = 0\}$  is a real vector bundle with one-dimensional fibers over the unit sphere.

Consider an almost complex deformation  $J_\lambda$  of the standard structure on  $\mathbb{B}_n$  and an elliptic prolongation  $T_\lambda$  of  $J_\lambda$  on  $\mathbb{B}_n \times \mathbb{R}^{2n}$ . Recall that the corresponding  $\bar{\partial}_{T_\lambda}$ -equation is given by the following system

$$(\mathcal{E}) \begin{cases} \bar{\partial} f + q_1(\lambda, f) \partial f = 0, \\ \bar{\partial} g + q_2(\lambda, f) \partial g + q_3(\lambda, f) g = 0. \end{cases}$$

Consider now the corresponding boundary problem:

$$(\mathcal{BP}_\lambda) \begin{cases} r(f, g, \zeta, \lambda) = 0, \zeta \in \partial \Delta, \\ \bar{\partial} f + q_1(\lambda, f) \partial f = 0, \\ \bar{\partial} g + q_2(\lambda, f) \partial g + q_3(\lambda, f) g = 0. \end{cases}$$

Combining Proposition 4 with the results of Section 2, we obtain the following

PROPOSITION 5. *For every sufficiently small positive  $\lambda$ , the set of solutions of  $(\mathcal{BP}_\lambda)$ , close enough to the disc  $(\widehat{f^0}, \widehat{g^0})$ , forms a smooth  $4n$ -dimensional submanifold  $V_\lambda$  in the space  $C^\alpha(\bar{\Delta})$  (for every noninteger  $\alpha > 1$ ).*

Moreover, in view of the assumption (ii) and of the linearity of  $(\mathcal{E})$  with respect to the fiber component  $g$ , we also have the following

COROLLARY 1. *The projections of discs from  $V_\lambda$  to the base  $(\mathbb{R}^{2n}, J_\lambda)$  form a  $(4n - 1)$ -dimensional subvariety in  $C^\alpha(\bar{\Delta})$ .*

Geometrically the solutions  $(f, g)$  of the boundary problem  $(\mathcal{BP}_\lambda)$  are such that the discs  $(f, \zeta^{-1}g)$  are attached to the conormal bundle of the unit sphere with respect to the standard structure. In particular, if  $\lambda = 0$  then every such disc satisfying  $f(0) = 0$  is linear.

#### 4. – Foliation and “Riemann map” associated with an elliptic prolongation of an almost complex structure

In this section we study the geometry of stationary discs in the unit ball after a small almost complex perturbation of the standard structure. The idea is simple since these discs are small deformations of the complex lines passing through the origin in the unit ball.

##### 4.1. – Foliation associated with an elliptic prolongation

Fix a vector  $v^0$  with  $\|v^0\| = 1$  and consider the corresponding stationary disc  $f^0 : \zeta \mapsto \zeta v^0$ . Denote by  $(\widehat{f^0}, \widehat{g^0})$  its lift to the conormal bundle of the unit sphere. Consider a smooth deformation  $J_\lambda$  of the standard structure on the unit ball  $\mathbb{B}_n$  in  $\mathbb{C}^n$ . For sufficiently small  $\lambda_0$  fix a prolongation  $T_\lambda$  of the structure  $J_\lambda$  on  $\mathbb{B}_n \times \mathbb{R}^{2n}$ , where  $\lambda \leq \lambda_0$ . Then the solutions of the associated boundary problem  $(\mathcal{B}P_\lambda)$  form a  $4n$ -parameter family of  $(J_0, T_\lambda)$ -holomorphic maps from  $\Delta \setminus \{0\}$  to  $\mathbb{C}^n \times \mathbb{C}^n$ . Given such a solution  $(f^\lambda, g^\lambda)$ , consider the disc  $(\widehat{f^\lambda}, \widehat{g^\lambda}) := (f^\lambda, \zeta^{-1} g^\lambda)$ . In the case where  $\lambda = 0$  this is just the lift of a stationary disc for the unit sphere to its conormal bundle. The set of solutions of the problem  $(\mathcal{B}P_\lambda)$  (considered in Section 3.1), close to  $(\widehat{f^0}, \widehat{g^0})$ , forms a smooth submanifold of real dimension  $4n$  in  $(\mathcal{C}^\alpha(\bar{\Delta}))^{4n}$  according to Theorem 5. Hence there is a neighborhood  $V_0$  of  $v^0$  in  $\mathbb{R}^{2n}$  and a smooth real hypersurface  $I_{v^0}^\lambda$  in  $V_0$  such that for every  $\lambda \leq \lambda_0$  and for every  $v \in I_{v^0}^\lambda$  there is one and only one solution  $(f_v^\lambda, g_v^\lambda)$  of  $(\mathcal{B}P_\lambda)$ , up to multiplication of the fiber component  $g_v^\lambda$  by a real constant, such that  $f_v^\lambda(0) = 0$  and  $df_v^\lambda(0)(\partial/\partial x) = v$ .

We may therefore consider the map

$$F_0^\lambda : (v, \zeta) \in I_{v^0}^\lambda \times \bar{\Delta} \mapsto (f_v^\lambda, g_v^\lambda)(\zeta).$$

This is a smooth map with respect to  $\lambda$  close to the origin in  $\mathbb{R}$ .

Denote by  $\pi$  the canonical projection  $\pi : \mathbb{B}_n \times \mathbb{R}^{2n} \rightarrow \mathbb{B}_n$  and consider the composition  $\widehat{F}_0^\lambda = \pi \circ F_0^\lambda$ . This is a smooth map defined for  $0 \leq \lambda < \lambda_0$  and such that

- (i)  $\widehat{F}_0^0(v, \zeta) = v\zeta$ , for every  $\zeta \in \bar{\Delta}$  and for every  $v \in I_{v^0}^\lambda$ .
- (ii) For every  $\lambda \leq \lambda_0$ ,  $\widehat{F}_0^\lambda(v, 0) = 0$ .
- (iii) For every fixed  $\lambda \leq \lambda_0$  and every  $v \in I_{v^0}^\lambda$  the map  $\widehat{F}_0^\lambda(v, \cdot)$  is a  $(J_0, J_\lambda)$ -holomorphic disc attached to the unit sphere.
- (iv) For every fixed  $\lambda$ , different values of  $v \in I_{v^0}^\lambda$  define different discs.

**DEFINITION 4.** We call the family  $(\widehat{F}_0^\lambda(v, \cdot))_{v \in I_{v^0}^\lambda}$  *canonical discs* associated with the boundary problem  $(\mathcal{B}P_\lambda)$ .

We stress that by a canonical disc we always mean a disc centered at the origin. The preceding condition (iv) may be restated as follows:

LEMMA 4. For  $\lambda < \lambda_0$  every canonical disc is uniquely determined by its tangent vector at the origin.

In the next subsection we glue the sets  $I_v^\lambda$ , depending on vectors  $v \in \mathbb{S}^{2n-1}$ , to define the global indicatrix of  $F^\lambda$ .

**4.2. – Indicatrix**

For  $\lambda < \lambda_0$  consider canonical discs in  $(\mathbb{B}_n, J_\lambda)$  centered at the origin and admitting lifts close to  $(\widehat{f^0}, \widehat{g^0})$ .

As above we denote by  $I_{v^0}^\lambda$  the set of tangent vectors at the origin of canonical discs whose lift is close to  $(\widehat{f^0}, \widehat{g^0})$ . Since these vectors depend smoothly on parameters  $v$  close to  $v^0$  and  $\lambda \leq \lambda_0$ ,  $I_{v^0}^\lambda$  is a smooth deformation of a piece of the unit sphere  $\mathbb{S}^{2n-1}$ . So this is a smooth real hypersurface in  $\mathbb{C}^n$  in a neighborhood of  $v^0$ . Repeating this construction for every vector  $v \in \mathbb{S}^{2n-1}$  we may find a finite covering of  $\mathbb{S}^{2n-1}$  by open connected sets  $U_j$  such that for every  $j$  the nearby stationary discs with tangent vectors at the origin close to  $v$  are given by  $\widehat{F_j^\lambda}$ . Since every nearby stationary disc is uniquely determined by its tangent vector at the origin, we may glue the maps  $\widehat{F_j^\lambda}$  to the map  $\widehat{F^\lambda}$  defined for every  $v \in \mathbb{S}^{2n-1}$  and every  $\zeta \in \bar{\Delta}$ . The tangent vectors of the constructed family of stationary discs form a smooth real hypersurface  $I^\lambda$  which is a small deformation of the unit sphere. This hypersurface is an analog of the indicatrix for the Kobayashi metric (more precisely, its boundary).

We point out that the local indicatrix  $I_{v^0}^\lambda$  for some fixed  $v^0 \in \mathbb{S}^{2n-1}$  is also useful (see Section 5).

**4.3. – Circled property and Riemann map**

If  $\lambda$  is small enough, the hypersurface  $I^\lambda$  is strictly pseudoconvex with respect to the standard structure. Another important property of the “indicatrix” is its invariance with respect to the linear action of the unit circle.

Let  $\lambda \leq \lambda_0$ ,  $v \in I^\lambda$  and  $f_v^\lambda := \widehat{F^\lambda}(v, \cdot)$ . For  $\theta \in \mathbb{R}$  we denote by  $f_{v,\theta}^\lambda$  the  $J_\lambda$ -holomorphic disc in  $\mathbb{B}_n$  defined by  $f_{v,\theta}^\lambda : \zeta \in \Delta \mapsto f_v^\lambda(e^{i\theta}\zeta)$ . We have:

LEMMA 5. For every  $0 \leq \lambda < \lambda_0$ , every  $v \in I^\lambda$  and every  $\theta \in \mathbb{R}$  we have:  $f_{v,\theta}^\lambda \equiv f_{e^{i\theta}v}^\lambda$ .

PROOF. Since  $f_v^\lambda$  is a canonical disc, the disc  $f_{v,\theta}^\lambda$  has a lift close to the lift of the disc  $\zeta \mapsto e^{i\theta}v\zeta$ . Then according to Lemma 3  $f_{v,\theta}^\lambda$  is a canonical disc close to the linear disc  $\zeta \mapsto e^{i\theta}v\zeta$ . Since the first jet of  $f_{v,\theta}^\lambda$  coincides with the first jet of  $f_{e^{i\theta}v}^\lambda$ , these two nearby stationary discs coincide according to Lemma 4. □

This statement implies that for any  $w \in I^\lambda$  the vector  $e^{i\theta}w$  is in  $I^\lambda$  as well.

It follows from the above arguments that there exists a natural parametrization of the set of canonical discs by their tangent vectors at the origin, that is by the points of  $I^\lambda$ . The map

$$\begin{aligned}\widehat{F}^\lambda : I^\lambda \times \Delta &\rightarrow \mathbb{B}_n \\ (v, \zeta) &\mapsto f_v^\lambda(\zeta)\end{aligned}$$

is smooth on  $I^\lambda \times \Delta$ . Moreover, if we fix a small positive constant  $\varepsilon_0$ , then by shrinking  $\lambda_0$  if necessary there is, for every  $\lambda < \lambda_0$ , a smooth function  $\widehat{G}^\lambda$  defined on  $I^\lambda \times \Delta$ , satisfying  $\|\widehat{G}^\lambda(v, \zeta)\| \leq \varepsilon_0|\zeta|^2$  on  $I^\lambda \times \Delta$ , such that for every  $\lambda < \lambda_0$  we have on  $I^\lambda \times \Delta$ :

$$(7) \quad \widehat{F}^\lambda(v, \zeta) = \zeta v + \widehat{G}^\lambda(v, \zeta).$$

Consider now the restriction of  $\widehat{F}^\lambda$  to  $I^\lambda \times [0, 1]$ . This is a smooth map, still denoted by  $\widehat{F}^\lambda$ . We have the following:

PROPOSITION 6. *There exists  $\lambda_1 \leq \lambda_0$  such that for every  $\lambda < \lambda_1$  the family  $(\widehat{F}^\lambda(v, r))_{(v,r) \in I^\lambda \times [0,1]}$  is a real foliation of  $\mathbb{B}_n \setminus \{0\}$ .*

PROOF. STEP 1. For  $r \neq 0$  we write  $w := rv$ . Then  $r = \|w\|$ ,  $v = w/\|w\|$  and we denote by  $\widetilde{F}^\lambda$  the function  $\widetilde{F}^\lambda(w) := \widehat{F}^\lambda(v, r)$ . For  $\lambda < \lambda_0$ ,  $\widetilde{F}^\lambda$  is a smooth map of the variable  $w$  on  $\mathbb{B}_n \setminus \{0\}$ , satisfying:

$$\widetilde{F}^\lambda(w) = w + \widetilde{G}^\lambda(w)$$

where  $\widetilde{G}^\lambda$  is a smooth map on  $\mathbb{B}_n \setminus \{0\}$  with  $\|\widetilde{G}^\lambda(w)\| \leq \varepsilon_0\|w\|^2$  on  $\mathbb{B}_n \setminus \{0\}$ . This implies that  $\widetilde{F}^\lambda$  is a local diffeomorphism at each point in  $\mathbb{B}_n \setminus \{0\}$ , and so that  $\widehat{F}^\lambda$  is a local diffeomorphism at each point in  $I^\lambda \times ]0, 1[$ . Moreover, the condition  $\|\widetilde{G}^\lambda(w)\| \leq \varepsilon_1\|w\|^2$  on  $\mathbb{B}_n \setminus \{0\}$  implies that  $\widetilde{G}^\lambda$  is differentiable at the origin with  $d\widetilde{G}^\lambda(0) = 0$ . Hence by the implicit function theorem there exists  $\lambda_1 < \lambda_0$  such that the map  $\widetilde{F}^\lambda$  is a local diffeomorphism at the origin for  $\lambda < \lambda_1$ . So there exists  $0 < r_1 < 1$  and a neighborhood  $U$  of the origin in  $\mathbb{C}^n$  such that  $\widehat{F}^\lambda$  is a diffeomorphism from  $I^\lambda \times ]0, r_1[$  to  $U \setminus \{0\}$ , for  $\lambda < \lambda_1$ .

STEP 2. We show that  $\widehat{F}^\lambda$  is injective on  $I^\lambda \times ]0, 1]$  for sufficiently small  $\lambda$ . Assume by contradiction that for every  $n$  there exist  $\lambda_n \in \mathbb{R}$ ,  $r_n, r'_n \in ]0, 1]$ ,  $v^n, w^n \in I^{\lambda_n}$  such that:

- $\lim_{n \rightarrow \infty} \lambda_n = 0$ ,  $\lim_{n \rightarrow \infty} r_n = r$ ,  $\lim_{n \rightarrow \infty} r'_n = r'$ ,
- $\lim_{n \rightarrow \infty} v^n = v \in \mathbb{S}^{2n-1}$ ,  $\lim_{n \rightarrow \infty} w^n = w \in \mathbb{S}^{2n-1}$

and satisfying

$$\widehat{F}^{\lambda_n}(v^n, r_n) = \widehat{F}^{\lambda_n}(w^n, r'_n)$$



for every  $n$ . Since  $\widehat{F}$  is smooth with respect to  $\lambda, v, r$ , it follows that  $\widehat{F^0}(v, r) = \widehat{F^0}(w, r')$  and so  $v = w$  and  $r = r'$ . If  $r < r_1$  then the contradiction follows from the fact that  $\widehat{F^\lambda}$  is a diffeomorphism from  $I^\lambda \times ]0, r_1[$  to  $U \setminus \{0\}$ . If  $r \geq r_1$  then for every neighborhood  $U_\infty$  of  $r v$  in  $\mathbb{B}_n \setminus \{0\}$ ,  $r_n v^n \in U_\infty$  and  $r'_n w^n \in U_\infty$  for sufficiently large  $n$ . Since we may choose  $U_\infty$  such that  $\widehat{F^\lambda}$  is a diffeomorphism from a neighborhood of  $(v, r)$  in  $I^\lambda \times ]r_1, 1[$  uniformly with respect to  $\lambda \ll 1$ , we still obtain a contradiction.

STEP 3. We show that  $\widehat{F^\lambda}$  is surjective from  $I^\lambda \times ]0, 1[$  to  $\mathbb{B}_n \setminus \{0\}$ . It is sufficient to show that  $\widehat{F^\lambda}$  is surjective from  $I^\lambda \times ]r_1, 1[$  to  $\mathbb{B}_n \setminus U$ . Consider the nonempty set  $E_\lambda = \{w \in \mathbb{B}_n \setminus U : w = \widehat{F^\lambda}(v, r) \text{ for some } (v, r) \in I^\lambda \times ]r_1, 1[ \}$ . Since the jacobian of  $\widehat{F^\lambda}$  does not vanish for  $\lambda = 0$  and  $\widehat{F^\lambda}$  is smooth with respect to  $\lambda$ , the set  $E_\lambda$  is open for sufficiently small  $\lambda$ . Moreover it follows immediately from its definition that  $E_\lambda$  is also closed in  $\mathbb{B}_n \setminus U$ . Thus  $E_\lambda = \mathbb{B}_n \setminus U$ .

These three steps prove the result. □

We can construct now the map  $\Psi_{T_\lambda}$  for  $\lambda < \lambda_1$ . For every  $z \in \mathbb{B}_n \setminus \{0\}$  consider the unique couple  $(v(z), r(z)) \in I^\lambda \times ]0, 1[$  such that  $f_v^\lambda$  is the unique canonical disc passing through  $z$  (its existence and unicity are given by Proposition 6) with  $f_{v(z)}^\lambda(0) = 0, df_{v(z)}^\lambda(0)(\partial/\partial x) = v(z)$  and  $f_{v(z)}^\lambda(r(z)) = z$ . The map  $\Psi_{T_\lambda}$  is defined by:

$$\begin{aligned} \Psi_{T_\lambda} : \bar{\mathbb{B}}_n \setminus \{0\} &\rightarrow \mathbb{C}^n \\ z &\mapsto r(z)v(z). \end{aligned}$$

DEFINITION 5. The map  $\Psi_{T_\lambda}$  is called the Riemann map associated with a prolongation  $T_\lambda$  of the almost complex structure  $J_\lambda$ .

This map is an analogue of the circular representation of a strictly convex domain introduced by L. Lempert [13]. The term ‘‘Riemann map’’ was used by S. Semmes [20] for a slightly different map where the vector  $v(z)$  is normalized (and so such a map takes values in the unit ball). In this paper we work with the indicatrix since this is more convenient for our applications.

The Riemann map  $\Psi_{T_\lambda}$  has the following properties:

PROPOSITION 7.

- (i) For every  $(v, \zeta) \in I^\lambda \times \Delta$  we have  $(\Psi_{T_\lambda} \circ f_v^\lambda)(\zeta) = \zeta v$  and so  $\log \|(\Psi_{T_\lambda} \circ f_v^\lambda)(\zeta)\| = \log |\zeta| + \log \|v\|$ .
- (ii) There exist constants  $0 < C' < C$  such that  $C'\|z\| \leq \|\Psi_{T_\lambda}(z)\| \leq C\|z\|$  on  $\mathbb{B}_n$ .

PROOF.

- (i) Let  $\zeta = e^{i\theta} r \in \Delta(0, r_0)$  with  $\theta \in [0, 2\pi[$ . Then  $f_v^\lambda(\zeta) = f_v^\lambda(e^{i\theta} r) = f_{e^{i\theta} v}^\lambda(r)$ . Hence we have  $(\Psi_{T_\lambda} \circ f_v^\lambda)(\zeta) = \Psi_{T_\lambda}(f_{e^{i\theta} v}^\lambda(r)) = e^{i\theta} v r = \zeta v$ .
- (ii) Let  $z \in \mathbb{B}_n \setminus \{0\}$ . Then according to equation (7) we have the inequality  $\|\Phi^\lambda(z)\| (1 - \varepsilon_1 \|\Psi_{T_\lambda}(z)\|) \leq \|z\|^2 \leq \Psi_{T_\lambda}(z) (1 + \varepsilon_1 \|\Psi_{T_\lambda}(z)\|)$ . Since  $\|\Psi_{T_\lambda}(z)\| \leq 1$  we obtain the desired inequality with  $c' = 1/1 + \varepsilon_1$  and  $c = 1/1 - \varepsilon_1$ . □

From the above analysis it follows the basic properties of the Riemann map.

PROPOSITION 8.

- (i) *The indicatrix  $I^\lambda$  is a compact circled smooth  $J_\lambda$ -strictly pseudoconvex hypersurface bounding a domain denoted by  $\Omega^\lambda$ .*
- (ii) *The Riemann map  $\Psi_{T_\lambda} : \mathbb{B}_n \setminus \{0\} \rightarrow \widehat{\Omega}^\lambda \setminus \{0\}$  is a smooth diffeomorphism.*
- (iii) *For every canonical disc  $f_v^\lambda$  we have  $\Psi_{T_\lambda} \circ f_v^\lambda(\zeta) = v\zeta$ .*

We note that the Riemann map possesses further important structure properties depending on the choice of a prolongation  $T_\lambda$  of an almost complex structure  $J_\lambda$ .

#### 4.4. – Local Riemann map

In Subsection 4.2 we introduced the notion of local indicatrix  $I_{v^0}^\lambda$  for  $v^0 \in \mathbb{S}^{2n-1}$ . We may localize the notion of the Riemann map, introducing a similar associated with the local indicatrix. Denote by  $\Omega_{v^0}^\lambda$  the set  $I_{v^0}^\lambda \times [0, 1[$ . The arguments used in the proof of Proposition 6 show that  $\widehat{F}^\lambda(\Omega_{v^0}^\lambda)$  is foliated by stationary discs centered at the origin. We may therefore define the Riemann map  $\Psi_{T_\lambda, v^0}$  on  $\widehat{F}^\lambda(\Omega_{v^0}^\lambda)$  by:

$$\Psi_{T_\lambda, v^0}(z) = r(z)v(z)$$

where  $v(z)$  is the tangent vector at the origin of the unique stationary disc  $f_{v(z)}^\lambda$  passing through  $z$  and  $f_{v(z)}^\lambda(r(z)) = z$ .

REMARK 1. We point out that the Riemann map can be defined in any sufficiently small deformation of the unit ball and satisfies all the same properties.

### 5. – Riemann map and local geometry of almost complex manifolds

As we have seen previously, a choice of an elliptic prolongation  $T_\lambda$  of an almost complex structure  $J_\lambda$  on a vector bundle over the ball (for  $\lambda$  small enough) allows to define the foliation by canonical discs and the Riemann map. Further properties of such a map depend on the choice of  $T_\lambda$  which is not unique. In this section we consider a natural definition of  $T_\lambda$  as the vertical lift of the almost complex structure  $J_\lambda$ . This construction is well-known in differential geometry on the tangent and cotangent bundles, see [12]. For reader's convenience we recall it.

#### 5.1. – Complete lift to the cotangent bundle

Our goal now is to construct a family of Lempert discs invariant with respect to biholomorphic transformations of an almost complex manifold with boundary.

We recall the definition of the complete lift of an almost complex structure  $J$  on  $M$  to the cotangent bundle  $T^*M$ . Set  $m = 2n$ . We use the following notations. Suffixes A, B, C, D take the values 1 to  $2m$ , suffixes  $a, b, c, \dots, h, i, j, \dots$  take the values 1 to  $m$  and  $\bar{j} = j + m, \dots$ . The summation notation for repeated indices is used. If the notation  $(\varepsilon_{AB}), (\varepsilon^{AB}), (F_B^A)$  is used for matrices, the suffix on the left indicates the column and the suffix on the right indicates the row. We denote local coordinates on  $M$  by  $(x^1, \dots, x^n)$  and by  $(p_1, \dots, p_n)$  the fiber coordinates.

Recall that the cotangent space  $T^*(M)$  of  $M$  possesses the *canonical contact form*  $\theta$  given in local coordinates by

$$\theta = p_i dx^i .$$

The cotangent lift  $\varphi^*$  of any diffeomorphism  $\varphi$  of  $M$  is contact with respect to  $\theta$ , that is  $\theta$  does not depend on the choice of local coordinates on  $T^*(M)$ .

The exterior derivative  $d\theta$  of  $\theta$  defines *the canonical symplectic structure* of  $T^*(M)$ :

$$d\theta = dp_i \wedge dx^i$$

which is also independent of local coordinates in view of the invariance of the exterior derivative. Setting  $d\theta = (1/2)\varepsilon_{CB} dx^C \wedge dx^B$  (where  $dx^{\bar{j}} = dp_j$ ), we have

$$(\varepsilon_{CB}) = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix} .$$

Denote by  $(\varepsilon^{BA})$  the inverse matrix, we write  $\varepsilon^{-1}$  for the tensor field of type (2,0) whose component are  $(\varepsilon^{BA})$ . By construction, this definition does not depend on the choice of local coordinates.

Let now  $E$  be a tensor field of type (1,1) on  $M$ . If  $E$  has components  $E_i^h$  and  $E_i^{*h}$  relative to local coordinates  $x$  and  $x^*$  respectively, then

$$p_a^* E_i^{*a} = p_a E_j^b \frac{\partial x^j}{\partial x^{*i}} .$$

If we interpret a change of coordinates as a diffeomorphism  $x^* = x^*(x) = \varphi(x)$  we denote by  $E^*$  the direct image of the tensor  $E$  under the action of  $\varphi$ . In the case where  $E$  is an almost complex structure (that is  $E^2 = -\text{Id}$ ), then  $\varphi$  is a biholomorphism between  $(M, E)$  and  $(M, E^*)$ . Any (1,1) tensor field  $E$  on  $M$  canonically defines a contact form on  $E^*M$  via

$$\sigma = p_a E_b^a dx^b .$$

Since

$$(\varphi^*)^*(p_a^* E_b^{*a} dx^{*b}) = \sigma ,$$

$\sigma$  does not depend on a choice of local coordinates (here  $\varphi^*$  is the cotangent lift of  $\varphi$ ). Then this canonically defines the symplectic form

$$d\sigma = p_a \frac{\partial E_b^a}{\partial x^c} dx^c \wedge dx^b + E_b^a dp_a \wedge dx^b .$$

The cotangent lift  $\varphi^*$  of a diffeomorphism  $\varphi$  is a symplectomorphism for  $d\sigma$ . We may write

$$d\sigma = (1/2)\tau_{CB}dx^C \wedge dx^B$$

where  $x^{\bar{i}} = p_i$ ; so we have

$$\begin{aligned}\tau_{ji} &= p_a \left( \frac{\partial E_i^a}{\partial x^j} - \frac{\partial E_j^a}{\partial x^i} \right) \\ \tau_{\bar{j}\bar{i}} &= E_i^j \\ \tau_{j\bar{i}} &= -E_j^i \\ \tau_{\bar{j}\bar{i}} &= 0\end{aligned}$$

We write  $\widehat{E}$  for the tensor field of type (1,1) on  $T^*(M)$  whose components  $\widehat{E}_B^A$  are given by

$$\widehat{E}_B^A = \tau_{BC} \varepsilon^{CA}.$$

Thus

$$\widehat{E}_i^h = E_i^h, \quad \widehat{E}_{\bar{i}}^h = 0$$

and

$$\widehat{E}_i^{\bar{h}} = p_a \left( \frac{\partial E_i^a}{\partial x^j} - \frac{\partial E_j^a}{\partial x^i} \right), \quad \widehat{E}_{\bar{i}}^{\bar{h}} = E_h^i.$$

In the matrix form we have

$$\widehat{E} = \begin{pmatrix} E_i^h & 0 \\ p_a \left( \frac{\partial E_i^a}{\partial x^j} - \frac{\partial E_j^a}{\partial x^i} \right) & E_h^i \end{pmatrix}.$$

By construction, the complete lift  $\widehat{E}$  has the following *invariance property*: if  $\varphi$  is a local diffeomorphism of  $M$  transforming  $E$  to  $E'$ , then the direct image of  $\widehat{E}$  under the cotangent lift  $\psi := \varphi^*$  is  $\widehat{E}'$ .

We point out that in general,  $\widehat{E}$  is not an almost complex structure, even if  $E$  is. However, the invariance condition

$$(8) \quad d\varphi^* \circ \widehat{E} = \widehat{E}'(\varphi^*) \circ d\varphi^*$$

gives a Beltrami-type equation quite similarly to the almost complex case.

## 5.2. – Structure properties of the Riemann map

Assume now that  $M \subset \mathbb{C}^n$  and let  $J_\lambda$  be an almost complex deformation of the standard structure on  $M$ . We denote by  $T_\lambda$  the elliptic prolongation  $\widehat{J}_\lambda$  of  $J_\lambda$  ( $\widehat{J}_\lambda$  is the vertical lift of  $J_\lambda$  to the cotangent bundle). So the Beltrami-type equation (8) is just the  $\bar{\partial}_{T_\lambda}$ -equation. In order to define a biholomorphically invariant family of corresponding canonical discs we need to consider an invariant boundary problem for the operator  $\bar{\partial}_{T_\lambda}$ .

We point out that the notion of the conormal bundle can be easily carried to the case of an almost complex manifold. Let  $i : T^*(M) \rightarrow T^*_{(1,0)}(M, J)$  be the canonical identification. Let  $D$  be a smoothly bounded domain in  $M$  with the boundary  $\Gamma$ . The conormal bundle  $\Sigma_J(\Gamma)$  of  $\Gamma$  is a real subbundle of  $T^*_{(1,0)}(M, J)|_\Gamma$  whose fiber at  $z \in \Gamma$  is defined by  $\Sigma_z(\Gamma) = \{\phi \in T^*_{(1,0)}(M, J) : \text{Re } \phi|_{H^J_{(1,0)}(\Gamma)} = 0\}$ .

DEFINITION 6. A continuous map  $f : \bar{\Delta} \rightarrow (\bar{D}, J)$ ,  $(J_0, J)$ -holomorphic on  $\Delta$ , is called a *stationary disc* for  $(D, J)$  (or for  $(\Gamma, J)$ ) if there exists a smooth map  $\hat{f} = (f, g) : \Delta \rightarrow T^*(M)$ ,  $\hat{f} \neq 0$  which is continuous on  $\bar{\Delta}$  and such that

- (i)  $\zeta \mapsto \hat{f}(\zeta)$  satisfies the  $\bar{\partial}_{T_\lambda}$ -equation on  $\Delta$ ,
- (ii)  $(i \circ (f, \zeta^{-1}g))(\partial\Delta) \subset \Sigma_J(\Gamma)$ .

We call  $\hat{f}$  a *lift* of  $f$  to the conormal bundle of  $\Gamma$ . Clearly, in view of our choice of  $T_\lambda$  the notion of a stationary disc is *invariant* in the following sense: if  $\phi$  is a  $C^1$  diffeomorphism between  $\bar{D}$  and  $\bar{D}'$  and a  $(J, J')$ -biholomorphism from  $D$  to  $D'$ , then for every stationary disc  $f$  in  $(D, J)$  the composition  $\phi \circ f$  is a stationary disc in  $(D', J')$ .

Let now  $D$  coincide with the unit ball  $\mathbb{B}_n$  equipped with an almost complex deformation  $J_\lambda$  of the standard structure. Then it follows by definition that stationary discs in  $(\mathbb{B}_n, J_\lambda)$  may be described as solutions of a nonlinear boundary problem  $(\mathcal{BP}_\lambda)$  associated with  $T_\lambda$ . The above techniques give the existence and efficient parametrization of the variety of stationary discs in  $(\mathbb{B}_n, J_\lambda)$  for  $\lambda$  small enough. This allows to apply the definition of the Riemann map and gives its existence. We sum up our considerations in the following

THEOREM 1. *Let  $J_\lambda, J'_\lambda$  be almost complex perturbations of the standard structure on  $\mathbb{B}_n$  and  $T_\lambda, T'_\lambda$  be their vertical lift to the cotangent bundle. The Riemann map  $\Psi_{T_\lambda}$  exists for sufficiently small  $\lambda$  and satisfies the following properties:*

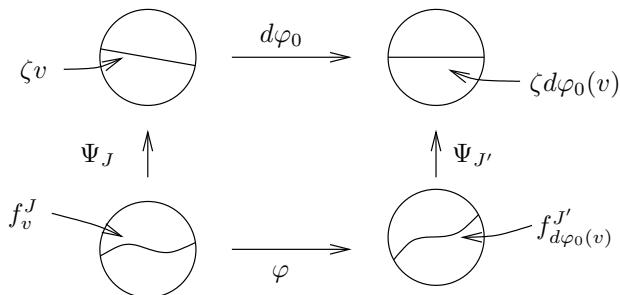
- (a)  $\Psi_{T_\lambda} : \mathbb{B}_n \setminus \{0\} \rightarrow \bar{\Omega}^\lambda \setminus \{0\}$  is a smooth diffeomorphism
- (b) The restriction of  $\Psi_{T_\lambda}$  on every stationary disc through the origin is  $(J_0, J_0)$  holomorphic (and even linear)
- (c)  $\Psi_{T_\lambda}$  commutes with biholomorphisms. More precisely for sufficiently small  $\lambda'$  and for every  $C^1$  diffeomorphism  $\varphi$  of  $\mathbb{B}_n$ ,  $(J_\lambda, J_{\lambda'})$ -holomorphic in  $\mathbb{B}_n$  and satisfying  $\varphi(0) = 0$ , we have

$$\varphi = (\Psi_{T_{\lambda'}})^{-1} \circ d\varphi_0 \circ \Psi_{T_\lambda}.$$

PROOF OF THEOREM 1. Conditions (a) and (b) are conditions (i) and (ii) of Proposition 8.

For condition (c), let  $\varphi : (\mathbb{B}_n, J) \rightarrow (\mathbb{B}_n, J')$  be a  $(J, J')$ -biholomorphism of class  $C^1$  on  $\mathbb{B}_n$  satisfying  $\varphi(0) = 0$ . We know that a disc  $f_v^J$  is a canonical disc for the almost complex structure  $J$  if and only if  $\varphi(f_v^J)$  is a canonical

disc for the almost complex structure  $J'$ . Since  $\varphi(f_v^J) = f_{d\varphi_0}^{J'}$  by definition,  $\Psi_J(f_v^J)(\zeta) = \zeta v$  and  $\Psi_{J'}(f_{d\varphi_0}^{J'})(\zeta) = \zeta d\varphi_0(v)$  by Proposition 8 (iii), condition (c) follows from the following diagram:



Theorem 1 gives the main structure properties of the Riemann map.

### 5.3. – Regularity of diffeomorphisms

Riemann maps are useful for the boundary study of biholomorphisms in almost complex manifolds. We have

**COROLLARY 2.** *If  $\lambda \ll 1$  and  $\varphi$  is a  $C^1$  diffeomorphism of  $\bar{\mathbb{B}}_n$ ,  $(J_\lambda, J_{\lambda'})$ -holomorphic in  $\mathbb{B}_n$  satisfying  $\varphi(0) = 0$ , then  $\varphi$  is of class  $C^\infty$  on  $\bar{\mathbb{B}}_n$ .*

**PROOF.** This follows immediately by Theorem 1 condition (c) since the Riemann map is smooth up to the boundary. □

Every almost complex structure is locally a small deformation of the standard structure. So we have the following partial generalization of Fefferman’s theorem:

**THEOREM 2.** *Let  $(M, J)$ ,  $(M, J')$  be almost complex manifolds and let  $D$  and  $D'$  be domains in  $M$  and  $M'$  respectively. Assume that  $\partial D$  and  $\partial D'$  contain smooth strictly pseudoconvex open pieces  $\Gamma$  and  $\Gamma'$ . If  $\varphi$  is a  $C^1$  diffeomorphism between  $D \cup \Gamma$  and  $D' \cup \Gamma'$ ,  $(J, J')$ -holomorphic on  $D$  such that  $\varphi(\Gamma) \subset \Gamma'$ , then  $\varphi$  is of class  $C^\infty$  on  $D \cup \Gamma$ .*

**PROOF.** Let  $q$  be a point of  $\Gamma$  and  $q' = \phi(q)$ . We may assume that  $J(q) = J_0$  and  $J'(q') = J_0$ . Since  $\Gamma$  and  $\Gamma'$  are strictly pseudoconvex, in local coordinates near  $q$  (resp.  $q'$ ) satisfying  $z(q) = 0$  (resp.  $z(q') = 0$ ) they are small deformations of the Siegel sphere  $\mathbb{H} = \{\text{Re}z^n + \|z\|^2 = 0\}$ . After the Caley map,  $\mathbb{H}$  is transformed to the unit sphere  $\mathbb{S}$  and  $\Gamma$  (resp.  $\Gamma'$ ) may be viewed as a small deformation of the sphere along the boundary of some stationary disc  $f^0$ ,  $f^0(0) = 0$  such that the corresponding disc on the Siegel domain  $\{\text{Re}z^n + \|z\|^2 < 0\}$  has direction at the center parallel to the holomorphic tangent space of  $\mathbb{H}$  at the origin. Denoting again by  $J$  resp.  $J'$  the images of our almost complex structures under the Caley map, we see that in a neighborhood of the disc  $f^0$  they may be viewed as an arbitrarily small deformations of

the standard structure. So we may consider as in Subsection 4.4 the Riemann map  $\Psi_{T_\lambda, v^0}$  where  $v^0$  is the tangent vector at the origin of the disc  $f^0$ . Since the property (c) of Theorem 1 obviously holds for local Riemann maps, we conclude.  $\square$

Theorem 2 admits the following global version.

**THEOREM 3.** *Let  $(M, J)$  and  $(M', J')$  be almost complex manifolds. Let  $D$  and  $D'$  be smoothly bounded strictly pseudoconvex domains in  $M$  and  $M'$  respectively. Every  $C^1$  diffeomorphism between  $\bar{D}$  and  $\bar{D}'$ ,  $(J, J')$ -holomorphic on  $D$ , is of class  $C^\infty$  on  $\bar{D}$ .*

The following reformulation of Theorem 2 may be considered as a geometrical version of the elliptic regularity for manifolds with boundary.

**THEOREM 4.** *Let  $M$  and  $M'$  be two  $C^\infty$  smooth real  $2n$ -dimensional manifolds,  $D \subset M$  and  $D' \subset M'$  be relatively compact domains. Suppose that there exists an almost complex structure  $J$  of class  $C^\infty$  on  $\bar{D}$  such that  $(D, J)$  is strictly pseudoconvex. Then a  $C^1$  diffeomorphism  $\phi$  between  $\bar{D}$  and  $\bar{D}'$  is of class  $C^\infty(\bar{D})$  if and only if the direct image  $J' := \phi^*(J)$  is of class  $C^\infty(\bar{D}')$  and  $(D', J')$  is strictly pseudoconvex.*

**5.4. – Rigidity and local equivalence problem**

Condition (c) of Theorem 1 implies the following partial generalization of Cartan’s theorem for almost complex manifolds:

**COROLLARY 3.** *If  $\lambda \ll 1$  and if  $\varphi$  is a  $C^1$  diffeomorphism of  $\bar{\mathbb{B}}_n$ ,  $(J_\lambda, J_{\lambda'})$ -holomorphic in  $\mathbb{B}_n$ , satisfying  $\varphi(0) = 0$  and  $d\varphi(0) = I$  then  $\varphi$  is the identity.*

This provides an efficient parametrization of the isotropy group of the group of biholomorphisms of  $(\mathbb{B}_n, J_\lambda)$ .

We can solve the local biholomorphic equivalence problem between almost complex manifolds in terms of the Riemann map similarly to [5], [16] (see the paper [18] by P. Libermann for a traditional approach to this problem based on Cartan’s equivalence method for  $G$ -structures). Let  $I^\lambda$  (resp.  $(I')^\lambda$ ) be the indicatrix of  $(\mathbb{B}_n, J_\lambda)$  (resp.  $(\mathbb{B}_n, J'_\lambda)$ ) bounding the domain  $\Omega^\lambda$  (resp.  $(\Omega')^\lambda$ ) and let  $\Psi_{T_\lambda}$  (resp.  $\Psi_{T'_\lambda}$ ) be the associated Riemann map. This induces the almost complex structure  $J_\lambda^* := d\Psi_{T_\lambda} \circ J_\lambda \circ d(\Psi_{T_\lambda})^{-1}$  (resp.  $(J'_\lambda)^* := d\Psi_{T'_\lambda} \circ J_\lambda \circ d(\Psi_{T'_\lambda})^{-1}$ ) on  $\Omega^\lambda$  (resp.  $(\Omega')^\lambda$ ). Then we have:

**THEOREM 5.** *The following conditions are equivalent:*

- (i) *there exists a  $C^\infty$  diffeomorphism  $\varphi$  of  $\bar{\mathbb{B}}_n$ ,  $(J_\lambda, J'_\lambda)$ -holomorphic on  $\mathbb{B}_n$  and satisfying  $\varphi(0) = 0$ ,*
- (ii) *there exists a  $J_0$ -linear isomorphism  $L$  of  $\mathbb{C}^n$ ,  $(J_\lambda^*, (J'_\lambda)^*)$ -holomorphic on  $\Omega^\lambda$  and such that  $L(\Omega^\lambda) = (\Omega')^\lambda$ .*

PROOF. If  $\varphi$  satisfies condition (i), the commutativity of the following diagram (in view of Theorem 1)

$$\begin{array}{ccc} (\Omega^\lambda, J_\lambda^*) & \xrightarrow{L=d\varphi_0} & ((\Omega')^\lambda, (J')_\lambda^*) \\ \uparrow \Psi_{T_\lambda} & & \uparrow \Psi'_{T_\lambda} \\ (\mathbb{B}_n, J_\lambda) & \xrightarrow{\varphi} & (\mathbb{B}_n, J'_\lambda) \end{array}$$

shows that  $L := d\varphi_0$  satisfies condition (ii). Conversely if  $L$  satisfies condition (ii) then the map  $\varphi := (\Psi'_{T_\lambda})^{-1} \circ L \circ \Psi_{T_\lambda}$  satisfies condition (i).  $\square$

In conclusion we point out that there are many open questions concerning the Riemann map on almost complex manifolds (contact properties, relation with the Monge-Ampère equation, ...). They will be studied in a forthcoming paper.

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LATP UMR 6632  
 C.M.I.  
 39, rue Joliot-Curie  
 13453 Marseille Cedex 13, France  
 coupet@cmi.univ-mrs.fr

LATP UMR 6632  
 C.M.I.  
 39, rue Joliot-Curie  
 13453 Marseille Cedex 13, France  
 gaussier@cmi.univ-mrs.fr

AGAT UMR 8524  
 U.S.T.L.  
 Cité Scientifique  
 59655 Villeneuve d'Ascq Cedex, France  
 sukhov@agat.univ-lille1.fr