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Some Remarks on the Theory of Elasticity for Compressible Neohookean Materials

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Abstract. In compressible Neohookean elasticity one minimizes functionals which are composed by the sum of the L^2 norm of the deformation gradient and a nonlinear function of the determinant of the gradient. Non–interpenetrability of matter is then represented by additional invertibility conditions. An existence theory which includes a precise notion of invertibility and allows for cavitation was formulated by Müller and Spector in 1995. It applies, however, only if some L^p-norm of the gradient with $p > 2$ is controlled (in three dimensions). We first characterize their class of functions in terms of properties of the associated rectifiable current. Then we address the physically relevant $p = 2$ case, and show how their notion of invertibility can be extended to $p = 2$. The class of functions so obtained is, however, not closed. We prove this by giving an explicit construction.

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1. – Introduction

The starting point of this paper is the following question: how does one address the problem of existence of minimizers in the case of compressible Neohookean materials? The model problem in this framework is minimizing energies like

$$E(u) := \int_\Omega |\nabla u|^2 + \varphi(\det \nabla u)\,, \tag{1}$$

where φ is a convex function with superlinear growth and approaching infinity at zero. The minimizers are sought among deformations u which map $\Omega \subset \mathbb{R}^3$ into $\mathbb{R}^3$ and satisfy some notion of invertibility and a Dirichlet boundary condition (the classical starting point is to look for minimizers in the class of

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orientation-preserving diffeomorphisms or in that of Bilipschitz maps which are orientation preserving).

The known existence theories, starting from the classical works of Morrey (see in particular Ball [4] and the bibliography in the second volume of [12]) give only partial answers for the case of interest here: the available works rely on relaxation techniques (see for example [3], [5], [13], [14], [1], and the works of Giaquinta, Modica and Souček cited below) but they do not attack the problem of how "bad" the domain of the relaxed functional could be. In particular one of the consequences of Malý's work [13], [14] is the following. For every function $u \in W^{1,2}$ we define $F(u)$ as the infimum of

$$\liminf_{n\to\infty} E(u_n)$$

among all sequences of orientation-preserving Bilipschitz maps (u_n) weakly converging in $W^{1,2}$ to u. If u is itself an orientation-preserving Bilipschitz map, then $F(u) = E(u)$. Hence a minimizer of F could be regarded as a weak solution of the classical problem of minimizing E among regular admissible elastic deformations.

The approach of Giaquinta, Modica and Souček (see the second volume of [12] for an overview and further references) is part of a more general program of applying Geometric Measure Theory to a wide class of problems in the Calculus of Variations. In our case the main idea of this program is to consider graphs instead of functions, hence to relax functionals in the framework of "Cartesian Currents" (which can be thought of as generalized graphs). The invertibility conditions can be translated into suitable properties of the graphs as currents and then existence can be obtained within the framework of generalized graphs using the machinery of Federer and Fleming.

On the other hand, works based on explicit exhibitions of function spaces closed under weak topologies can attack problems like showing existence of minimizers of

$$E^p(u) := \int_\Omega |\nabla u|^p + \varphi(\det \nabla u) \tag{2}$$

for p strictly bigger than 2, but the proofs fail when $p = 2$. The key problem in the latter case is to exhibit an appropriate weak notion of invertibility which is closed under the weak $W^{1,2}$ topology when one controls $\det \nabla u$. For $p > 3$ this was addressed by Ciarlet and Nečas [7].

In this paper we focus our attention on the work of Müller and Spector [15] (inspired by previous ideas of Šverák [17]) and on those of Giaquinta, Modica and Souček [12]. The main idea of Müller and Spector is to give a condition of invertibility which strongly relies on topological arguments (called condition INV by the authors). This theory is tailored to energies of the form E^p with $p > 2$ and deals with the more general problem of treating cavitations. Among the maps which satisfy condition INV the authors shows the existence of a minimum of $E^p + \lambda\,\mathrm{Per}(u(\Omega))$, where Per denotes the perimeter and $u(\Omega)$ is the image of u in the sense of Geometric Measure Theory (see Definition 2.7). Anyaway, for the problem at hand we can use their results in the following way. After choosing a smooth diffeomorphism $g : \Omega \to \Omega' \subset \mathbb{R}^3$ one defines the class $\mathcal{A}_{nc} \subset W^{1,p}(\Omega, \mathbb{R}^3)$ as the maps $u \in W^{1,p}(\Omega, \mathbb{R}^3)$ such that

$$u \text{ satisfies condition INV} \qquad u|_{\partial\Omega} = g \qquad \mathrm{Per}(u(\Omega)) = \mathrm{Per}(g(\Omega))\,.$$

This class contains all the smooth diffeomorphisms of Ω into Ω' which coincide with g at the boundary and the results of [15] can be used to prove that E^p has a minimizer in $\mathcal{A}_{nc}$.

Our first result is that the point of view of Müller and Spector and the one of Giaquinta, Modica and Souček are closely related, as already suggested by the first authors in their work. Namely, the admissible classes of functions given in [15] can be described as the classes of those functions whose graph is a rectifiable current with some precise properties (see Theorem 5.1). In particular the class $\mathcal{A}_{nc}$ corresponds to those maps u which are injective almost everywehere and whose graphs are currents with no boundary in $\Omega \times \mathbb{R}^3$. Hence for functionals like (2) when $p > 2$ the closedness of the classes considered by Müller and Spector can be seen as a byproduct of the closedness of the respective class of rectifiable currents.

What can be said when $p = 2$? In this case we can give a definition of admissible maps which is the strict analog of that of Müller and Spector in the case $p > 2$. Again we can characterize this class of functions in terms of their graphs. However neither the techniques of Müller and Spector, nor the use of Geometric Measure Theory can prove the closedness of the class under the topology induced by the functional. In Section 6 of this work we prove that this is not merely a technical problem: we exhibit a sequence of functions satisfying all the conditions given by Müller and Spector, which are equibounded in energy and which converge weakly to a function which is not in their class. The limiting map appears to be very pathological from the point of view of elasticity: this pathology could be interpreted in physical terms as “interpenetration of matter”. Moreover we underline the fact that the sequence of functions exhibited consists of orientation-preserving Bilipschitz maps. Since every reasonable class of admissible deformations has to include Bilipschitz maps, this means that if one wants to use the direct methods of Calculus of Variations one has to admit our pathology when building an existence theory.

From the point of view of Cartesian Currents an interesting consequence is that new types of singularities need to be involved in the relaxation procedure. These singularities are different from the ones due to “cavitation” (i.e. the opening of holes in some points). This point is quite delicate and we shall discuss it in the final section.

The paper is organized as follows. In Section 2 we give some preliminary definitions and notations. In Section 3 we introduce the definition of condition INV, extended to maps in $W^{1,2} \cap L^\infty$, and we prove some properties of the admissible maps. Focussing on maps in $W^{1,2} \cap L^\infty$ is not a severe restriction in view of the applications we have in mind. Indeed if $\mathcal{A}$ is a natural extension of the class of smooth diffemorphisms of Ω taking certain boundary conditions g, we expect that every map $u \in \mathcal{A}$ takes values in $g(\Omega)$. Indeed both the $\mathcal{A}$ which can be constructed using Cartesian Currents and Müller–Spector theory satisfy this expectation.

In Section 4 we describe the distributional determinant of the admissible maps using the properties proved in Section 3. This description will be crucial for proving in Section 5 the characterization of the classes of functions in terms of properties of their graphs. In Section 6 we exhibit the sequence of Bilipschitz maps with the "bad" behavior and in Section 7 we make some remarks on the consequences of such a behavior.

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2. – Preliminaries

In this section we establish the basic notation and we give some definitions which will be used throughout the paper. First of all, using the notation of [15] we introduce

DEFINITION 2.1 (Cofactor matrix). Given an $n \times n$ matrix we call $\Lambda_{n-1}A$ the matrix of cofactors of A (which obeys $(\Lambda_{n-1}A) \cdot A^T = (\det A) \cdot \mathrm{Id}$).

We observe that $\Lambda_{n-1}A$ is the transpose of the adjoint matrix to A, and that for gradient fields it is divergence-free, i.e.

$$\operatorname{div} \Lambda_{n-1}\nabla u = \partial_i(\Lambda_{n-1}\nabla u)_{ji} = 0 \qquad \text{distributionally}$$

for every $u \in W^{1,n-1}$. In the following B will denote an open ball in $\mathbb{R}^k$, $B(x,r)$ the open ball of radius r centered on x and Ω a bounded open set. Given an open set A, ∂A will be its topological boundary, $\overline{A}$ its topological closure and $\mathbf{1}_A$ its characteristic function. Moreover we denote by $|A|$ the Lebesgue measure of any measurable set A and we use the notation $\mathcal{H}^k$ for the k-dimensional Hausdorff measure. We call the density of A in $x \in \mathbb{R}^n$ the limit

$$\lim_{\rho\to 0} \frac{|B(x,\rho) \cap A|}{|B(x,\rho)|}$$

whenever it exists and we denote this number by $D(A,x)$. Given a Radon measure μ and a Borel set A we define the Radon measure $\mu \llcorner A$ by $\mu \llcorner A(C) := \mu(C \cap A)$.

Next we give a brief list of concepts and more technical objects of Geometric Measure Theory. For proofs and for a more detailed exposition we refer to Federer [10] and to the first volume of Giaquinta, Modica and Souček [12].

We say that A is a *Caccioppoli set* if its characteristic function is a function of bounded variation. Moreover we denote by $\mathrm{Per}(A)$ the total variation of the distributional derivative of $\mathbf{1}_A$. The following is a well-known theorem

THEOREM 2.2 (Reduced boundary). *Let $A \subset \mathbb{R}^n$ be a Caccioppoli set. Then there exists a rectifiable $n-1$ dimensional set (which is called* reduced boundary *and is denoted by $\partial^* A$) such that*

(i) *for every $x \in \partial^* A$ the sets defined by $(A-x)/\rho$ converge locally in measure to a half space when $\rho \downarrow 0$ (hence $D(A, x) = 1/2$);*
(ii) *for $\mathcal{H}^{n-1}$ a.e. $x \in \mathbb{R}^n \setminus \partial^* A$ the density of A in x is either 1 or 0;*
(iii) *the distributional derivative $D\mathbf{1}_A$ is equal to $\nu\mathcal{H}^{n-1} \llcorner \partial^* A$, where ν is a unit vector normal to $\partial^* A$ (which is called* outward normal).

DEFINITION 2.3 (a.e. injectivity). A measurable function $f : \Omega \to \mathbb{R}^n$ is called *a.e. injective* if there is a measurable set $A \subset \Omega$ such that $|\Omega \setminus A| = 0$ and $f|_A$ is injective.

Now let us fix a Sobolev map $u \in W^{1,p}(\Omega)$. Using standard arguments involving maximal functions one can show that for every ε there is a closed set A such that $|\Omega \setminus A| \leq \varepsilon$ and u is Lipschitz on A. At this point using Rademacher's theorem and Whitney's theorem, we can show that for every ε there is a closed set A such that $|\Omega \setminus A| \leq \varepsilon$ and $u|_A$ is C^1. Hence we define

DEFINITION 2.4 (Approximate differentiability). Let u be a measurable function with domain Ω. We say that u is approximately differentiable in $x \in \Omega$ if there exists a closed set $A \subset \Omega$ such that $D(A, x) = 1$ and the restriction of u to A is a C^1 function. Moreover we call the differential of $u|_A$ approximate differential of u in x, and denote it by $apDu(x)$. Finally we will denote by Ω_d the set of points where u is approximately differentiable.

REMARK 2.5. The previous definition (though it is sometimes used in the literature) is slightly weaker than the classical one of Federer's book. Indeed, using Theorem 3.1.8 in [10], Rademacher's theorem and Whitney's theorem one can see that if u is approximately differentiable in A according to Federer's definition, then it is approximately differentiable a.e. in A according to Definition 2.4. This is enough for our purposes and we believe that this choice makes theorems and proofs easier.

REMARK 2.6. If u is a function in some Sobolev space and we select a pointwise representative for ∇u, then this function coincides a.e. with $apDu$. Hence we will use both notations without distinction.

DEFINITION 2.7 (Geometrical image). Let $u : \Omega \to \mathbb{R}^n$ be a function which is approximately differentiable almost everywhere. Given a set $A \subset \Omega$ we call *geometrical image of A through u* the set given by $u(\Omega_d \cap A)$, and we denote it by $\mathrm{im}_{\mathrm{G}}(u, A)$.

The following theorem gives a version of the classical area formula for approximately differentiable functions.

THEOREM 2.8 (Area formula). *Let us suppose that u maps Ω into $\mathbb{R}^n$ and it is injective on a measurable set $A \subset \Omega$. Then*

$$|\mathrm{im}_G(u, A)| = \int_{A\cap\Omega_d} |\det(apDu(x))|dx .$$

Moreover we will use the following technical lemma, which is easily checked for functions which are restrictions of C^1 functions, and can be immediately extended to approximately differentiable functions using Definition 2.4.

LEMMA 2.9. *Suppose that u maps Ω into $\mathbb{R}^n$ and that $x \in \Omega_d$ is such that* $\det(apDu(x)) > 0$, *and let A be a measurable subset of Ω. Then, the following holds*:

(i) *if* $D(A, x) = 1$ *then* $D(\mathrm{im}_G(u, A), u(x)) = 1$;
(ii) *if* $D(A \cap \{(x' - x) \cdot \nu \geq 0\}, x) = 1/2$, *then*

$$D(u(A) \cap \{(y - u(x)) \cdot \Lambda_2(u)\nu \geq 0\}, u(x)) = \frac{1}{2} .$$

A k-dimensional current T in $\mathbb{R}^N$ is defined as a linear functional on the space of C^∞ k-dimensional differential forms in $\mathbb{R}^N$. For the boundary, the product and all the standard operations on currents we adopt the usual definitions.

A k-dimensional manifold $M \subset \mathbb{R}^N$ with a given orientation can be associated to the current given by the integration of forms on M. The same can be said for a Lipschitz k-dimensional manifold (since it possesses a tangent plane in $\mathcal{H}^k$-a.e. point) and for every measurable subset of it, with Borel-measurable orientation.

We can associate a current to any approximately differentiable function $u : \Omega \to \mathbb{R}^n$ whose minors are integrable. First we choose an orthonormal basis $e_1, \ldots e_k$ for $\mathbb{R}^k \supset \Omega$. Then we split Ω (possibly neglecting a subset of Lebesgue measure zero) into a countable union of closed sets F_i such that $u|_{F_i}$ is a C^1 function. We regard $G_i := \{(x, u(x)) : x \in F_i\}$ as a closed subset of a C^1 manifold and we choose for every $x \in F_i$ the k-tuples of vectors

$$\{(e_1, \nabla u(x) \cdot e_1), \ldots , (e_k, \nabla u(x) \cdot e_k)\}$$

as orientation of the tangent plane to G_i in $(x, u(x))$. To every G_i we associate the current T_i induced by this orientation. At this point one is tempted to associate $\sum_i T_i$ to u. In general this is not possible because $\sum_i T_i(\omega)$ could be a divergent series for some form $\omega \in C_c^\infty$. However, when the minors of $apDu$ are L^1 functions, $\sum_i T_i$ actually defines a current (see below) and we denote it by G_u.

DEFINITION 2.10 (Minors). Suppose that L is an $n \times k$ matrix. If α, β are two k-tuples of indices $\alpha_1, \ldots , \alpha_k$, $\beta_1, \ldots , \beta_k$ then we call $G_\alpha^\beta(L)$ the matrix given by the intersection of rows $\beta_1, \ldots, \beta_k$ and columns $\alpha_1, \ldots, \alpha_k$ of L. Moreover we call $M^\beta_\alpha(L)$ the determinant of $G^\beta_\alpha(L)$.

Let now $u : \mathbb{R}^k_x \supset \Omega \to \mathbb{R}^n_y$ be a C^1 function and fix two systems of coordinates for $\mathbb{R}^k_x$ and $\mathbb{R}^n_y$, namely $x_1, \ldots, x_k$ and $y_1, \ldots y_n$. We then identify ∇u with its matrix representation in these systems. Finally let us fix a k-dimensional form $f(x, y)dx_{\alpha_1} \wedge \ldots \wedge dx_{\alpha_j} \wedge dy_{\beta_1} \wedge \ldots \wedge dy_{\beta_i}$, with $i + j = k$ (in the following we will use the shorthand $f dx_\alpha \wedge dy_\beta$). For any $\overline{\alpha} = (\overline{\alpha}_1, \ldots, \overline{\alpha}_i)$ such that $(\overline{\alpha}, \alpha)$ is an even permutation of $\{1, \ldots, k\}$ one can show that

$$G_u(f dx_\alpha \wedge dy_\beta) = \int_\Omega f(x, u(x)) M^\beta_{\overline{\alpha}}(\nabla u(x))dx \,. \tag{3}$$

With standard techniques analogous formulas can be written when u is merely an approximately differentiable function such that $M^\beta_{\overline{\alpha}}(\nabla u(x))$ is absolutely integrable. The product structure of $\mathbb{R}^k_x \times \mathbb{R}^n_y$ gives a natural splitting of any differential form ω, which induces a splitting on currents by duality.

Definition 2.11 (Splitting of currents). Let us write the differential form ω as $\sum_{\alpha,\beta} f_{\alpha\beta} dx_\alpha \wedge dy_\beta$. Then for every integer h we define

$$\omega^{(h)} := \sum_{\alpha,\,\beta \text{ s.t. the length of } \beta \text{ is } h} f_{\alpha\beta} dx_\alpha \wedge dy_\beta \,. \tag{4}$$

Given a current T in $\mathbb{R}^k_x \times \mathbb{R}^n_y$ we define $(T)_h$ by

$$(T)_h(\omega) := T(\omega^{(h)}) \,.$$

We notice that if u maps $\mathbb{R}^k$ into $\mathbb{R}^k$ then G_u is k-dimensional current. Hence ∂G_u is a $k - 1$ dimensional current and $(\partial G_u)_h$ is a $k - 1$-dimensional current which is different from zero only on forms of type (4). A typical example of a current which behaves like $(\partial G_u)_h$ is given by the product of a $(k-1-h)$-dimensional surface $M_x \subset \mathbb{R}^k_x$ with an h-dimensional surface $N_y \subset \mathbb{R}^k_y$. Of course when u is smooth then $(\partial G_u)_h = 0$ for every h. Intuitively, when $\partial G_u \not\equiv 0$ some holes or some fractures occur in the graph of u: the currents $(\partial G_u)_h$ measure in a certain sense the degree of verticality of such holes.

When $u \in W^{1,p}(\mathbb{R}^k, \mathbb{R}^k)$ we have that $(\partial G_u)_h = 0$ for every $h \leq (p - 1)$. This can be proved by approximating u strongly in $W^{1,p}$ with C^∞ functions u_n and observing that $(\partial G_{u_n})_h$ converges weakly to $(\partial G_u)_h$ (since in the computations only products of up to h derivatives of u_n are involved). We refer to pages 238–247 of the first volume of [12] for a thorough discussion.

3. – Condition INV

In this section we recall the definition of condition INV given by Müller and Spector in [15] following ideas of Šverák in [17] and we extend it to the case of functions in $W^{1,2} \cap L^\infty$. Moreover we will follow the proofs contained in [15] of some properties which we will use for the "graph" characterization of the next section. We start with the integral definition of the degree for $W^{1,2}$ functions, which is nowadays quite well known in the literature (compare with [6], see also the book [11]).

Definition 3.1 (Degree for maps in $W^{1,2} \cap L^\infty$). Let us suppose that $u \in W^{1,2}(\partial B, \mathbb{R}^3) \cap L^\infty(\partial B, \mathbb{R}^3)$. Then we define $\deg(u, \partial B, \cdot)$ as the only L^1 function which satisfies the identity

$$\int_{\mathbb{R}^3} \deg(u, \partial B, y) \operatorname{div} g(y) dy := \int_{\partial B} (g \circ u) \cdot \Lambda_2 Du \cdot \nu \, d\mathcal{H}^2 , \tag{5}$$

for every C^∞ vector field g (ν denotes the outer unit normal to B).

For the sake of simplicity in the following we will use the notation $\Lambda_2(u)$ for $\Lambda_2 Du$ and we will suppose that the target space of a function is $\mathbb{R}^3$ when not specified.

Remark 3.2. Let $u \in C^\infty(\partial B, \mathbb{R}^3)$, extend it to a smooth map $v : B \to \mathbb{R}^3$ and suppose that y is a regular value of v. Then the classical definition of degree for $v : B \to \mathbb{R}^3$ is

$$\deg(v, B, y) = \sum_{x \in v^{-1}(y) \cap B} \operatorname{sgn} \det Dv(x) . \tag{6}$$

This definition is then extended to any $y \notin v(\partial B)$ using the invariance of the RHS of (6) under homotopies. The same invariance can be used to show that $\deg(v, B, \cdot)$ is independent of the extension v and depends only on u. Using (6), Sard's theorem and the identity

$$\operatorname{div}((g \circ u)\Lambda_2(u)) = \det \nabla u \ (\operatorname{div} g) \circ u$$

we can integrate by parts to get

$$\int_{\mathbb{R}^3} \deg(v, B, y) \operatorname{div} g(y) dy = \int_{\partial B} (g \circ u) \cdot \Lambda_2(u) \cdot \nu d\mathcal{H}^2 . \tag{7}$$

This proves that the Definition (5) agrees with the classical one (6) when u is smooth.

Remark 3.3. We now show that $\deg(u, \partial B, \cdot)$ is well defined. In view of the previous remark, $\deg(u, \partial B, \cdot)$ is well defined if u is smooth. In order to prove the same for general u's it is sufficient to show that if $\operatorname{div} g = \operatorname{div} h$ then the RHS of equation (5) is the same for g and h. This is equivalent to proving that the RHS vanishes if $\operatorname{div} g = 0$. In order to do that take a sequence of functions u_n in $C^\infty(B)$ which are equibounded and converge to u strongly in $W^{1,2}(\partial B)$. Since $\Lambda_2(u_n) \to \Lambda_2(u)$ strongly in L^1, and $g \circ u_n \overset{*}{\rightharpoonup} g \circ u$ in L^∞,

$$\lim_{n\to\infty} \int (g \circ u_n) \cdot \Lambda_2(u_n) \cdot \nu \, dy = \int (g \circ u) \cdot \Lambda_2(u) \cdot \nu \, dy \,. \tag{8}$$

The left hand side vanishes thanks to (7) and this yields the desired property.

Note that if u in the previous remark were continuous and $u_n \to u$ uniformly the pointwise convergence of $\deg(u_n, \partial B, \cdot)$ could be proved classically using the invariance of degree under homotopies: this is the way $\deg(v, B, \cdot)$ is defined classically for continuous v's. When $u \in W^{1,p}(\partial B)$ and $p > 2$ the Sobolev embedding yields that u is continuous. Thus we can extend it to a continuous map $v : B \to \mathbb{R}^3$ and define $\deg(u, \partial B, \cdot) = \deg(v, B, \cdot)$. This is the definition used by Müller and Spector and, in view of the previous discussion, coincides with our when $u \in W^{1,p}$, $p > 2$.

PROPOSITION 3.4 (Basic properties of deg). deg *is an integer-valued function of bounded variation. Moreover for a.e. ball B we have*

$$\int \deg(u, \partial B, y) \operatorname{div} g(y) dy = \int_{u(\partial B \cap \Omega_d)} g(y) \cdot \tilde{\nu}(y) \, d\mathcal{H}^2(y) \,, \tag{9}$$

where

$$\tilde{\nu}(y) = \sum_{x \in u^{-1}(y) \cap \partial B \cap \Omega_d} \frac{\Lambda_2(u)(x) \cdot \nu(x)}{|\Lambda_2(u)(x) \cdot \nu(x)|}$$

($\nu(x)$ *is the outer unit normal to* B).

PROOF. From the definition it follows that

$$\left| \int \deg(u, \partial B, y) \operatorname{div} g(y) dy \right| \leq \|\nabla u\|^2_{L^2(\partial B)} \|g\|_\infty \,. \tag{10}$$

To prove the first statement take a sequence of functions u_n in $C^\infty(B)$ which are equibounded and converge to u in $W^{1,2}(\partial B)$. Then, from Remark 3.2 there is a large ball B' such that $\deg(u_n, \partial B, y) = 0$ if $y \in \mathbb{R}^3 \setminus B'$. Moreover, by equation (10) and Remark 3.2, we get that the total variation of $\nabla \deg(u_n)$ is equibounded. Poincaré inequality applied to the ball B' yields uniform control of the L^1 norm of $\deg(u_n)$. Then, by the compactness theorem for functions of bounded variation there is $\phi \in BV(\mathbb{R}^3, \mathbb{Z})$ such that

$$\deg(u_n, \partial B, y) \to \phi$$

strongly in L^1. Further, equation (8) shows that $\deg(u, \partial B, y) = \phi(y)$ in the sense of distributions, hence it is a BV function and it is integer-valued. Equation (9) is the area formula from Corollary 3.2.20 of [10]. □

One easy corollary of the previous statement is that for every $u \in W^{1,2}(\partial B) \cap L^\infty(\partial B)$, the set

$$A_{u,B} := \{y \,|\, \deg(u, \partial B, y) \neq 0\} \tag{11}$$

is a Caccioppoli set.

Definition 3.5 (Topological image). For $u \in W^{1,2}(\partial B) \cap L^\infty(\partial B)$, the topological image of B under u, $\mathrm{im_T}(u, B)$, is the set of points where the density of $A_{u,B}$ is 1.

The topological image, defined here as in [17] and [15], can be seen as the set "enclosed" by $u(\partial B)$. Indeed when $u \in W^{1,p}(\mathbb{R}^3, \mathbb{R}^3)$, $p > 2$ and $x \in \mathbb{R}^3$, then $u|_{\partial B(x,r)}$ is continuous for almost every radius $r > 0$. Thus the degree of Definition 3.1 coincides with the classical one and the topological image of u is a bounded open set of $\mathbb{R}^3$ whose boundary is contained in $u(\partial B(x, r))$. If $u|_{\partial B}$ is also injective, then the topogical image is the bounded connected component of $\mathbb{R}^3 \setminus u(\partial B)$. Hence it is very natural to request that maps allowed in elasticity map the interior of balls inside the topological image of the respective spheres: this is what Müller and Spector call *condition* INV.

Definition 3.6 (INV). We say that $u \in W^{1,2}(\Omega) \cap L^\infty(\Omega)$ satisfies property INV_L in the ball $B(a, r) \subset \Omega$ if

(i) its trace on ∂B is in $W^{1,2} \cap L^\infty$;
(ii) for a.e. $x \in B(a, r)$, $u(x) \in \mathrm{im_T}(u, B(a, r))$;
(iii) for a.e. $x \in \Omega \setminus B(a, r)$, $u(x) \notin \mathrm{im_T}(u, B(a, r))$.

We say that $u \in W^{1,2}(\Omega) \cap L^\infty$ satisfies property INV if for every $a \in \Omega$ there is $r_a > 0$ such that for $\mathcal{L}^1$-a.e. $r \in (0, r_a)$ property INV_L holds in $B(a, r)$.

In the next lemmas we will follow the proofs of Müller and Spector (based on the work of Šverák) with slight modifications which allow us to include the case $u \in W^{1,2} \cap L^\infty$. The propositions will yield the positivity of the distributional determinant of the maps which satisfy condition INV (this description is given in the next section). This property and the invariance of condition INV under orientation-preserving diffeomorphisms of the target space will be the main ingredients in the characterization of Section 5.

Remark 3.7. In the following we will consider only maps such that $\det Du > 0$ a.e. For such maps, we denote by Ω_d the set where u is approximately differentiable and $\det apDu > 0$. This affects also Definition 2.7 (geometrical image).

The next two lemmas hold (with the same proof) for functions u such that $\det Du \neq 0$ a.e., for uniformity we state them with the stronger assumption $\det Du > 0$ a.e.

Lemma 3.8. *Let* $u \in W^{1,2}(\Omega) \cap L^\infty$, *with* $\det Du > 0$ *a.e., and choose* $B \subset \Omega$ *such that condition* INV_L *holds. Then* $\mathrm{im_G}(u, B) \subset \mathrm{im_T}(u, B)$, *and* $\mathrm{im_G}(u, \mathbb{R}^3 \setminus B) \subset \mathbb{R}^3 \setminus \mathrm{im_T}(u, B)$.

PROOF. Let $A = \{x \in B \cap \Omega_d : u(x) \in \text{im}_T(u, B)\}$. By condition INV_L, $B \setminus A$ is a null set, and for any $x \in B \cap \Omega_d$ we get $D(A, x) = 1$. By Lemma 2.9 $D(u(A), u(x)) = 1$, and since $u(A) \subset \text{im}_T(u, B)$ by definition, $D(\text{im}_T(u, B), u(x)) = 1$. Hence $u(x) \in \text{im}_T(u, B)$. The converse is proved analogously. □

This lemma easily implies

LEMMA 3.9. *Let $u \in W^{1,2}(\Omega) \cap L^\infty$. Suppose that condition* INV *holds and that* $\det Du > 0$ *a.e. Then, $u|_{\Omega_d}$ is injective.*

PROOF. Fix x, $y \in \Omega_d$. By condition INV we can choose $r > 0$ such that the two balls $B(x, r)$ and $B(y, r)$ are disjoint, contained in Ω and satisfy condition INV_L. Therefore Lemma 3.8 yields $u(x) \in \text{im}_G(u, B(x, r)) \subset \text{im}_T(u, B(x, r))$ and $u(y) \in \text{im}_G(u, B(y, r)) \subset \mathbb{R}^3 \setminus \text{im}_T(u, B(x, r))$. □

LEMMA 3.10. *Suppose that $u \in W^{1,2}(\Omega) \cap L^\infty$ satisfies condition* INV *and that* $\det Du > 0$ *a.e. Let $z \in \Omega$. Then for a.e. $r \in (0, \text{dist}(z, \partial\Omega))$ there exists a bounded Caccioppoli C such that*

$$\partial^* C = u(\Omega_d \cap \partial B(z, r)) \quad \textit{up to } \mathcal{H}^2\textit{–null sets} \tag{12}$$

$$\deg(u, \partial B(z, r)) = \mathbf{1}_C\,. \tag{13}$$

PROOF. Fix a ball $B = B(z, r)$ such that

(i) condition INV_L is satisfied for B;
(ii) $\mathcal{H}^2((\Omega \setminus \Omega_d) \cap \partial B) = 0$.

We observe that if $z \in \Omega$ then $B(z, r)$ satifsies (i) and (ii) for $\mathcal{L}^1$–a.e. $r \in (0, \text{dist}(z, \partial\Omega))$. Moreover, for B satisfying (i) and (ii) and for $\mathcal{H}^2$–a.e. $x \in \partial B$, the approximate differential of $u|_{\partial B}$ at x coincides with the restriction of $apDu(x)$ to the tangent plane to ∂B. From Proposition 3.4 we have

$$\int \deg(u, \partial B, y) \operatorname{div} g(y) dy = \int_{u(\Omega_d \cap \partial B)} g \cdot \tilde{\nu}\, d\mathcal{H}^2\,, \tag{14}$$

where

$$\tilde{\nu}(y) = \sum_{x \in u^{-1}(y) \cap \partial B \cap \Omega_d} \frac{\Lambda_2(u)(x) \cdot \nu(x)}{|\Lambda_2(u)(x) \cdot \nu(x)|}\,. \tag{15}$$

Thanks to Lemma 3.8 u is injective on Ω_d and thus by (ii) $u^{-1}(y) \cap \partial B \cap \Omega_d$ consists of a single point for $\mathcal{H}^2$–a.e. $y \in u(\partial B \cap \Omega_d)$. Hence $|\tilde{\nu}| = 1$. This fact combined with the theory of BV functions (see for example [2]) gives that

(i) $\deg(u, \partial B, \cdot)$ is approximately continuous $\mathcal{H}^2$–a.e. on $\mathbb{R}^3 \setminus u(\partial B \cap \Omega_d)$;
(ii) $\tilde{\nu}$ is perpendicular to $u(\partial B \cap \Omega_d)$ $\mathcal{H}^2$–a.e.;

(iii) in $\mathcal{H}^2$–a.e. $x \in u(\partial B \cap \Omega_d)$ $\deg(u, \partial B, \cdot)$ has a left and right trace d^+ and d^- with respect to $\tilde{\nu}$;
(iv) $d^+ - d^- = 1$ $\mathcal{H}^2$–a.e. on $u(\partial B \cap \Omega_d)$.

Consider the sets

$$\begin{aligned} U_k &:= \{y : \deg(u, \partial B, y) \geq k\} \qquad \text{for integers } k > 0 \\ U_k &:= \{y : \deg(u, \partial B, y) \leq k\} \qquad \text{for integers } k < 0\,. \end{aligned}$$

Note that every U_k is a Caccioppoli set and it is bounded. We now use (i)-(iv) to prove that $|U_k| = 0$ for $k \neq 1$. Indeed assume $|U_k| > 0$. Then $\mathcal{H}^2(\partial^* U_k) > 0$. If $x \in \partial^* U_k$ then (since deg is integer valued) x cannot be a point of approximate continuity. Thus in view of (i) $\partial^* U_k \subset u(\partial B \cap \Omega_d)$, up to $\mathcal{H}^2$–negligible sets. Hence there exists $x \in \partial B \cap \Omega_d \cap u^{-1}(\partial^* U_k)$ such that $u(x)$ satisfies (ii), (iii) and (iv). Let $A = \Omega_d \setminus B$. By point (ii) of Lemma 2.9 we get that

$$u(A) \cap \{y \in \mathbb{R}^3 : (y - u(x)) \cdot \tilde{\nu}(u(x)) \leq 0\}$$

has density 1/2 at $u(x)$. Moreover by condition INV_L, $\deg(u, \partial B, \cdot)$ vanishes on $u(A)$. Thus the "left trace" $d^-(u(x))$ is zero, and by (iv) the right trace is one. Hence $k = 1$.

Since $|U_k| = 0$ for $k \neq 1$, the range of deg is contained in $\{0, 1\}$ and hence $\deg = \mathbf{1}_{U_1}$. The set $C := U_1$ then satisfies also (12) by (14). □

DEFINITION 3.11 (Good radii). Given $a \in \Omega$, we call R_a the set of $r > 0$ such that $B(a, r) \subset \Omega$,

(i) condition INV_L is satisfied for $B(a, r)$;
(ii) $\mathcal{H}^2(\partial B(a, r) \setminus \Omega_d) = 0$.

LEMMA 3.12. *Let* $u \in W^{1,2}(\Omega) \cap L^\infty$. *Suppose that condition* INV *holds and that* $\det Du > 0$ *a.e. Then, for any* $a, b \in \Omega$ *and any* $r \in R_a$, $s \in R_b$,

(i) $\mathrm{im_T}(u, B(a, r)) \cap \mathrm{im_T}(u, B(b, s)) = \emptyset$ *if* $B(a, r) \cap B(b, s) = \emptyset$;
(ii) $\mathrm{im_T}(u, B(a, r)) \subset \mathrm{im_T}(u, B(b, s))$ *whenever* $B(a, r) \subset B(a, s)$.

PROOF. Let us first prove (i). Let $A = \mathrm{im_T}(u, B(a, r))$, and $C = \mathbb{R}^3 \setminus \mathrm{im_T}(u, B(b, s))$. Thanks to Lemma 3.10, except for an $\mathcal{H}^2$ null set, $\partial^* C$ equals $u(\Omega_d \cap \partial B(b, s))$. Take $x \in \Omega_d \cap \partial B(b, s)$. Then, $x \notin B(a, r)$, and by Lemma 3.8, $u(x) \notin A = \mathrm{im_T}(u, B(a, r))$. Thus the essential boundary of C is contained in $\mathbb{R}^3 \setminus A$. Analogously we can prove that the essential boundary of A does not intersect C. Thus if A and C were open sets, an elementary topological argument would prove that $A \subset C$. The technical details needed to prove the same result for general Caccioppoli sets is given in Lemma A.1 of the Appendix. Point (ii) is obtained analogously. □

Before concluding this section, we define $F(x)$ as the "topological image of a point x". More precisely,

Definition 3.13 (Topological image of a point). We define

$$F(x) := \bigcap_{r \in R_x} \mathrm{im}_{\mathrm{T}}(u, B(x,r)) . \tag{16}$$

By Lemma 3.9, $u(x) \in F(x)$ for every $x \in \Omega_d$.

4. – Distributional determinant

Definition 4.1 (Distributional Det). Let $u \in W^{1,2}(\Omega) \cap L^\infty(\Omega)$. We define $\operatorname{Det} Du$ as the divergence in the sense of distributions of the vector field given by $u \cdot \Lambda_2(u)/3$.

The goal of this section is the following theorem, which characterizes the distributional determinant of the class of functions considered by Müller and Spector. As in the previous section we follow their proofs with minor adjustments to include the case $p = 2$.

Theorem 4.2. *Let* $u \in W^{1,2}(\Omega) \cap L^\infty$. *Suppose that condition* INV *holds, that* $\det Du > 0$ *a.e., and* $\mathrm{Per}(\mathrm{im}_{\mathrm{G}}(u,\Omega)) < \infty$. *Then,*

$$\operatorname{Det} Du = \det Du + \sum_{x_i \in C_u} \mathcal{L}^3(F(x_i)) \delta_{x_i} \tag{17}$$

where C_u is the set of points x such that $\mathcal{L}^3(F(x)) > 0$. Further,

$$\sum_{x_i} \mathrm{Per}(F(x_i)) \leq \mathrm{Per}(\mathrm{im}_{\mathrm{G}}(u,\Omega)) . \tag{18}$$

Before proving Theorem 4.2 we give some partial results.

Lemma 4.3. *Let* $u \in W^{1,2}(\Omega) \cap L^\infty$. *Suppose that condition* INV *holds, and that* $\det Du > 0$ *a.e. Then,*

(i) $\operatorname{Det} Du \geq 0$, *hence it is a Radon measure*;
(ii) *the absolutely continuous part of* $\operatorname{Det} Du$ *with respect to* $\mathcal{L}^3$ *has density* $\det Du$;
(iii) *for every $a \in \Omega$ and for a.e. $r \in R_a$,*

$$(\operatorname{Det} Du)(B(a,r)) = \mathcal{L}^3(\mathrm{im}_{\mathrm{T}}(u, B(a,r))) . \tag{19}$$

Proof. Take $\phi \in C_0^\infty(B(0,1))$ radially symmetric, nonnegative, monotone in the radial direction and such that $\int \phi = 1$. Define the standard sequence of mollifiers as

$$\phi_\varepsilon(x) := \varepsilon^{-3} \phi\left(\frac{x}{\varepsilon}\right) .$$

First of all we prove positivity of $(\phi_\epsilon * \operatorname{Det} Du)(x)$ for every $x \in \Omega$ such that $\operatorname{dist}(x, \partial\Omega) \geq \varepsilon$. Let $\phi(x) = f(|x|)$, with $f' \leq 0$. Then, from Definition 4.1

$$\begin{aligned}(\phi_\varepsilon * \operatorname{Det} Du)(x) &= -\frac{1}{3} D\phi_\varepsilon * [u \cdot (\Lambda_2(u))](x)\\ &= -\frac{1}{3}\int_{B(x,\varepsilon)} D\phi_\varepsilon(x-z)u(z)(\Lambda_2(u))(z)dz\\ &= -\frac{1}{3}\int_0^\varepsilon \varepsilon^{-4} f'\left(\frac{r}{\varepsilon}\right) dr \int_{\partial B(x,r)} u \cdot (\Lambda_2(u))\nu \, d\mathcal{H}^2 ,\end{aligned}$$

where ν is the outward unit normal to $B(x,r)$. From Definition 3.1 and Lemma 3.10

$$\frac{1}{3}\int_{\partial B(x,r)} u \cdot (\Lambda_2(u))\nu \, d\mathcal{H}^2 = \int \deg(u, \partial B, y)dy = \mathcal{L}^3(\operatorname{im}_{\mathrm{T}}(u, B)) \geq 0 .$$

This concludes the proof of point (i). Point (ii) is a direct consequence of Lemma 4.7 of [9]. By standard arguments on Radon measures,

$$(\operatorname{Det} Du)(B(a,r)) = \sup_{\delta>0} \int \Phi_\delta(|x-a|)\, d \operatorname{Det} Du(x)$$

where

$$\Phi_\delta(s) = \begin{cases} 1 & \text{if } s \leq r - \delta \\ (r-s)/\delta & \text{if } r-\delta \leq s \leq r \\ 0 & \text{if } s \geq r . \end{cases}$$

A computation similar to the one above, gives, for δ sufficiently small,

$$\begin{aligned}\int \Phi_\delta(|x-a|)\, d \operatorname{Det} Du(x) &= \int_{r-\delta}^r \frac{1}{3\delta} ds \int_{\partial B(a,s)} u \cdot \Lambda_2(u) \cdot \nu \, d\mathcal{H}^2\\ &= \int_{r-\delta}^r \frac{1}{\delta}\mathcal{L}^3(\operatorname{im}_{\mathrm{T}}(u, B(a,s)))ds\\ &= \mathcal{L}^3(\operatorname{im}_{\mathrm{T}}(u, B(a,r))\end{aligned}$$

for $\mathcal{L}^1$-a.e. $r \in R_a$. □

REMARK 4.4. It is convenient to redefine the sets R_a excluding the zero-measure part in which equation (19) does not hold.

PROOF OF THEOREM 4.2. Let us split $\operatorname{Det} Du$ into the absolutely continuous part with respect to the Lebesgue measure μ_{ac} and the singular part μ_s. By standard arguments of measure theory we can find a countable collection of Dirac masses δ_{x_i} such that

$$\mu_s = \sum k_i \delta_{x_i} + \mu_c ,$$

where μ_c contains no atoms. From the previous lemma we know that $\mu_{ac} = \det Du$. We have to prove that $\mu_c = 0$, $\{x_i\} = \{x : \mathcal{L}^3(F(x)) > 0\}$, and $k_i = \mathcal{L}^3(F(x_i))$.

Let $A = \mathrm{im}_{\mathrm{G}}(u, \Omega)$. Fix $a \in \Omega$, $r \in R_a$, and let $C = \mathrm{im}_{\mathrm{T}}(u, B(a,r))$. The previous lemma shows that $\mathcal{L}^3(C \setminus A) = \mu_s(B(a,r))$. Now we want to show that

$$\mathcal{L}^3(C \setminus A)^{2/3} \leq c_3 \mathcal{H}^2(\partial^* A \cap C) \tag{20}$$

where $c_3 = (36\pi)^{-1/3}$ is the isoperimetric constant.

The set $C \setminus A$ is a Caccioppoli set, hence the previous equation is proved if we can show that

$$\mathcal{H}^2(\partial^*(C \setminus A)) \leq \mathcal{H}^2(\partial^* A \cap C)\,. \tag{21}$$

Take $y \in \partial^*(C \setminus A)$. Clearly $D(A, y) < 1$ and $D(C, y) > 0$. We now show that $D(C, y) < 1$ only for a $\mathcal{H}^2$-null set. Indeed, if $D(C, y) < 1$ then y belongs to $\partial^* C$, which up to a $\mathcal{H}^2$-null set coincides with $u(\Omega_d \cap \partial B(a, r))$. By Lemma 2.9, $D(A, u(x)) = 1$ for every $x \in \Omega_d$, which contradicts the statement $D(A, y) < 1$. Therefore $D(C, y) = 1$, which in turn implies $D(A, y) = 1/2$, hence $y \in \partial^* A$. By Definition 3.5, $y \in C$, and equation (21) is proved.

Now we use equation (20) to prove that $\mu_c = 0$. By definition we have $\lim_{r\downarrow 0} \mu_c(B(a,r)) = 0$ for every a. Fix $\varepsilon > 0$ and consider the family of closed balls

$$\mathcal{F}_\varepsilon = \{\overline{B(a,r)} : a \in \Omega,\ r \in R_a,\ \mu_c(\partial B(a,r)) = 0,\ \mu_c(\overline{B(a,r)}) < \varepsilon\}\,.$$

By the Besicovitch covering theorem (e.g. Theorem 2.19 of [2]) there is a sequence of pairwise disjoint, closed balls $\overline{B_i} \in \mathcal{F}_\varepsilon$ such that

$$\mu_c(\Omega) = \mu_c\left(\bigcup_{i=1}^{\infty} \overline{B_i}\right) = \sum_{i=1}^{\infty} \mu_c(B_i)\,.$$

By (19) and (20),

$$\sum_{i=1}^{\infty} [\mu_c(B_i)]^{2/3} \leq \sum_{i=1}^{\infty} [\mathcal{L}^3(\mathrm{im}_{\mathrm{T}}(u, B_i) \setminus A)]^{2/3} \leq c_3 \mathcal{H}^2(\partial^* A)\,.$$

We conclude that $\mu_c(\Omega) \leq c_3 \varepsilon^{1/3} \mathcal{H}^2(\partial^* A)$ and since ε was arbitrary, this gives $\mu_c = 0$.

To show that $k_i = \mathcal{L}^3(F(x_i))$, we observe that

$$\begin{aligned} k_i &= \lim_{r \to 0} \mu_s(B(x_i, r)) \\ &= \lim_{r\to 0} \mathcal{L}^3(\mathrm{im}_{\mathrm{T}}(u, B(x_i, r)) \setminus \mathrm{im}_{\mathrm{G}}(u, B(x_i, r)))\,. \end{aligned}$$

Since by the area formula $\mathcal{L}^3(\mathrm{im}_{\mathrm{G}}(u, B(x_i, r)))$ converges to zero, we get

$$k_i = \lim_{r\to 0} \mathcal{L}^3(\mathrm{im}_{\mathrm{T}}(u, B(x_i, r))) = \mathcal{L}^3\left(\bigcap_{r>0} \mathrm{im}_{\mathrm{T}}(u, B(x_i, r))\right) = \mathcal{L}^3\left(F(x_i)\right)$$

where monotonicity and the definition of $F(x)$ have been used. The same arguments shows that $\{x_i\} = \{x : \mathcal{L}^3(F(x)) > 0\}$. $\square$

The following remark will be crucial for the next section.

REMARK 4.5 (Invariance). If we compose u with a diffeomorphism $H : B(0, R) \to \Omega' \subset\subset \mathbb{R}^3$, where $B(0, R) \supset \mathrm{im_G}(u, \Omega)$, then $H \circ u$ satisfies all the hypotheses of the previous theorem, and

$$\mathrm{Det}\, D(H \circ u) = (\det \nabla H) \circ u \det Du \mathcal{L}^3 + \sum_{x_i \in C_u} \delta_{x_i} \int_{F(x_i)} \det \nabla H(y) dy\,.$$

5. – Graphs and currents

The goal of this section is to prove the following theorem.

THEOREM 5.1 (Currents versus INV). *Suppose $u \in W^{1,2}(\Omega) \cap L^\infty$, with* $\det Du > 0$ *a.e. Then the following two conditions are equivalent*:

(i) *u satisfies condition* INV *and* $\mathrm{im_G}(u, B)$ *is a Caccioppoli set in every ball where u satisfies* INV_L;
(ii) *there exists a countable number of bounded Caccioppoli sets F_i and of points $x_i \in \Omega$ such that*

$$\partial G_u = -\sum_i \{x_i\} \times \partial^* F_i$$
$$\partial G_u \text{ has locally finite mass in } \Omega$$
$$u \text{ is injective a.e. and } |\mathrm{im_G}(u, \Omega) \cap F_i| = 0\,.$$

In the previous statement there is a slight abuse of notation: indeed $\{x_i\} \times \partial^* F_i$ is a set and not a current. However, after having fixed an orientation for $\mathbb{R}^3$ we can orient $\partial^* F_i$ in such a way that in every point the orienting couple and the outward unit normal form a triple of vectors oriented as $\{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$. This induces a two dimensional current in $\mathbb{R}^3$ and by mapping $\mathbb{R}^3$ into $\{x_i\} \times \mathbb{R}^3$ it gives a rectifiable 2-dimensional current in $\Omega \times \mathbb{R}^3$. We identify $\{x_i\} \times \partial^* F_i$ with this current.

PROOF OF (i) $\Rightarrow$ (ii). We will prove that if (i) holds then (ii) is true if we take $\{x_i\} = C_u = \{x : \mathcal{L}^3(F(x)) > 0\}$ and $F_i = F(x_i)$. From Remark 3.2.3.3 in [12] (page 245) we have

$$(\partial G_u)_{(k)} \llcorner \Omega \times \mathbb{R}^3 = 0$$

for $k = 0$ and $k = 1$. Hence we have to compute $\langle G_u, d\omega \rangle$ for ω of the form

$$\omega = h_1(x, y) dy_2 \wedge dy_3 + h_2(x, y) dy_3 \wedge dy_1 + h_3(x, y) dy_1 \wedge dy_2$$

where $h \in C_0^\infty(\Omega \times \mathbb{R}^3)$. Namely, we want to show that

$$\langle \partial G_u, \omega \rangle = \langle G_u, d\omega \rangle = -\sum_{x_i \in C_u} \int_{\partial^* F(x_i)} h(x_i, y) \cdot \nu_i \, d\mathcal{H}^2(y) \tag{22}$$

where ν_i is the outer normal to $F(x_i)$. Now, we notice that the vector space generated by functions of the form $\phi(x)g(y)$ is dense in C_0^∞ in the topology induced by C^k seminorms. Hence it is sufficient to prove (22) for

$$\omega = \phi(x)[g_1(y)dy_2 \wedge dy_3 + g_2(y)dy_3 \wedge dy_1 + g_3(y)dy_1 \wedge dy_2]$$

where ϕ has compact support in Ω. We first show that, if $\operatorname{div} g = 0$, then $\langle G_u, d\omega \rangle = 0$. In order to do that, we observe that

$$\langle G_u, d\omega \rangle = \int (\operatorname{div} g) \circ u \, \phi \det Du + \int g \circ u \cdot \Lambda_2(u) \cdot \nabla\phi \, .$$

The first term in the RHS is zero. To show that also the second one vanishes we take a sequence of C^∞ functions u_n converging to u strongly in $W^{1,2}$ and weak-$*$ in L^∞. Then, the same calculation and the fact that the boundary of G_{u_n} is zero gives $\langle G_{u_n}, d\omega \rangle = 0$, and hence

$$\int g \circ u_n \cdot \Lambda_2(u_n) \cdot \nabla\phi = 0 \, .$$

For $n \to \infty$, $\Lambda_2(u_n) \to \Lambda_2(u)$ strongly in L^1, and $g \circ u_n \stackrel{*}{\rightharpoonup} g \circ u$ in L^∞, and so $\langle G_u, d\omega \rangle = 0$ whenever $\operatorname{div} g = 0$.

We now prove (22) in the general case. By linearity, we can assume that $\operatorname{div} g \geq k > 0$. Now, using a result of Dacorogna and Moser [8], we can find a diffeomorphism $H : \Omega \to \mathbb{R}^3$ such that

$$\operatorname{div} g = \det \nabla H = \frac{1}{3} \operatorname{div}(H \cdot \Lambda_2(H)) \, .$$

Let us define the form ν as

$$\begin{aligned} \nu := \frac{1}{3}\phi(x)[&(H \cdot \Lambda_2(H))_1(y)dy_2 \wedge dy_3 \\ &+ (H \cdot \Lambda_2(H))_2(y)dy_3 \wedge dy_1 + (H \cdot \Lambda_2(H))_3(y)dy_1 \wedge dy_2] \, , \end{aligned}$$

where $(H \cdot \Lambda_2(H))_i$ denotes the i-th component of the vector $H \cdot \Lambda_2(H)$. Since we have shown that $\langle G_u, d\omega \rangle$ only depends on the divergence of g, we have $\langle G_u, d\omega \rangle = \langle G_u, d\nu \rangle$. Then, a computation analogous to the previous one gives

$$\langle G_u, d\nu \rangle = \int (\det \nabla H) \circ u \, \phi \det Du + \frac{1}{3} \int ((H \cdot \Lambda_2(H)) \circ u) \cdot \Lambda_2(u) \cdot \nabla\phi \, .$$

We observe that $(H \cdot \Lambda_2(H)) \circ u \cdot \Lambda_2(u) = (H \circ u) \cdot \Lambda_2(H \circ u)$. Then, integrating by parts we get

$$\frac{1}{3} \int ((H \cdot \Lambda_2(H)) \circ u) \cdot \Lambda_2(u) \cdot \nabla \phi = - \int \phi \operatorname{Det} D(H \circ u) \,.$$

Finally, in view of Remark 4.5, we have

$$\begin{aligned} \langle G_u, d\omega \rangle &= - \sum_{x_i \in C_u} \phi(x_i) \int_{F(x_i)} \det \nabla H(y) dy \\ &= - \sum_{x_i \in C_u} \phi(x_i) \int_{F(x_i)} \operatorname{div} g(y) dy \\ &= \sum_{x_i \in C_u} \phi(x_i) \int_{\partial^* F(x_i)} g(y) \cdot \nu_i(y) d\mathcal{H}^2(y) \,. \end{aligned}$$ □

In order to prove the converse implication we first need some definitions.

DEFINITION 5.2. Let $M \subset \Omega \times \mathbb{R}^3_y$ be a 3-dimensional smooth oriented manifold (possibly with boundary). Given a C^∞ open set $A \subset\subset \Omega$, for every point $y \in \mathbb{R}^3$ we set $M_y := (A \times \{y\}) \cap M$. For every $x \in M_y$ we call $T(x)$ the tangent plane to M in x with its orientation and we call $T_p(x)$ the projection of $T(x)$ on $\mathbb{R}^3_y$ with the induced orientation. We set

$$\chi(x) := \begin{cases} 0 & \text{if } \dim(T_p(x)) < 3 \\ 1 & \text{if } \dim(T_p(x)) = 3 \text{ and } T_p(x) \text{ is oriented as } \mathbb{R}^3_y \\ -1 & \text{otherwise}. \end{cases}$$

We define

$$\deg(M, A, y) := \sum_{x \in M_y} \chi(x) \,.$$

Exactly in the same way, using approximate tangent planes, we can define $\deg(T, A, y)$ if T is a rectifiable current.

REMARK 5.3. It is easy to see that the degree is well defined and

$$\int_{M \cap \{A \times \mathbb{R}^3_y\}} g(y) dy_1 \wedge dy_2 \wedge dy_3 = \int_{\mathbb{R}^3} g(y) \deg(M, A, y) dy \,. \tag{23}$$

Hence, if $(\partial A \times \mathbb{R}^3_y) \cap M$ is a 2-dimensional manifold then

$$\begin{aligned} \int_{\mathbb{R}^3} \operatorname{div} h \deg(M, A, y) dy &= \int_{M \cap \{A \times \mathbb{R}^3_y\}} \operatorname{div} h \, dy_1 \wedge dy_2 \wedge dy_3 \\ &= \int_{\partial(A \times \mathbb{R}^3) \cap M} h_1 dy_2 \wedge dy_3 + h_2 dy_3 \wedge dy_1 + h_3 dy_1 \wedge dy_2 \,. \end{aligned}$$

Moreover if A is a ball B and $(A\times\mathbb{R}^3_y)\cap M$ is the trace of a function $u:\Omega\to\mathbb{R}^3$ then this last term is equal to

$$\int_{\partial B} h\circ u\cdot\Lambda_2(u)\cdot\nu\, d\mathcal{H}^2$$

and we can write

$$\int_{\mathbb{R}^3}\operatorname{div} h\deg(M,B,y)dy=\int_{\mathbb{R}^3}\operatorname{div} h\deg(u,\partial B,y)dy\,.$$

Since every smooth function g can be written as the divergence of a vector field we conclude that $\deg(M,B,y)=\deg(u,\partial B,y)$ a.e.

Proof of (ii) $\Rightarrow$ (i) in Theorem 5.1. What happens in the previous remark if we try to replace the manifold M with a rectifiable current T? Equation (23) remains true. Moreover if we fix a point x a classical theorem of Federer (the theorem of slicing 4.2.1 in [10]) says that for $\mathcal{L}^1$ a.e. r such that $\operatorname{dist}(\Omega,x)>r>0$ we have:

(i) the intersection of the rectifiable set which supports T with $\partial B(x,r)\times\mathbb{R}^3$ gives a 2-dimensional rectifiable set S_r;
(ii) $\mathcal{H}^2$ a.e. $y\in S_r$ has an orientation induced naturally by the orientation of the tangent space to $\partial B\times\mathbb{R}^3$ and the one of the approximate tangent space to T;
(iii) the orientation of (ii) on S_r induces a rectifiable current, which we also call S_r with a slight abuse of notation;
(iv) if we take in $\mathbb{R}^3\times\mathbb{R}^3$ the current T_r given by the restriction of T to $B(x,r)\times\mathbb{R}^3$ we have that $\partial T_r=S_r$.

Keeping this in mind we notice that the current given by $T=G_u+\sum_i\{x_i\}\times F_i$ is rectifiable and has no boundary in $\Omega\times\mathbb{R}^n$. The current T is supported on the rectifiable set R given by $u(\Omega_d)\cup\bigcup_i\{x_i\}\times F_i$. Now let us fix an $x\in\Omega$. For $\mathcal{L}^1$-a.e. radius r such that $B(x,r)\subset\Omega$, we have that $(\partial B(x,r)\times\mathbb{R}^3)\cap R$ is given by the rectifiable set $u(\Omega_d\cap\partial B(x,r))$. Hence (for a.e.) r ∂T_r is given by the current induced by the $W^{1,2}\cap L^\infty$ function $u|_{\partial B(x,r)}$.

Reasoning as in Remark 5.3 it is not difficult to check that

$$\deg(T,B(x,r),y)=\deg(u,\partial B(x,r),y)$$

for a.e. y. Hence, since every approximate tangent plane to T is oriented in the same way, it is easy to see that for every point $z\in B(x,r)$, $\deg(u,\partial B(x,r),u(z))$ is positive. Moreover, since for a.e. point $z\notin B(x,r)$ we have $(B(x,r)\times u(z))\cap T=\emptyset$ we can conclude that $\deg(u,\partial B(x,r),u(z))=0$ for a.e. $z\in\Omega\setminus B(x,r)$. This implies that u satisfies condition INV.

Finally, let us fix a ball B such that u satisfies INV_L on it. Of course we have that

$$\operatorname{im}_{\mathrm{G}}(u,B)=\{y|\deg(T,B,y)>0\}\setminus\bigcup_{x_i\in B}F_i\,.$$

Since $\deg(T, B, \cdot) = \deg(u, \partial B, \cdot)$ we have that $\{y | \deg(B, T, y) > 0\} = \mathrm{im_T}(u, B)$. As we have seen in Section 3, the last set is a set of bounded variation. Moreover

$$\sum_{x_i \in B} \mathrm{Per}(F_i)$$

is equal to the mass of ∂G_u in $B \times \mathbb{R}^3$, hence it is finite. This means that $\mathrm{im_G}(u, B)$ is the difference between two Caccioppoli sets and completes the proof. □

6. – Examples

In this section we prove the following

THEOREM 6.1 (Bad sequence). *There is a sequence* $(u_n) \subset W^{1,2}(\Omega, \mathbb{R}^3)$ *such that*:

(i) u_n *is Bilipschitz*;
(ii) *every* u_n *satisfies condition* INV;
(iii) $u_n \rightharpoonup u$ *in* $W^{1,2}$ *and* u *does not satisfy condition* INV;
(iv) $\sup_n \int_\Omega \psi(\det \nabla u_n) < \infty$ *for some convex and superlinear* ψ *approaching infinity at* 0.

The limit u is illustrated in figure 1. The whole construction is cylindrically symmetric with respect to the x-axis, therefore only a section is plotted, and the approximate determinant is positive a.e. The half-ball a, centered in the origin, is mapped into the ball $\mathrm{im_G}(u, a) = B((1/2, 0, 0), 1/2)$. The half-ball e, centered in $(1, 0, 0)$, is mapped into $\mathrm{im_G}(u, e) = B((1, 0, 0), 1) \setminus \mathrm{im_G}(u, a)$. The topological image of e, $\mathrm{im_T}(u, e) = B((1, 0, 0), 1)$, also contains $\mathrm{im_G}(u, a)$. If B is any ball of positive radius centered in $(1, 0, 0)$, then its topological image contains $B((1/2, 0, 0), 1/2)$. If B' is any ball of positive radius centered in the origin, then its geometrical image is a set of positive measure contained in $B((1/2, 0, 0), 1/2)$. If their radii are sufficiently small B and B' are disjoint and hence condition INV is violated by u.

REMARK 6.2. The same example shows that the condition $\mathrm{Det}\, Du \geq 0$ is not preserved under weak convergence for $p = 2$, in three spatial dimensions. Indeed, by (iv) each u_n has nonnegative distributional determinant, whereas

$$\mathrm{Det}\, Du = \det Du + \frac{\pi}{6}\delta_P - \frac{\pi}{6}\delta_O .$$

To see this, we compute the topological images of the two points $O = (0, 0, 0)$ and $P = (1, 0, 0)$. As $\rho \to 0$, the image of a sphere of radius ρ centered in the origin O is composed by the boundaries of two regions, which touch in $(1, 0, 0)$. The first converges to the boundary of $B((1/2, 0, 0), 1/2)$ (i.e. $\partial \mathrm{im_G}(u, a)$), with negative orientation; the second is the boundary of a small subset of $\mathrm{im_G}(u, d)$, which is negligible in the limit. Hence the distributional determinant contains a negative Dirac at O. Analogously, the image of a sphere of radius ρ centered in $P = (1, 0, 0)$ converges to the same sphere $\partial B((1/2, 0, 0), 1/2)$, with positive orientation, hence $\mathrm{Det}\, Du$ contains a positive Dirac at P.

To understand how u can be reached as the limit of a sequence of functions satisfying condition INV, it is instructive to mention the following fact (well known in the literature on harmonic maps as bubbling off).

LEMMA 6.3. *There is a sequence of diffeomorphisms $v_n : S^2 \to S^2$ s.t.*

$$\sup_n \int_{S^2} |\nabla v_n|^2 < \infty$$

and $v_n \rightharpoonup v$ in $W^{1,2}$, where v a constant function.

PROOF. The construction is cylindrically symmetric, and is based on expanding the usual projection of the sphere onto the complex plane. In polar coordinates, this gives

$$(v_\theta, v_\phi)(\theta, \phi) = \left(2 \arctan n \tan \frac{\theta}{2}, \phi\right) .$$

It is clear that $v_\theta \to \pi$ as $n \to \infty$. The L^2 norm of cylindrically invariant functions is given by (see Appendix B)

$$||\nabla v||^2_{L^2(S^2)} = 2\pi \int_{-1}^{1} (\partial_\theta v_\theta)^2 + \left(\frac{\sin v_\theta}{\sin \theta}\right)^2 d\cos\theta$$

and by direct substitution we obtain

$$||\nabla v||^2_{L^2(S^2)} = 4\pi \int_{-1}^{1} \left(\frac{2n}{(n^2+1) + \cos\theta(1-n^2)}\right)^2 d\cos\theta = 8\pi .$$

This concludes the proof. □

Before presenting the actual construction of u_n, we show a simple method for checking the determinant constraint (i.e. condition (iv) in Theorem 6.1). One possible way would of course be to explicitly construct the superlinear function ψ. It is however easier to use the following lemma

LEMMA 6.4. *If the sequence of Bilipschitz maps $u_n : \Omega \to \mathbb{R}^n$ has the following properties*:

(i) *for any δ there exists ε such that, if $\omega \subset \Omega$ is measurable and $|\omega| \le \varepsilon$, then $|u_n(\omega)| < \delta$;*

(ii) *for any δ, there exists ε such that, if $|\omega| \le \varepsilon$, then $|u_n^{-1}(\omega)| < \delta$.*

Then, there is a convex function $\phi : \mathbb{R} \to \mathbb{R}$ *with*

$$\lim_{t\to\infty} \frac{\phi(t)}{t} = \lim_{t\to 0} \phi(t) = \infty \tag{24}$$

such that

$$\sup_n \int_\Omega \phi(\det \nabla u_n) < \infty\,. \tag{25}$$

PROOF. Since every u_n is Bilipschitz we have

$$|u_n(\omega)| = \int_\omega \det \nabla u_n(x) dx\,.$$

Hence the first condition implies the equiintegrability of $\det \nabla u_n$ and using Dunford-Pettis theorem for weakly compact L^1 sequences (see for example [2] for a proof) we conclude that there exists a positive, increasing, convex and superlinear function ψ_1 such that $\int \psi_1(\det \nabla u_n)$ is equibounded. Using condition (ii) in the same way we find an analogous ψ_2 such that $\int \psi_2((\det \nabla u_n)^{-1})$ is equibounded. Since $\psi_2(x)$ is increasing, $\psi_2(x^{-1})$ is still convex. It follows that $\psi_1(x) + \psi_2(1/x)$ gives the desired function. □

We now turn to the relevant construction, which is illustrated in figures 1 and 2. The unwrapping of the sphere mentioned in Lemma 6.3 is used in the central section, denoted by c in figure 2. For finite n, sections of c at constant x are mapped into spheres which are contained between the image of a and the one of e. In the limit, those spheres shrink to a point, the origin.

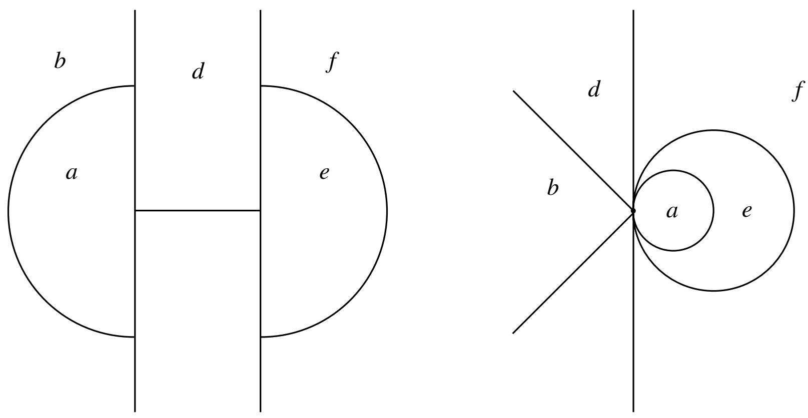

Fig. 1. Representation of the limit function u. Left panel: subdivision of the domain. Right panel: image. The letters denote the images of the various pieces of the domain.

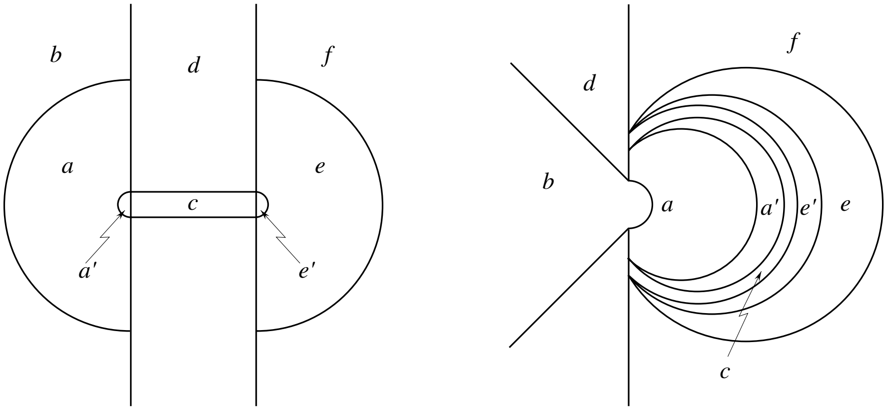

Fig. 2. Representation of a function of the sequence u_n, which converge to the one represented in Figure 1. Left panel: subdivision of the (x, y) plane used in the construction in Theorem 6.1. Right panel: image. The letters denote the images of the various pieces of the domain.

PROOF OF THEOREM 6.1. We work in cylindrical coordinates, and construct the sequence u_n using piecewise smooth functions. We shall consider $\Omega = B(0, 3)$, examples for other domains can be obtained by rescaling and translating the construction. We first construct $v_n(x, y)$, which coincides with the restriction of u_n to the $\{z = 0,\ y > 0\}$ half-plane, then extend it cylindrically,

$$u_n(x, y\cos\phi, y\sin\phi) = v_x^n(x, y)e_x + v_y^n(x, y)(\cos\phi e_y + \sin\phi e_z)\,.$$

The construction of v_n is based on the subdivision of the domain shown in figure 2. In the limit u, the region a, which corresponds to a half-sphere, is mapped into the sphere $u(a) = B(1/2, 1/2)$, and the region e is mapped into $u(e) = B(1, 1) \setminus B(1/2, 1/2)$. The regions b and d are mapped into the $x < 0$ half-space, the region f is mapped into the rest of the $x > 0$ half-space. The remaining regions all disappear in the limit. For finite n, the regions a', c and e' (of total volume of order n^{-2}) ensure that the maps are Bilipschitz. The corresponding images also have small volume, and join continuously $u_n(a)$ with $u_n(e)$.

Let (r, θ) be polar coordinates centered in $(0, 0)$, and $(\bar{r}, \bar{\theta})$ polar coordinates centered in $(1, 0)$. We consider first the inner regions a, a', c, e' and e, and afterwards the outer ones, b, d and f. For simplicity of notation we denote here v_n by v, and use $\varepsilon := 1/n$ as small parameter. We also use the same symbols a, b, etc. to denote both the three-dimensional pieces and their two-dimensional sections.

We start from $a := \{\varepsilon \le r \le 1\,, \pi/2 \le \theta \le \pi\}$, and set

$$v_\theta := \pi - \theta \tag{26}$$

$$v_r := \frac{1-r}{1-\varepsilon}(\cos v_\theta + 2\varepsilon) + \varepsilon\frac{r-\varepsilon}{1-\varepsilon} \tag{27}$$

where v_r and v_θ are polar components of $(v_x, v_y) = (v_r \cos v_\theta, v_r \sin v_\theta)$. For $r = \varepsilon$ this reduces to $v_r = \cos v_\theta + 2\varepsilon$. In region $a' := \{0 \le r \le \varepsilon\,, \pi/2 \le \theta \le \pi\}$ we set

$$v_\theta := 2\arctan\left(\frac{r\sin\theta}{-r\cos\theta+\varepsilon}\right) \tag{28}$$

$$v_r := (1+\varepsilon)\cos v_\theta + 2\varepsilon - r\cos\theta\,. \tag{29}$$

In the strip $0 \le x \le 1$ we use Cartesian coordinates in the (x, y)-plane. In $c := \{0 \le x \le 1\,,\ 0 \le y \le \varepsilon\}$,

$$v_\theta := 2\arctan(y/\varepsilon)\,,$$
$$v_r := (1+\varepsilon)\cos v_\theta + 2\varepsilon + x\varepsilon^2\,.$$

We finally come to the region $x \ge 1$. Here we use the polar coordinates $(\bar r, \bar\theta)$ centered in $(1, 0)$, and only need to consider $0 \le \bar\theta \le \pi/2$. In $e' := \{0 \le \bar r \le \varepsilon\,, 0 \le \bar\theta \le \pi/2\}$ we set

$$v_\theta := 2\arctan\left(\frac{\bar r\sin\bar\theta}{\bar r\cos\bar\theta+\varepsilon}\right)\,,$$
$$v_r := (1+\varepsilon)\cos v_\theta + 2\varepsilon + \bar r\cos\bar\theta + \varepsilon^2\,.$$

Finally, in $e := \{\varepsilon \le \bar r \le 1\,, 0 \le \bar\theta \le \pi/2\}$, we set

$$v_\theta := \bar\theta$$
$$v_r := 2\varepsilon + \varepsilon\bar r + (1+\varepsilon+\bar r)\cos v_\theta\,.$$

We now come to the outer region. Since the function here has a smooth dependence on ε, we only shortly sketch the construction, without giving all details. We start from region $b := \{x \le 0\,, x^2 + y^2 \ge 1\}$. The construction is most simply understood by composing two Bilipschitz functions. Let c_ε be the map defined, in polar coordinates, by

$$(r, \theta) \to (r - 1 + \varepsilon, (\theta + \pi)/2)\,.$$

Let ψ_ε be a Bilipschitz function which maps $\{x \le 0\,, |y| \le \varepsilon + |x|\} \setminus B(0, \varepsilon)$ into $\{x \le 0\,, |y| \le \varepsilon + |x|\} \cup B(0, \varepsilon)$ and satisfies

(i) $\psi_\varepsilon(r, \theta) = (r, \theta)$ on the lines $\theta = \pi \pm \pi/4$, $r \ge \varepsilon$;
(ii) $\psi_\varepsilon(r, \theta) = (r, \pi - \theta)$ on the half-circumference $r = \varepsilon$, $\theta \in (\pi/2, 3\pi/2)$;
(iii) ψ_ε equals the identity outside $B(0, 2\varepsilon)$.

We finally set $v = \psi_\varepsilon \circ c_\varepsilon$. Region f is analogous.

We now consider region $d := \{0 \le x \le 1\,, y \ge \varepsilon\}$. Let g_ε be uniformly Bilipschitz functions that map d into $(0, 3) \times (0, \infty)$, with $g_\varepsilon(0, 1) = (0, 0)$,

$g_\varepsilon(0,\varepsilon) = (1,0)$, $g_\varepsilon(1,\varepsilon) = (2,0)$, $g_\varepsilon(1,1) = (3,0)$ and affine in the segments joining these points. Now we consider the additional function h given by

$$h_x := -y\cos(\varphi(x))$$
$$h_y := \omega(x) + y\sin(\varphi(x))$$

where $\omega(x)$ is still to be determined and

$$\varphi(x) := \frac{\pi}{4}\left(1 + \frac{x}{3}\right).$$

We finally set $v = h \circ g_\varepsilon$, and choose ω so that the trace on the $x = 0$ axis agrees with the one obtained from the interior.

We leave to the reader to check that all pieces match continuously and are orientation-preserving, and focus directly on the estimate of $|\nabla u_n|^2 + \phi(\det \nabla u_n)$.

The estimate of the L^2 norm of the gradients is done with the help of the expressions in polar/cylindrical coordinates given in Appendix B. We start from region a. From equation (30) we see that if the partial derivatives $\partial_{(r,\theta)} u_{(r,\theta)}$ are uniformly bounded, all terms are immediately controlled, except for the last one. The last one is also uniformly controlled, since in this region $u_\theta / \sin\theta = 1$. In region a', we get $|u_{r,\theta}| + |u_{\theta,\theta}| \le c$, and $|u_{r,r}| + |u_{\theta,r}| \le c/\varepsilon$. Since $|r| \le \varepsilon$, there is nothing to be checked. Similar arguments apply to region c, by considering the formula for cylindrical coordinates in equation (31). Finally, regions e and e' are completely analogous to a and a'.

We now come to the determinant. This is easily done using Lemma 6.4. Indeed, let us check the first property. Fix $\eta \in (0,1)$, and let Ω_η be the set of points of distance less than η from the segment joining $(0,0,0)$ and $(1,0,0)$. It is clear that for ε small enough (as compared to η) the first property is satisfied outside of Ω_η. This implies the first hypothesis of the lemma, for all $\delta > 2|u_n(\Omega_\eta)|$. Since

$$\lim_{\eta \downarrow 0} (\sup_n |u_n(\Omega_\eta)|) = 0$$

we only have to choose η small enough. The second hypothesis of Lemma 6.4 can be proved analogously.

This concludes the proof of the theorem. □

7. – Final Remarks

We now give a brief summary of the results of the previous sections and of their implications. Let us fix a bounded open set $\Omega \subset \mathbb{R}^3$ (sufficiently regular) and call $\mathcal{A}$ the class of maps $u \in W^{1,2}(\Omega) \cap L^\infty$ which satisfy condition INV and such that $\det Du > 0$ a.e. After fixing a diffeomorphism $g : \overline{\Omega} \to \overline{\Omega}' \subset \mathbb{R}^3$ we introduce the following class of functions

$$\mathcal{A}_{nc} := \{u \in \mathcal{A} | \mathrm{im}_{\mathrm{G}}(u, \Omega) = \mathrm{im}_{\mathrm{T}}(u, \Omega), u|_{\partial\Omega} = g|_{\partial\Omega}\}$$
$$\mathcal{A}_c := \{u \in \mathcal{A} | \operatorname{Per}(\mathrm{im}_{\mathrm{G}}(u, \Omega)) < \infty, u|_{\partial\Omega} = g|_{\partial\Omega}\} .$$

Following Sivaloganathan and Spector [16] one could also define "no cavitation" set as $\mathcal{A}'_{nc} = \{u \in \mathcal{A} | \det Du = \operatorname{Det} Du, u|_{\partial\Omega} = g|_{\partial\Omega}\}$. By Lemma 4.3, this definition is equivalent to the previous one.

Of course $\mathcal{A}_{nc} \subset \mathcal{A}_c$. Moreover it is easy to build a map which is in $\mathcal{A}_c \setminus \mathcal{A}_{nc}$ (see the first examples in the pioneering work of Ball [4]). Basically the first class does not allow for the opening of holes, whereas the second one does: the union of such holes gives the set $\mathrm{im}_{\mathrm{T}} \setminus \mathrm{im}_{\mathrm{G}}$. In the first class there exists a minimum for the functional (2) when $p > 2$ and in the second one there exists a minimum of the modified functional

$$E'(u) = \int_\Omega (|\nabla u|^p + \varphi(\det \nabla u))dx \; + \operatorname{Per}(\mathrm{im}_{\mathrm{G}}(u, \Omega)) .$$

We have proved that the first problem is equivalent to minimization of the same functional in the class of functions u such that their graphs are currents which have no boundary in $\Omega \times \mathbb{R}^3$ and $\det \nabla u \geq 0$ a.e. The second one is equivalent to minimization of the energy

$$\int_\Omega (|\nabla u|^p + \varphi(\det \nabla u))dx \; + \|\partial G_u\|(\Omega \times \mathbb{R}^3)$$

among all the maps u such that:

(i) their graphs are rectifiable currents with boundary given by

$$\partial G_u \llcorner \Omega \times \mathbb{R}^3 = \sum_i \{x_i\} \times \partial F_i$$

where every F_i is a Caccioppoli set;

(ii) $G_u + \sum \{x_i\} \times F_i$ has degree 1 or 0 as generalized graph in a.e. y in the target.

In Section 6 we have shown that the domain of the first problem is not closed under the $W^{1,2}$ topology. Hence we have a sequence of good functions u_n which are converging to a function u such that the boundary ∂G_u is nontrivial in $\Omega \times \mathbb{R}^3$. Moreover u does not even fall in the class $\mathcal{A}_c$. Indeed, if we take the sequence of currents given by G_{u_n} we notice that they are converging to a current T which is composed by G_u plus a nontrivial current V. Being the limit of a sequence of currents with no boundary in $\Omega \times \mathbb{R}^3$ we have that $G_u + V$ has no boundary as well.

With respect to the sequence constructed in the proof of Theorem 6.1, it is not difficult to check that the boundary of G_u by the current naturally induced by the oriented manifold

$$\{A\} \times S^2 - \{B\} \times S^2$$

where A and B are the two points given by $(0,0,0)$ and $(1,0,0)$ (more formally this current can be denoted by $\delta_A \times [[S^2]] - \delta_B \times [[S^2]]$). We notice that the "hole" opened in $(1,0,0)$ has the "wrong" sign and cannot be interpreted as the boundary given by the opening of a cavity. Hence the "vertical current" V must be a cylinder which connects $\{A\} \times S^2$ and $\{B\} \times S^2$ (this singularity is often called "dipole" in the literature on harmonic maps).

We conclude that in order to use Giaquinta, Modica and Souček's theory one has to deal with functions which create singularities which are more complicated than cavities.

Appendix A

Lemma A.1. *Let $A, C \subset \mathbb{R}^n$ be two Caccioppoli sets and let us denote by A^* and C^* the sets of points with density* 1 *with respect to A and C. Suppose that*:

(i) *at least one of them is not $\mathbb{R}^n$*;
(ii) $\mathcal{H}^{n-1}(\partial^* A \cap C^*) = \mathcal{H}^{n-1}(\partial^* C \cap A^*) = 0$.

Then $|A \cap C| = 0$ and hence $A^* \cap C^* = \emptyset$.

Proof. We notice that $A \cap C$ is a Caccioppoli set and that $x \in \partial^*(A \cap C)$ if and only if

$$\lim_{r \to 0} \frac{|A \cap C \cap B(x,r)|}{\omega_n r^n} = \frac{1}{2} .$$

Furthermore we notice that condition (ii) implies that for $\mathcal{H}^{n-1}$ a.e. $x \in \mathbb{R}^n$ either one of the sets A and C has density 0 in x, or they have both density 1. It follows that $\mathcal{H}^{n-1}(\partial^*(A \cap C)) = 0$ and so either $A \cap C$ is a set of Lebesgue measure zero, or is $\mathbb{R}^n$ minus a set of measure zero. The second possibility is excluded by condition (i) and this ends the proof. □

Appendix B. $W^{1,2}$ norms in cylindrical and spherical coordinates

Let (r, θ, ϕ) be spherical coordinates, with

$$\mathbf{r} = r(\mathbf{e}_1 \cos\theta + \sin\theta(\mathbf{e}_2 \cos\phi + \mathbf{e}_3 \sin\phi)) ,$$

and (u_r, u_θ, u_ϕ) be spherical components of $\mathbf{u}$ (here we use boldface for vectors). Assume $u_\phi = \phi$, with u_r and u_θ independent on ϕ. Then, a simple calculation gives

$$\int_\Omega |\nabla \mathbf{u}|^2 = 2\pi \int_\Omega \left[(\partial_r u_r)^2 + \left(\frac{\partial_\theta u_r}{r}\right)^2 + (u_r \partial_r u_\theta)^2 + \left(\frac{u_r \partial_\theta u_\theta}{r}\right)^2 + \left(\frac{u_r \sin u_\theta}{r \sin\theta}\right)^2 \right] r^2 \sin\theta \, dr d\theta . \tag{30}$$

Now consider cylindrical coordinates (x, y, ϕ), such that $\mathbf{r} = x\mathbf{e}_1 + y(\mathbf{e}_2 \cos\phi + \mathbf{e}_3 \sin\phi)$, but still express u in spherical components. Then,

$$\int_\Omega |\nabla \mathbf{u}|^2 = 2\pi \int_\Omega \left[(\partial_x u_r)^2 + (\partial_y u_r)^2 + \left(\frac{\partial_\theta u_r}{y}\right)^2 + (u_r \partial_y u_\theta)^2 + \left(\frac{u_r \partial_\theta u_\theta}{y}\right)^2 + \left(\frac{u_r \sin u_\theta}{y \sin\theta}\right)^2 \right] y dx dy . \tag{31}$$

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