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Universal Solutions of a Nonlinear Heat Equation on $\mathbb{R}^N$

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Abstract. In this paper, we study the relationship between the long time behavior of a solution $u(t,x)$ of the nonlinear heat equation $u_t - \Delta u + |u|^\alpha u = 0$ on $\mathbb{R}^N$ (where $\alpha > 0$) and the asymptotic behavior as $|x| \to \infty$ of its initial value u_0. In particular, we show that if the sequence of dilations $\lambda_n^{2/\alpha} u_0(\lambda_n \cdot)$ converges weakly to $z(\cdot)$ as $\lambda_n \to \infty$, then the rescaled solution $t^{1/\alpha} u(t, \cdot\sqrt{t})$ converges uniformly on $\mathbb{R}^N$ to $\mathcal{U}(1)z$ along the subsequence $t_n = \lambda_n^2$, where $\mathcal{U}(t)$ is an appropriate flow. Moreover, we show there exists an initial value U_0 such that the set of all possible z attainable in this fashion is a closed ball B of a weighted L^∞ space. The resulting "universal" solution is therefore asymptotically close along appropriate subsequences to all solutions with initial values in B. These results are restricted to positive solutions in the case $\alpha < 2/N$.

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1. – Introduction

This paper is concerned with the long time behavior of solutions of the nonlinear heat equation

$$u_t - \Delta u + |u|^\alpha u = 0\,, \tag{1.1}$$

in $(0,\infty)\times\mathbb{R}^N$, where $\alpha > 0$. More precisely, we study the relationship between the asymptotic behavior in space of the initial value and the asymptotic behavior in time of the resulting solution. As is well known, given any $u_0 \in C_0(\mathbb{R}^N)$, there exists a unique solution $u \in C([0,\infty), C_0(\mathbb{R}^N))$ of (1.1) with the initial condition $u(0,x) = u_0(x)$, which we denote by

$$u(t) = \mathcal{S}(t)u_0\,, \tag{1.2}$$

where $u(t) = u(t,\cdot)$. More generally, a function $u \in C((0,\infty), C_0(\mathbb{R}^N))$ is a solution of (1.1) if

$$u(t) = \mathcal{S}(t-s)u(s)\,, \tag{1.3}$$

for all $0 < s < t$.

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The dilation structure of solutions of (1.1), and in particular self-similar solutions, play a key role in the description of the asymptotic behavior of solutions. Recall that if $u \in C((0,\infty), C_0(\mathbb{R}^N))$ is a solution of (1.1), then so is $\Gamma_\lambda u$, for all $\lambda > 0$, where the space-time dilation operators Γ_λ are given by

$$\Gamma_\lambda u(t,x) = \lambda^{\frac{2}{\alpha}} u(\lambda^2 t, \lambda x) . \tag{1.4}$$

Furthermore, if the solution u has initial value u_0, either in the sense of $C_0(\mathbb{R}^N)$ or in some more general sense, then $\Gamma_\lambda u$ has initial value $D_\lambda u_0$, where the space dilation operators D_λ, for $\lambda > 0$, are given by

$$D_\lambda u_0(x) = \lambda^{\frac{2}{\alpha}} u_0(\lambda x) . \tag{1.5}$$

Note that

$$D_\lambda [u(\lambda^2 t)] = [\Gamma_\lambda u](t) , \tag{1.6}$$

for all $\lambda > 0$ and $t > 0$. In particular,

$$D_\lambda \mathcal{S}(\lambda^2 t) = \mathcal{S}(t) D_\lambda . \tag{1.7}$$

A solution u of (1.1) is self-similar if $\Gamma_\lambda u = u$, for all $\lambda > 0$, or equivalently if

$$u(t,x) = t^{-\frac{1}{\alpha}} f(x/\sqrt{t}) = D_{\frac{1}{\sqrt{t}}} f(x) , \tag{1.8}$$

where $f(x) = u(1,x)$ is called the profile of u. It follows that if the self-similar solution u of (1.1) has initial value u_0, then $D_\lambda u_0 = u_0$ for all $\lambda > 0$, i.e. u_0 is homogeneous of degree $-2/\alpha$. Moreover, at least formally,

$$|x|^{\frac{2}{\alpha}} u_0(x) = \lim_{t\to 0} |x|^{\frac{2}{\alpha}} u(t,x) = \lim_{t\to 0} \left(\frac{|x|}{\sqrt{t}}\right)^{\frac{2}{\alpha}} f\left(\frac{x}{\sqrt{t}}\right) = \zeta\left(\frac{x}{|x|}\right) ,$$

for some function ζ defined on S^{N-1}.

Rigorously, it is known (see [3], [4], [16]) that if $\zeta \in C(S^{N-1})$, and if $\zeta \ge 0$ in case $\alpha < 2/N$, then there exists a unique (positive, if $\alpha < 2/N$) self-similar solution $u_\zeta \in C((0,\infty), C_0(\mathbb{R}^N))$ of (1.1), with profile f_ζ, such that

$$\lim_{|x|\to\infty} |x|^{\frac{2}{\alpha}} f_\zeta(x) - \zeta\left(\frac{x}{|x|}\right) = 0 ,$$

and

$$\lim_{t\to 0} |x|^{\frac{2}{\alpha}} u_\zeta(t,x) - \zeta\left(\frac{x}{|x|}\right) = 0 , \tag{1.9}$$

the limit in (1.9) being uniform for $|x| \ge \epsilon$, for all $\epsilon > 0$.

These self-similar solutions are related to the asymptotic behavior of $u(t) = \mathcal{S}(t)u_0$ in the following way. Suppose $u_0 \in C_0(\mathbb{R}^N)$ is asymptotically homogeneous in space in the sense that

$$\lim_{|x|\to\infty} |x|^{\frac{2}{\alpha}} u_0(x) - \zeta\left(\frac{x}{|x|}\right) = 0\,, \tag{1.10}$$

or equivalently,

$$\lim_{\lambda\to\infty} |x|^{\frac{2}{\alpha}} D_\lambda u_0(x) = \zeta\left(\frac{x}{|x|}\right), \tag{1.11}$$

where the limit in (1.1) holds uniformly for $|x| \geq \epsilon$, for all $\epsilon > 0$. If $\alpha < 2/N$, assume in addition that $u_0 \geq 0$, $u_0 \not\equiv 0$. It follows that $\mathcal{S}(t)u_0$ is asymptotically self-similar in that

$$\sup_{x\in\mathbb{R}^N} (t + |x|^2)^{\frac{1}{\alpha}} |\mathcal{S}(t)u_0(x) - u_\zeta(t,x)| \underset{t\to\infty}{\longrightarrow} 0\,,$$

or equivalently, after a rescaling,

$$\sup_{x\in\mathbb{R}^N} (1 + |x|^2)^{\frac{1}{\alpha}} |D_{\sqrt{t}}\mathcal{S}(t)u_0(x) - f_\zeta(x)| \underset{t\to\infty}{\longrightarrow} 0\,. \tag{1.12}$$

We refer the reader to Theorem 1.2 and Lemma 3.6 in Cazenave, Dickstein, Escobedo and Weissler [4] for the above formulation, earlier results of this type having been obtained by Kamin and Peletier [16], Escobedo and Kavian [10], Escobedo, Kavian and Matano [11], Herraiz [15], and Kwak [17]. See Gmira and Veron [13] for results in the case where u_0 decays more slowly than in (1.10), and Gmira and Veron [13], Kamin and Peletier [16], Herraiz [15] and Wayne [20] for some additional information when $\zeta \equiv 0$ in (1.10).

The purpose of this paper is to show that the relationship between the spatially asymptotic behavior of the initial value u_0 and the long time asymptotic behavior of the resulting solution u of (1.1) is much more general than the fact that (1.11) implies (1.12). More precisely, instead of assuming (1.11), we wish to consider initial values u_0 for which different limits might be realized along different sequences $(\lambda_n)_{n\geq 0}$, with $\lambda_n \to \infty$. Consequently, we also allow, instead of (1.12), the possibility of different limits of $D_{\sqrt{t_n}}\mathcal{S}(t_n)u_0(x)$ along different sequences as $t_n \to \infty$. To carry out this idea, we reformulate (1.12) using the invariance properties of solutions of (1.1). It follows from (1.7), by setting $t = 1$ and replacing λ^2 by t, that

$$D_{\sqrt{t}}\mathcal{S}(t) = \mathcal{S}(1)D_{\sqrt{t}}\,. \tag{1.13}$$

Thus, the fact that (1.11) imples (1.12) means that if

$$D_\lambda u_0 \underset{\lambda\to\infty}{\longrightarrow} |x|^{-\frac{2}{\alpha}}\zeta\left(\frac{x}{|x|}\right), \tag{1.14}$$

in some appropriate sense, then

$$\mathcal{S}(1)D_\lambda u_0 \underset{\lambda\to\infty}{\longrightarrow} u_\zeta(1)\,, \tag{1.15}$$

in some corresponding sense. Note that by (1.9), u_ζ is the self-similar solution with initial value $|\cdot|^{-\frac{2}{\alpha}}\zeta(\cdot/|\cdot|)$.

The generalization is now clear: we would like to replace convergence as $\lambda\to\infty$ in both (1.14) and (1.15) by convergence along a sequence $\lambda_n\to\infty$. In other words, we need to prove a finite time (i.e. at $t=1$) continuous dependence result for solutions of (1.1), where initial values as singular as $|\cdot|^{-\frac{2}{\alpha}}$ must be allowed. Furthermore, any function z obtained as a limit of $D_{\lambda_n}u_0$ must also be allowed.

Consider the Banach space $\mathcal{W}$ defined by

$$\mathcal{W} = \{u\in L^1_{\text{loc}}(\mathbb{R}^N\setminus\{0\});\ |\cdot|^{\frac{2}{\alpha}}u(\cdot)\in L^\infty(\mathbb{R}^N)\}\,, \tag{1.16}$$

endowed with its natural norm

$$\|u\|_{\mathcal{W}} = \|\,|\cdot|^{\frac{2}{\alpha}}u(\cdot)\|_{L^\infty}\,. \tag{1.17}$$

If $u_0\in C_0(\mathbb{R}^N)$ satisfies (1.10), then $u_0\in\mathcal{W}$. Also, $|\cdot|^{-\frac{2}{\alpha}}\zeta(\cdot/|\cdot|)\in\mathcal{W}$, where $\zeta\in C(S^{N-1})$. Finally, the dilations operators D_λ are isometries of $\mathcal{W}$, and so limits of $D_{\lambda_n}u_0$ for $u_0\in\mathcal{W}$ will be in $\mathcal{W}$. Clearly then, the space $\mathcal{W}$ is a natural choice for the initial values.

The interplay of the dilation operators D_λ with various topologies on $\mathcal{W}$ was extensively studied in Section 2 of our previous work [5]. We recall here the basic definitions and some elementary facts. Since the Banach space $\mathcal{W}$ is isometrically isomorphic to $L^\infty(\mathbb{R}^N)$, it is the dual of a Banach space isometrically isomorphic to $L^1(\mathbb{R}^N)$. It follows that for any $M>0$, the sets

$$\mathcal{B}_M = \{u\in\mathcal{W};\ \|u\|_{\mathcal{W}}\le M\}\quad\text{and}\quad \mathcal{B}_M^+ = \{u\in\mathcal{B}_M;\ u\ge 0\}\,, \tag{1.18}$$

endowed with the weak* topology of $\mathcal{W}$ are compact. Since $L^1(\mathbb{R}^N)$ is separable, the weak* topology on $\mathcal{B}_M$ is metrizable. We denote by d^*_M a corresponding metric, so that $(\mathcal{B}_M,\mathrm{d}^*_M)$ is a compact metric space (hence complete and separable), for all $M>0$.

We mention that Kamin and Peletier [16] used a weak formulation of equation (1.1), and therefore a weak formulation of the condition that the initial value be asymptotically homogeneous. Their condition is equivalent to convergence in (1.14) in the sense of $(\mathcal{B}_M,\mathrm{d}^*_M)$.

In addition, let

$$|||u_0|||_{\mathcal{W}} = \sum_{n\ge 1}2^{-n}\operatorname*{ess\,sup}_{|x|>\frac{1}{n}}|x|^{\frac{2}{\alpha}}|u_0(x)|\le\|u_0\|_{\mathcal{W}}\,. \tag{1.19}$$

This norm is not complete on $\mathcal{W}$ (since it is not equivalent to the norm $\|\cdot\|_{\mathcal{W}}$). On the other hand, if d_M is the metric on $\mathcal{B}_M$ induced by $|||\cdot|||_{\mathcal{W}}$, then $(\mathcal{B}_M, \mathrm{d}_M)$ is a complete metric space, which is not compact. This topology is clearly stronger than the weak* topology and weaker than the norm topology on $\mathcal{B}_M$. Furthermore, the convergence of $D_\lambda u_0$ to $|\cdot|^{\frac{2}{\alpha}}\zeta$ in formula (1.1) is with respect to the norm $|||\cdot|||_{\mathcal{W}}$.

Given $u_0 \in \mathcal{W}$, we consider $M \geq \|u_0\|_{\mathcal{W}}$ and we set

$$\Omega(u_0) = \{z \in \mathcal{B}_M; \exists \lambda_n \to \infty \text{ such that } \mathrm{d}_M^*(D_{\lambda_n} u_0, z) \to 0\} = \bigcap_{\mu>0} \overline{\bigcup_{\lambda>\mu} \{D_\lambda u_0\}}, \tag{1.20}$$

$$\Omega_1(u_0) = \{z \in \mathcal{B}_M; \exists \lambda_n \to \infty \text{ such that } \mathrm{d}_M(D_{\lambda_n} u_0, z) \to 0\} = \bigcap_{\mu>0} \overline{\bigcup_{\lambda>\mu} \{D_\lambda u_0\}}, \tag{1.21}$$

where the closure in (1.20) is in $(\mathcal{B}_M, \mathrm{d}_M^*)$, and the closure in (1.21) is in $(\mathcal{B}_M, \mathrm{d}_M)$. It is clear that the above definitions are independent of $M \geq \|u_0\|_{\mathcal{W}}$. Moreover, $\Omega(u_0)$ is a nonempty, closed, connected subset of the compact metric space $(\mathcal{B}_M, \mathrm{d}_M^*)$ and $\Omega_1(u_0)$ is a (perhaps empty) closed subset of the complete metric space $(\mathcal{B}_M, \mathrm{d}_M)$.

Since if $\alpha \leq 2/N$, the elements of $\mathcal{W}$ are not always locally integrable, we need to consider solutions of (1.1) whose initial values are attained in the following rather weak sense.

DEFINITION 1.1. Given $u_0 \in \mathcal{W}$, we denote by $\Sigma(u_0)$ the set of solutions $u \in C((0,\infty), C_0(\mathbb{R}^N))$ of (1.1) such that $u(t) \to u_0$ in $L^1_{\mathrm{loc}}(\mathbb{R}^N \setminus \{0\})$ as $t \downarrow 0$.

It follows from Corollary 2.4 below that there exists C such that if $u_0 \in \mathcal{W}$ and $u \in \Sigma(u_0)$, then

$$u(t) \in \mathcal{W} \cap C_0(\mathbb{R}^N)$$

for all $t > 0$, and

$$|u(t,x)| \leq C(t + |x|^2)^{-\frac{1}{\alpha}}(1 + \|u_0\|_{\mathcal{W}}). \tag{1.22}$$

Using the norm $\|\cdot\|_{\mathcal{W}\cap C_0}$ on $\mathcal{W} \cap C_0(\mathbb{R}^N)$ given by

$$\|u\|_{\mathcal{W}\cap C_0} = \|(1 + |\cdot|^2)^{\frac{1}{\alpha}} u(\cdot)\|_{L^\infty}, \tag{1.23}$$

which is equivalent to its natural norm as the intersection of two Banach spaces, we may rewrite (1.22) in the form

$$\|D_{\sqrt{t}} u(t)\|_{\mathcal{W}\cap C_0} \leq C(1 + \|u_0\|_{\mathcal{W}}). \tag{1.24}$$

The above estimates motivate the following definition.

DEFINITION 1.2. Let $u_0 \in \mathcal{W}$. Given $u \in \Sigma(u_0)$, we define

$$\omega(u) = \{f \in C_0(\mathbb{R}^N);\ \exists t_n \to \infty \text{ s.t. } \|D_{\sqrt{t_n}} u(t_n) - f\|_{L^\infty} \underset{n\to\infty}{\longrightarrow} 0\}$$

$$= \bigcap_{T>0} \overline{\bigcup_{t>T} \{D_{\sqrt{t}} u(t)\}}, \tag{1.25}$$

$$\omega_1(u) = \{f \in C_0(\mathbb{R}^N);\ \exists t_n \to \infty \text{ s.t. } \|D_{\sqrt{t_n}} u(t_n) - f\|_{\mathcal{W}\cap C_0} \underset{n\to\infty}{\longrightarrow} 0\}$$

$$= \bigcap_{T>0} \overline{\bigcup_{t>T} \{D_{\sqrt{t}} u(t)\}}, \tag{1.26}$$

where the closure in (1.25) is in the L^∞ norm and the closure in (1.26) is in the $\mathcal{W} \cap C_0$ norm.

Note that the definition of $\omega_1(u)$ is a direct generalization of formula (1.12). Clearly $\omega_1(u) \subset \omega(u)$, and the inclusion can be strict (see Remarks 3.12 and 4.18). In particular, it can happen that $\omega_1(u)$ is empty (see Corollaries 3.13 and 4.19). It is straightforward to check that $\omega(u)$ is a nonempty, closed, connected subset of the Banach space $C_0(\mathbb{R}^N)$ and that $\omega_1(u_0)$ is a (perhaps empty) closed subset of the Banach space $\mathcal{W} \cap C_0$. (See Proposition 2.7.)

Using the dilation property (1.6), we can immediately re-write the above definitions in the following equivalent forms.

$$\omega(u) = \{f \in C_0(\mathbb{R}^N);\ \exists \lambda_n \to \infty \text{ s.t. } \|\Gamma_{\lambda_n} u(1) - f\|_{L^\infty} \underset{n\to\infty}{\longrightarrow} 0\}$$

$$= \bigcap_{\mu>0} \overline{\bigcup_{\lambda>\mu} \{\Gamma_\lambda u(1)\}}, \tag{1.27}$$

$$\omega_1(u) = \{f \in C_0(\mathbb{R}^N);\ \exists \lambda_n \to \infty \text{ s.t. } \|\Gamma_{\lambda_n} u(1) - f\|_{\mathcal{W}\cap C_0} \underset{n\to\infty}{\longrightarrow} 0\}$$

$$= \bigcap_{\mu>0} \overline{\bigcup_{\lambda>\mu} \{\Gamma_\lambda u(1)\}}, \tag{1.28}$$

where the closure in (1.27) is in the L^∞ norm and the closure in (1.28) is in the $\mathcal{W} \cap C_0$ norm. Note that, in view of (1.24), we could replace in the definition (1.25) or (1.27) of $\omega(u)$ the convergence in $L^\infty(\mathbb{R}^N)$ by the convergence for the norm $\|(1 + |\cdot|^2)^\gamma u(\cdot)\|_{L^\infty}$, for $0 \le \gamma < 1/\alpha$. In other words,

$$\omega(u) = \{f \in C_0(\mathbb{R}^N);\ \exists \lambda_n \to \infty \text{ s.t. } \|(1 + |\cdot|^2)^\gamma (\Gamma_{\lambda_n} u(1) - f)\|_{L^\infty} \underset{n\to\infty}{\longrightarrow} 0\},$$

for any $0 \le \gamma < 1/\alpha$.

The principal results of this paper are explicit expressions for $\omega(u)$ and $\omega_1(u)$ in terms of $\Omega(u_0)$ and $\Omega_1(u_0)$, where $u \in \Sigma(u_0)$. As suggested by formulas (1.12) and (1.15), these relationships involve the time 1 flow for (1.1) applied to elements of $\Omega(u_0)$ or $\Omega_1(u_0)$. Since $\Omega(u_0)$ and $\Omega_1(u_0)$ may contain singular functions, we need to consider separately the cases $\alpha \ge 2/N$ and $\alpha < 2/N$.

It turns out that if $\alpha \geq 2/N$, then $\Sigma(u_0)$ contains precisely one solution for every $u_0 \in \mathcal{W}$. In other words, the nonlinear operators $\mathcal{S}(t)$ extend in a natural and unique way to $\mathcal{W}$. If $\alpha > 2/N$, this is straightforward to prove since $\mathcal{W} \subset L^1(\mathbb{R}^N) \cap L^q(\mathbb{R}^N)$ for some $1 < q < \infty$. In the case $\alpha = 2/N$, this follows from arguments due to Brezis and Friedman [2]. (See Proposition 3.1 below.) We have the following result.

THEOREM 1.3. *Suppose $\alpha \geq 2/N$ and let $\mathcal{S}(t)$ be the unique extension to $\mathcal{W}$ of $\mathcal{S}(t)$ defined on $\mathcal{W} \cap C_0(\mathbb{R}^N)$. Let $u_0 \in \mathcal{W}$ and let $\Omega(u_0)$ and $\Omega_1(u_0)$ be defined by* (1.20) *and* (1.21), *respectively. Also, let $u \in \Sigma(u_0)$ and let $\omega(u)$ and $\omega_1(u)$ be defined by* (1.25) *and* (1.26), *respectively. It follows that $\omega(u) = \mathcal{S}(1)\Omega(u_0)$. Moreover, if $\Omega_1(u_0) \neq \emptyset$, then $\omega_1(u) = \overline{\mathcal{S}(1)\Omega_1(u_0)}$, where the closure is in $\mathcal{W} \cap C_0(\mathbb{R}^N)$.*

If $\alpha < 2/N$, the situation is more delicate due to lack of uniqueness in some cases, and we need to restrict ourselves to nonnegative initial values and nonnegative solutions. It follows from results of Marcus and Véron [18] that if $u_0 \in \mathcal{W}$ and $u_0 \geq 0$, there exists a unique solution $u \in \Sigma(u_0)$ such that $u \geq 0$, $u(t) \to u_0$ in $L^1_{\mathrm{loc}}(\mathbb{R}^N \setminus \{0\})$, and $\|u(t)\|_{L^1} \to \infty$ as $t \downarrow 0$. (See Proposition 4.3 below.) In this case, we set

$$\mathcal{U}(t)u_0 = u(t)\,. \tag{1.29}$$

We note that if $u_0 \in \mathcal{W} \cap C_0(\mathbb{R}^N)$, $u_0 \geq 0$, then $\mathcal{U}(t)u_0 \neq \mathcal{S}(t)u_0$. (See Remark 4.6 below). On the other hand, (see Proposition 4.9) it turns out that if $u_0 \in \mathcal{W}$, $u_0 \geq 0$, then $\omega(u)$ and $\omega_1(u)$ are independent of $u \in \Sigma(u_0)$ with $u \geq 0$ and $u \not\equiv 0$. In particular, $\omega(\mathcal{S}(\cdot)u_0) = \omega(\mathcal{U}(\cdot)u_0)$ and $\omega_1(\mathcal{S}(\cdot)u_0) = \omega_1(\mathcal{U}(\cdot)u_0)$ for all $u_0 \in \mathcal{W} \cap C_0(\mathbb{R}^N)$ such that $u_0 \geq 0$, $u_0 \not\equiv 0$. Our result in this case is the following.

THEOREM 1.4. *Suppose $\alpha < 2/N$ and let $\mathcal{U}(t)$ be given by* (1.29). *Let $u_0 \in \mathcal{W}$, $u_0 \geq 0$, and let $\Omega(u_0)$ and $\Omega_1(u_0)$ be defined by* (1.20) *and* (1.21), *respectively. Also, let $u \in \Sigma(u_0)$, $u \geq 0$, $u \not\equiv 0$, and let $\omega(u)$ and $\omega_1(u)$ be defined by* (1.25) *and* (1.26), *respectively. It follows that $\omega(u) = \mathcal{U}(1)\Omega(u_0)$. Moreover, if $\Omega_1(u_0) \neq \emptyset$, then $\omega_1(u) = \overline{\mathcal{U}(1)\Omega_1(u_0)}$, where the closure is in $\mathcal{W} \cap C_0(\mathbb{R}^N)$.*

Theorem 1.3 is a particular case of Theorems 3.10 and 3.16 and Theorem 1.4 is a particular case of Theorems 4.15 and 4.21. The proofs rely on formulas (1.27) and (1.28) and the continuous dependence properties of the operators $\mathcal{S}(1)$ and $\mathcal{U}(1)$ given in Propositions 3.4 and 4.8. More precisely, if $\alpha \geq 2/N$, $u_0 \in \mathcal{W}$, and $u(t) = \mathcal{S}(t)u_0$, then

$$\Gamma_{\lambda_n} u(1) = \mathcal{S}(1) D_{\lambda_n} u_0\,,$$

and if $\alpha < 2/N$, $u_0 \in \mathcal{W}$, $u_0 \geq 0$ and $u(t) = \mathcal{U}(t)u_0$, then

$$\Gamma_{\lambda_n} u(1) = \mathcal{U}(1) D_{\lambda_n} u_0\,.$$

In addition, $\mathcal{S}(1)$ and $\mathcal{U}(1)$ are continuous from the compact metric spaces $(\mathcal{B}_M, \mathrm{d}_M^*)$ and $(\mathcal{B}_M^+, \mathrm{d}_M^*)$, respectively, into $C_0(\mathbb{R}^N)$, and from $(\mathcal{B}_M, \mathrm{d}_M)$ and $(\mathcal{B}_M^+, \mathrm{d}_M)$, respectively, into $\mathcal{W} \cap C_0(\mathbb{R}^N)$. The lack of compactness of $(\mathcal{B}_M, \mathrm{d}_M)$ and $(\mathcal{B}_M^+, \mathrm{d}_M)$ is partially compensated by detailed knowledge of the structure of the sets $\omega_1(u)$ as given in Propositions 3.8 and 4.13.

Theorems 1.3 and 1.4 enable us to transfer some of the more delicate asymptotic properties of elements of $\mathcal{W}$ in terms of corresponding long time asymptotic properties of solutions of (1.1). For example, it was shown in [5] that for all $M > 0$, there exist $U_0 \in \mathcal{B}_M \cap C^\infty(\mathbb{R}^N)$ and $U_0^+ \in \mathcal{B}_M^+ \cap C^\infty(\mathbb{R}^N)$ such that $\Omega(U_0) = \mathcal{B}_M$ and $\Omega(U_0^+) = \mathcal{B}_M^+$. The resulting solutions, respectively if $\alpha \geq 2/N$, or if $\alpha < 2/N$, have encoded in them the long time asymptotic behavior of every other solution with initial value in $\mathcal{B}_M$, respectively $\mathcal{B}_M^+$. As a result, we refer to these solutions as "universal". See Corollary 3.11, Remark 3.12, Corollary 4.17 and Remark 4.18 below for a more precise statement. In particular, "universal" solutions are the opposite of asymptotically self-similar solutions. The latter ultimately settle down into a unique asymptotic form, and the former move about among all possible asymptotic forms in a certain class. It is worth mentioning that if $\alpha \geq 2/N$, then a "universal initial value" U_0 as above cannot be in $L^1(\mathbb{R}^N)$. Indeed, it is known [13] that if $\alpha \geq 2/N$ and $u_0 \in L^1(\mathbb{R}^N)$, then $\mathcal{S}(t)u_0$ is asymptotic for large time to a multiple of the Gauss kernel. On the other hand, if $\alpha < 2/N$, then every continuous universal initial value U_0^+ is in $L^1(\mathbb{R}^N)$.

A related property is that $\mathcal{S}(t_0)$ for a fixed t_0, combined with an appropriate rescaling, generates a chaotic discrete dynamical system on a certain compact subset of $C_0(\mathbb{R}^N)$. See Corollaries 3.14 and 4.20 below.

Since the $\mathcal{W} \cap C_0$ norm is stronger than the L^∞ norm, the structure of $\omega_1(u)$ is much more restricted than that of $\omega(u)$. It turns out (Propositions 3.8 and 4.13) that if $\omega_1(u) \neq \emptyset$, then it is the closure in $\mathcal{W} \cap C_0(\mathbb{R}^N)$ of any single orbit it contains, and $\omega(u)$ is the closure in $C_0(\mathbb{R}^N)$ of $\omega_1(u)$. In particular, if $f \in \omega_1(u)$, where f is the profile of a self-similar solution, then necessarily u is asymptotically self-similar in the sense of (1.12). Clearly then, $\omega_1(U) \neq \omega(U)$, where U is a universal solution as described above. While we do not know if $\omega_1(U) = \emptyset$ in this case, we do know there exist solutions u for which $\omega_1(u) = \emptyset$ (see Corollaries 3.13 and 4.19). Also, there exist solutions u for which $\emptyset \neq \omega_1(u) \subset \omega(u)$, $\omega_1(u) \neq \omega(u)$ and $\omega_1(u)$ is not compact (see Remarks 3.18 and 4.22). On the other hand, there is an interesting class of solutions u for which $\omega_1(u) = \omega(u)$ (see Propositions 3.19 and 4.23). Finally, we remark that, if $\alpha > 2/N$, there exists $u_0 \in \mathcal{W} \cap C^\infty(\mathbb{R}^N)$ with $\Omega_1(u_0) = \emptyset$ but $\omega_1(u_0) \neq \emptyset$. Thus the condition $\Omega_1(u_0) \neq \emptyset$ in the last statement of Theorem 1.13 is necessary. (See Remark 3.17) If $\alpha \leq 2/N$, this is an open question.

The fundamental ideas in this paper should apply in principle to any equation satisfying an invariance relationship such as (1.4). Indeed, our previous work [5] treated the linear heat equation. In the present paper, in addition to equation (1.1), we have similar results for a Ginzburg-Landau type equation.

In a subsequent paper [6], we present analogous results for solutions of the Navier-Stokes system. See also Vazquez and Zuazua [19] for related work on the porous medium equation, among others.

The plan of this paper is as follows. In Section 2 we prove a fundamental convergence theorem (Theorem 2.2) for solutions in the class $\Sigma(u_0)$, with $u_0 \in \mathcal{W}$, and give some other continuity properties of solutions. This enables us to prove some basic properties of the sets $\omega(u)$ and $\omega_1(u)$. Sections 3 and 4 contain detailed studies of $\omega(u)$ and $\omega_1(u)$ in the cases $\alpha \geq 2/N$ and $\alpha < 2/N$, respectively, including the proofs of Theorems 1.3 and 1.4. While the arguments in Section 3, at least when $\alpha > 2/N$, are for the most part straightforward adaptations of Section 3 in [5], Section 4 contains several new elements and depends in an essential way on a delicate uniqueness result of Marcus and Véron [18]. Section 5 treats, in the case $\alpha > 2/N$, initial values in $\mathcal{W}$ that have a faster decay. In this case the resulting solutions are asymptotically linear and their large time behavior is as described in Section 3 of [5]. Section 6 gives results analogous to those in Section 3 (but limited to data small in $\mathcal{W}$) for a Ginzburg-Landau type equation. This includes, as a particular case, the nonlinear heat equation with the opposite sign from (1.1).

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2. – Continuity properties of solutions

We begin this section with an elementary result about the action of the operators $\mathcal{S}(t)$ on $\mathcal{W} \cap C_0(\mathbb{R}^N)$.

PROPOSITION 2.1. *Let $\alpha > 0$. The operators $\mathcal{S}(t)$ are Lipschitz continuous $\mathcal{W} \cap C_0(\mathbb{R}^N) \to \mathcal{W} \cap C_0(\mathbb{R}^N)$, uniformly in $t \in [0, T]$, for all $0 < T < \infty$.*

PROOF. We recall that if $\mathcal{S}(t)$ is defined by (1.2), then it follows from Lemma 3.1 in [4] that

$$|\mathcal{S}(t)u_0 - \mathcal{S}(t)v_0| \leq 2\mathcal{S}(t)\frac{|u_0 - v_0|}{2} \leq e^{t\Delta}|u_0 - v_0|\,, \tag{2.1}$$

for all $t \geq 0$ and all $u_0, v_0 \in C_0(\mathbb{R}^N)$. Note that $|f - g| \leq \varphi\|f - g\|_{\mathcal{W}\cap C_0}$ with $\varphi(x) = (1 + |x|^2)^{-\frac{1}{\alpha}}$. It follows from (2.1) that $|\mathcal{S}(t)f - \mathcal{S}(t)g| \leq e^{t\Delta}|f - g| \leq e^{t\Delta}\varphi\|f - g\|_{\mathcal{W}\cap C_0}$, and the result follows from Lemma 8.1 in [4]. □

Since we need to consider solutions of (1.1) with initial values in $\mathcal{W}$, the previous result falls far short of our needs. The following theorem is the main tool we use in proving both the existence and the continuous dependence properties of solutions of equation (1.1) with initial values in $\mathcal{W}$.

THEOREM 2.2. *Let* $u_0 \in \mathcal{W}$ *and* $(u_0^n)_{n\geq 0} \subset \mathcal{W}$. *For every* $n \geq 0$, *let* $u^n \in \Sigma(u_0^n)$. *If* $u_0^n \to u_0$ *in* $\mathcal{W}$ *weak** *as* $n \to \infty$, *then there exist a subsequence* $(u^{n_k})_{k\geq 0}$ *and a solution* $u \in \Sigma(u_0)$ *such that* $u^{n_k} \underset{k\to\infty}{\longrightarrow} u$ *in* $C([\tau,\infty), C_0(\mathbb{R}^N))$, *for every* $\tau > 0$.

If, in addition, $\operatorname{ess\,sup}_{|x|>\rho} |x|^{\frac{2}{\alpha}} |u_0^n(x) - u_0(x)| \underset{n\to\infty}{\longrightarrow} 0$ *for some* $\rho > 0$, *then* $\|u^{n_k}(\tau) - u(\tau)\|_{\mathcal{W}\cap C_0} \underset{k\to\infty}{\longrightarrow} 0$, *for all* $\tau > 0$.

Our proof of Theorem 2.2 depends on two technical results (Lemmas 2.3 and 2.6) which are possibly already known. Since we could not find similar statements in the literature, we give the proofs for completeness.

LEMMA 2.3. *Let* $u_0 \in \mathcal{W}$ *and let* $u \in C((0,\infty), C_0(\mathbb{R}^N)) \cap H^1_{\mathrm{loc}}((0,\infty)\times\mathbb{R}^N)$ *satisfy* $u(t) \to u_0$ *in* $L^1_{\mathrm{loc}}(\mathbb{R}^N \setminus \{0\})$ *as* $t \downarrow 0$ *and* $u_t - \Delta u + |u|^\alpha u \leq 0$. *It follows that*

$$u(t,x) \leq A(t + |x|^2)^{-\frac{1}{\alpha}}\,, \tag{2.2}$$

for all $t > 0$, $x \in \mathbb{R}^N$, *where* $A = \max\{\|u_0\|_{\mathcal{W}}, [4(\alpha+2)/\alpha^2]^{\frac{1}{\alpha}}\}$. *Moreover, given* $M, \rho, t_0, \varepsilon > 0$, *there exists* $\Lambda = \Lambda(M,\rho,t_0,\varepsilon)$ *such that if* $\|u_0\|_{\mathcal{W}} \leq M$ *and* $|x|^{\frac{2}{\alpha}} u_0(x) \leq \varepsilon$ *for* $|x| > \rho$, *then*

$$|x|^{\frac{2}{\alpha}} u(t_0, x) \leq 2\varepsilon\,, \tag{2.3}$$

for all $|x| \geq \Lambda$.

PROOF. We proceed in three steps.

STEP 1. $u(t,x) \leq A|x|^{-\frac{2}{\alpha}}$. Fix $0 < r_1 < r_2$ and let $\omega = \{x \in \mathbb{R}^N;\ r_1 < |x| < r_2\}$. Set $\varphi = v + w$ with $v(x) = A(|x| - r_1)^{-\frac{2}{\alpha}}$ and $w(x) = Br_2^{\frac{2}{\alpha}}(r_2^2 - |x|^2)^{-\frac{2}{\alpha}}$, for $x \in \omega$. Since v is radially decreasing, we easily see that

$$-\Delta v + v^{\alpha+1} \geq A\left(A^\alpha - \frac{2(\alpha+2)}{\alpha^2}\right)(|x| - r_1)^{-\frac{2(\alpha+1)}{\alpha}} \geq 0\,.$$

Also,

$$\begin{aligned}-\Delta w + w^{\alpha+1} &= Br_2^{\frac{2}{\alpha}}(r_2^2 - |x|^2)^{-\frac{2(\alpha+1)}{\alpha}}\left[-\frac{4N}{\alpha}(r_2^2 - |x|^2) - \frac{8(\alpha+2)}{\alpha^2}|x|^2 + B^\alpha r_2^2\right]\\ &\geq Br_2^{\frac{2(\alpha+1)}{\alpha}}(r_2^2 - |x|^2)^{-\frac{2(\alpha+1)}{\alpha}}\left[-\frac{4N}{\alpha} - \frac{8(\alpha+2)}{\alpha^2} + B^\alpha\right] \geq 0\end{aligned}$$

by choosing

$$B^\alpha \geq \frac{4N}{\alpha} + \frac{8(\alpha+2)}{\alpha^2}\,.$$

Since $\varphi^{\alpha+1} \geq v^{\alpha+1} + w^{\alpha+1}$, we deduce that

$$-\Delta\varphi + \varphi^{\alpha+1} \geq 0\,,$$

in ω. We claim that

$$u(t,x) \leq \varphi(x)\,, \tag{2.4}$$

for $x \in \omega$ and $t > 0$. The result then follows by letting $r_2 \to \infty$ then $r_1 \downarrow 0$ in (2.4). We now prove the Claim (2.4). Setting $z = u - \varphi$, we observe that

$$z_t - \Delta z + (|u|^\alpha u - \varphi^{\alpha+1}) \leq 0\,, \tag{2.5}$$

in $L^2((0,\infty), H^{-1}(\omega))$. Consider a nondecreasing function $\theta \in C^\infty(\mathbb{R}) \cap L^\infty(\mathbb{R})$ such that $\theta(s) = 0$ for $s \leq 0$ and $|\theta'|$ is bounded and fix $0 < \sigma < \tau$. Since $\varphi(x) = \infty$ on $\partial\omega$, we see that $\theta(z(t))$ is supported in a fixed compact subset of ω for all $t \in [\sigma, \tau]$. Therefore, it follows from (2.5) that

$$\frac{d}{dt}\int_\omega \Theta(z(t)) \leq -\int_\omega \theta'(z)|\nabla z|^2 - \int_\omega (|u|^\alpha u - \varphi^{\alpha+1})\theta(z) \leq 0\,,$$

a.e. on (σ, τ), where $\Theta(s) = \int_0^s \theta(\mu)\,d\mu$. Therefore,

$$\int_\omega \Theta(z(\tau)) \leq \int_\omega \Theta(z(\sigma))\,. \tag{2.6}$$

Since $u(\sigma) \to u_0 \leq \varphi$ in $L^1(\omega)$ as $\sigma \downarrow 0$, and since $\Theta(s)$ is globally Lipschitz and vanishes for $s \leq 0$, we see that

$$\int_\omega \Theta(z(\sigma)) \underset{\sigma\downarrow 0}{\longrightarrow} 0\,. \tag{2.7}$$

By (2.6) and (2.7), we have

$$\int_\omega \Theta(z(\tau)) = 0\,,$$

for all $\tau > 0$. We may clearly choose θ such that $\Theta(s) > 0$ for $s > 0$, and we deduce that $z(\tau) \leq 0$ for all $\tau > 0$. This means that $u(\tau) \leq \varphi$.

STEP 2. Proof of (2.2). If $\psi(t,x) = (t + |x|^2)^{-\frac{1}{\alpha}}$, then

$$\begin{aligned}\psi_t - \Delta\psi &= \frac{1}{\alpha}\left(2N - 5 - \frac{4}{\alpha}\right)(t+|x|^2)^{-\frac{\alpha+1}{\alpha}} + \frac{4(\alpha+1)}{\alpha^2}t(t+|x|^2)^{-\frac{2\alpha+1}{\alpha}} \\ &\geq -4\frac{\alpha+1}{\alpha^2}(t+|x|^2)^{-\frac{\alpha+1}{\alpha}}\,.\end{aligned}$$

Also, $\psi^{\alpha+1} \ge (t+|x|^2)^{-\frac{\alpha+1}{\alpha}}$ so that, setting $\varphi^\varepsilon = (A+\varepsilon)\psi$ with $\varepsilon > 0$,

$$\varphi^\varepsilon_t - \Delta\varphi^\varepsilon + (\varphi^\varepsilon)^{\alpha+1} \ge (A+\varepsilon)\left((A+\varepsilon)^\alpha - \frac{4(\alpha+1)}{\alpha^2}\right)(t+|x|^2)^{-\frac{\alpha+1}{\alpha}} \ge 0\,.$$

Therefore, φ^ε is a supersolution of (1.1). Given $\delta > 0$, it follows from Step 1 (and the fact that $u(\delta) \in C_0(\mathbb{R}^N)$) that there exists $\tau > 0$ such that $\varphi^\varepsilon(\tau) \ge u(\delta)$. Therefore, $u(t+\delta) \le \varphi^\varepsilon(t+\tau) \le \varphi^\varepsilon(t)$. Letting $\delta \downarrow 0$, we deduce that $u(t) \le \varphi^\varepsilon(t)$ and the result follows by letting $\varepsilon \downarrow 0$.

STEP 3. Proof of (2.3). It follows from Step 1 that

$$u(t,x) \le A\rho^{-\frac{2}{\alpha}} \quad \text{for} \quad |x| = \rho \quad \text{and} \quad t > 0\,. \tag{2.8}$$

Let $v_0 \in C_0(\mathbb{R}^N)$, $v_0 \ge 0$ satisfy $v_0(x) = \varepsilon|x|^{-\frac{2}{\alpha}}$ for $|x| > \rho$ and let $v(t) = \mathcal{S}(t)v_0$. It follows that $v > 0$ and (see Proposition 5.5 in [4])

$$|x|^{\frac{2}{\alpha}} v(t,x) \underset{|x|\to\infty}{\longrightarrow} \varepsilon\,, \tag{2.9}$$

for all $t > 0$. Let $z_0(x) = e^{-|x|^2}$ and let $z(t) = \mathcal{S}(t)z_0$. We have $z > 0$ and (see Proposition 5.5 in [4])

$$|x|^{\frac{2}{\alpha}} z(t,x) \underset{|x|\to\infty}{\longrightarrow} 0\,, \tag{2.10}$$

for all $t > 0$. We observe that, since $\min_{\substack{0\le t\le t_0\\ |x|=\rho}} z(t,x) > 0$, it follows from (2.8) that we may choose $K = K(M,\rho,t_0) \ge 1$ sufficiently large so that

$$Kz(t,x) \ge u(t,x) \quad \text{for} \quad |x| = \rho \quad \text{and} \quad 0 < t < t_0\,. \tag{2.11}$$

Since $K \ge 1$, Kz is a supersolution of (1.1). Finally, let $r_2 > \rho$ and let w be as in Step 1, so that w is a also supersolution of (1.1) on $\{\rho < |x| < r_2\}$. Therefore, $\psi = v + Kz + w$ satisfies $\psi_t - \Delta\psi + \psi^{\alpha+1} \ge 0$ in $(0,t_0) \times \{\rho < |x| < r_2\}$, and by (2.11) $\psi(t,x) \ge u(t,x)$ for $0 < t < t_0$ and $|x| = \rho, r_2$. Since $\psi(0,x) \ge v_0(x) \ge u_0$ on $\{\rho < |x| < r_2\}$, we deduce (arguing as in Step 1) that $u(t_0,x) \le \psi(t_0,x)$ for $\rho < |x| < r_2$. Letting $r_2 \to \infty$, we see that $u(t_0,x) \le v + Kz$ for $|x| > \rho$. The result now follows from (2.9) and (2.10). □

COROLLARY 2.4. *If $u_0 \in \mathcal{W}$ and $u \in \Sigma(u_0)$, then*

$$|u(t,x)| \le A(t+|x|^2)^{-\frac{1}{\alpha}}\,, \tag{2.12}$$

with A as in (2.2). *In particular,*

$$\|D_{\sqrt{t}}u(t)\|_{\mathcal{W}\cap C_0} \le A\,, \tag{2.13}$$

with the notation of (1.5) *and* (1.23).

REMARK 2.5. Suppose $u_0 \in \mathcal{W}$ and let $u \in \Sigma(u_0)$. It follows from Corollary 2.4 that $u \in L^\infty((0,\infty) \times \{x \in \mathbb{R}^N : |x| > \epsilon\})$, for all $\epsilon > 0$, so that by Hölder's inequality $u(t) \to u_0$ in $L^p_{\mathrm{loc}}(\mathbb{R}^N \setminus \{0\})$ as $t \downarrow 0$ for all $p < \infty$. If, in addition, $u_0 \in C(\mathbb{R}^N \setminus \{0\})$, then it follows from Lemma 2.2 in [4] that $u(t) \to u_0$ in $C(\overline{\mathcal{O}})$ as $t \downarrow 0$ for all $\mathcal{O} \subset\subset \mathbb{R}^N \setminus \{0\}$.

LEMMA 2.6. *Let $\mathcal{O}$ be a smooth, bounded domain of $\mathbb{R}^N$, let $C, T > 0$ and set $Q = (0,T) \times \mathcal{O}$. Suppose $(u^n)_{n\geq 0} \subset L^\infty(Q) \cap C([0,T], L^1(\mathcal{O}))$, and assume there exists $u \in L^\infty(Q) \cap C((0,T], L^1(\mathcal{O}))$ such that $u^n \to u$ in $L^\infty((\varepsilon,T) \times \mathcal{O})$ for every $0 < \varepsilon < T$. Suppose further that every u^n is C^1 in $t \in (0,T)$ and C^2 in $x \in \mathcal{O}$ and satisfies $|\partial_t u_n - \Delta u_n| + |u_n| \leq C$ in Q. If $u^n(0) \to \varphi$ in $\mathcal{D}'(\mathcal{O})$ as $n \to \infty$, then $\|u(t) - \varphi\|_{L^1(\mathcal{O})} \to 0$ as $t \downarrow 0$.*

PROOF. By the maximum principle,

$$v^n_- \leq u^n \leq v^n_+ , \tag{2.14}$$

where $v^n_\pm$ is the solution of

$$\begin{cases} \partial_t v^n_\pm - \Delta v^n_\pm = \pm C & \text{in} \quad Q, \\ v^n_\pm = \pm C & \text{in} \quad (0,T) \times \partial\mathcal{O}, \\ v^n_\pm(0,x) = \varphi_n(x) & \text{in} \quad \mathcal{O}, \end{cases}$$

where $\varphi_n = u^n(0)$. Note that $\|\varphi_n\|_{L^\infty(\mathcal{O})} \leq C$ since $\|u^n\|_{L^\infty(Q)} \leq C$ and $u^n \in C([0,T], L^1(\mathcal{O}))$. If we denote by $\mathcal{T}(t)$ the heat semigroup with Dirichlet boundary condition in $\mathcal{O}$, then

$$v^n_\pm(t) = \pm C - \mathcal{T}(t)(\pm C - \varphi_n) \pm C \int_0^t \mathcal{T}(s)1\, ds . \tag{2.15}$$

Since $\pm C - \varphi_n$ converges to $\pm C - \varphi$ in $L^\infty(\mathcal{O})$ weak*, it follows easily from the compactness properties of $(\mathcal{T}(t))_{t\geq 0}$ that $\mathcal{T}(t)(\pm C - \varphi_n) \to \mathcal{T}(t)(\pm C - \varphi)$ in $L^\infty(\mathcal{O})$ (strong) for every $t > 0$. Therefore, we deduce from (2.14)-(2.15) that

$$v_- \leq u \leq v_+ , \tag{2.16}$$

where

$$v_\pm(t) = \pm C - \mathcal{T}(t)(\pm C - \varphi) \pm C \int_0^t \mathcal{T}(s)1\, ds . \tag{2.17}$$

We re-write (2.17) in the form

$$v_\pm(t) - \varphi = (\pm C - \varphi) - \mathcal{T}(t)(\pm C - \varphi) \pm C \int_0^t \mathcal{T}(s)1\, ds .$$

On one hand, the integral converges to 0 in $L^\infty(\mathcal{O})$ as $t \downarrow 0$. On the other hand, since $\pm C - \varphi \in L^\infty(\mathcal{O})$, it is clear that $\mathcal{T}(t)(\pm C - \varphi) \to \pm C - \varphi$ as $t \downarrow 0$ in $L^1(\mathcal{O})$. Therefore, $\|v_\pm(t) - \varphi\|_{L^1(\mathcal{O})} \to 0$ as $t \downarrow 0$. Since by (2.16), $v_- - \varphi \le u - \varphi \le v_+ - \varphi$, the result follows. □

PROOF OF THEOREM 2.2. It follows from Corollary 2.4 that there exists A independent of n such that

$$|u^n(t,x)| \le A(t + |x|^2)^{-\frac{1}{\alpha}}, \tag{2.18}$$

for all $x \in \mathbb{R}^N$, $t > 0$. Fix $\tau > 0$. We deduce from (2.18) that the $u^n(\tau/2)$ are uniformly bounded in $L^p(\mathbb{R}^N)$ for all $\max\{1, N\alpha/2\} < p \le \infty$. By standard smoothing effect, we see that the $u^n(\tau)$ are uniformly bounded in $W^{1,\infty}(\mathbb{R}^N)$. Using again (2.18) with $t = \tau$, we conclude that $\bigcup_{n\ge 0}\{u^n(\tau)\}$ is relatively compact in $C_0(\mathbb{R}^N)$. By continuous dependence in $C_0(\mathbb{R}^N)$ for (1.1), it follows that $\bigcup_{n\ge 0}\{u^n(\cdot)\}$ is relatively compact in $C([\tau, T], C_0(\mathbb{R}^N))$ for all $T > \tau$, the limit points being solutions of (1.1). Since, by (2.18), $\|u^n(t)\|_{L^\infty} \to 0$ as $t \to \infty$, uniformly in $n \ge 0$, we may let $T = \infty$ in the previous property. By letting $\tau \downarrow 0$ and using a diagonal procedure, we see that there exist a subsequence $(u^{n_k})_{k\ge 0}$ and a solution $u \in C((0,\infty), C_0(\mathbb{R}^N))$ of (1.1) such that $u^{n_k} \underset{k\to\infty}{\longrightarrow} u$ in $C([\tau,\infty), C_0(\mathbb{R}^N))$, for every $\tau > 0$. To see that $u \in \Sigma(u_0)$, we consider $\mathcal{O} \subset\subset \mathbb{R}^N \setminus \{0\}$. It follows from (2.18) that there exists a constant C such that $|u^n(t,x)| \le C$ for all $t > 0$, $x \in \mathcal{O}$ and $n \ge 0$. We conclude using Lemma 2.6.

Finally, we prove the last statement. Given $k \ge 0$, let $w^k = |u^{n_k} - u|/2$. It follows from Kato's parabolic inequality that

$$w^k_t - \Delta w^k + \frac{1}{2}\big|\,|u^{n_k}|^\alpha u^{n_k} - |u|^\alpha u\big| \le 0\,.$$

Since $\big|\,|u^{n_k}|^\alpha u^{n_k} - |u|^\alpha u\big| \ge 2^{-\alpha}|u^{n_k} - u|^{\alpha+1}$, we deduce that

$$w^k_t - \Delta w^k + (w^k)^{\alpha+1} \le 0\,. \tag{2.19}$$

We note that $w^k(t) \to |u_0^{n_k} - u_0|/2$ in $L^1_{\text{loc}}(\mathbb{R}^N \setminus \{0\})$ as $t \downarrow 0$, so that

$$w^k(0,x) \le M|x|^{-\frac{2}{\alpha}} \quad \text{and} \quad \operatorname*{ess\,sup}_{|x|>\rho} |x|^{\frac{2}{\alpha}} w^k(0,x) \le \varepsilon_k\,, \tag{2.20}$$

with $\varepsilon_k \to 0$ as $k \to \infty$ and $M = \sup_{n\ge 0}\|u_0^n\|_{\mathcal{W}}$. Fix $\tau, \varepsilon > 0$. It follows from (2.19), (2.20) and Lemma 2.3 that there exist $\Lambda \ge \rho$ and $k_0 \ge 0$ such that

$$w^k(\tau,x) \le \varepsilon(1 + |x|^2)^{-\frac{1}{\alpha}} \quad \text{for} \quad k \ge k_0 \quad \text{and} \quad |x| \ge \Lambda\,. \tag{2.21}$$

On the other hand, it follows from convergence in $C_0(\mathbb{R}^N)$ that, by possibly choosing k_0 larger, $w^k(\tau,x) \le \varepsilon(1 + \Lambda^2)^{-\frac{1}{\alpha}}$ for $k \ge k_0$ and $|x| < \Lambda$, so that we deduce from (2.21) that $\|w^k(\tau)\|_{\mathcal{W}\cap C_0} \le \varepsilon$ for $k \ge k_0$. The result follows, since $\varepsilon > 0$ is arbitrary. □

As noted in the introduction, Corollary 2.4 is the first key estimate needed in the study of $\omega(u)$ and $\omega_1(u)$. In particular, we can make the following immediate observations.

PROPOSITION 2.7. *Assume* $\alpha > 0$. *Let* $u_0 \in \mathcal{W}$ *and* $u \in \Sigma(u_0)$. *Let* $\omega(u)$ *and* $\omega_1(u)$ *be as in Definition* 1.2.

(i) $\omega(u)$ *and* $\omega_1(u)$ *are closed subsets respectively of* $C_0(\mathbb{R}^N)$ *and* $\mathcal{W} \cap C_0(\mathbb{R}^N)$.
(ii) $\omega(u)$ *and* $\omega_1(u)$ *are bounded sets in* $\mathcal{W} \cap C_0(\mathbb{R}^N)$.
(iii) $\bigcup_{t>0}\{D_{\sqrt{t}}u(t)\}$ *is a relatively compact subset of* $C_0(\mathbb{R}^N)$.
(iv) $\omega(u)$ *is a nonempty, compact, connected subset of* $C_0(\mathbb{R}^N)$.

PROOF.

(i) This is an immediate consequence of the definition.
(ii) This follows from Corollary 2.4.
(iii) It follows from (1.3) and (1.7) that

$$D_{\sqrt{1+s}}\mathcal{S}(s)[D_{\sqrt{t}}u(t)] = D_{\sqrt{(1+s)t}}u((1+s)t)\,, \tag{2.22}$$

for all $t > 0$, $s \geq 0$. In particular,

$$\bigcup_{t>0}\{D_{\sqrt{2}}\mathcal{S}(1)D_{\sqrt{t}}u(t)\} = \bigcup_{t>0}\{D_{\sqrt{2t}}u(2t)\} = \bigcup_{t>0}\{D_{\sqrt{t}}u(t)\}\,. \tag{2.23}$$

By Corollary 2.4, we know that $\bigcup_{t>0}\{D_{\sqrt{t}}u(t)\}$ is bounded in $\mathcal{W} \cap C_0(\mathbb{R}^N)$ and thus in $L^p(\mathbb{R}^N)$ for all $p > N\alpha/2$, $p \geq 1$. By standard smoothing properties, we deduce that the set $\bigcup_{t>0}\{D_{\sqrt{2}}\mathcal{S}(1)D_{\sqrt{t}}u(t)\}$ is relatively compact in $L^\infty_{\mathrm{loc}}(\mathbb{R}^N)$. Since it is also bounded in $\mathcal{W}$ by (2.23), it is relatively compact in $C_0(\mathbb{R}^N)$ and the result follows.
(iv) The set $\omega(u)$ is nonempty because of (iii). Furthermore, $\omega(u)$ is a closed subset of the compact set $\overline{\bigcup_{t>0}\{D\sqrt{t}u(t)\}}$, where the closure is in $C_0(\mathbb{R}^N)$, hence compact. Finally, since the map $t \mapsto D\sqrt{t}u(t)$ is continuous $(0,\infty) \to C_0(\mathbb{R}^N)$, it follows from the definition that $\omega(u)$ is connected in $C_0(\mathbb{R}^N)$. □

A more subtle property of the sets $\omega(u)$ and $\omega_1(u)$ is that they are invariant under translation of the solution u. At first glance this seems trivial: how could two translates of the same solution exhibit different asymptotic behaviors? On the other hand, the result is not completely obvious since in general $D_{\sqrt{t}}u(t) \neq D_{\sqrt{t}}u(s+t)$ for $s > 0$. Nonetheless, the result is true. To prove it, we first show that solutions $u \in \Sigma(u_0)$ are locally Hölder continuous as functions into $\mathcal{W} \cap C_0(\mathbb{R}^N)$, uniformly for u_0 in a bounded set of $\mathcal{W}$.

LEMMA 2.8. *Let* $\alpha > 0$. *There exists a constant* C *such that*

$$\|u(t) - u(s)\|_{\mathcal{W}\cap C_0} \leq C(1 + \|u_0\|_{\mathcal{W}})\sqrt{t-s}\,, \tag{2.24}$$

for all $u_0 \in \mathcal{W}$, $u \in \Sigma(u_0)$ *and all* $1/2 \leq s \leq t \leq 1$.

PROOF. It follows from Corollary 2.4 that

$$\|u(\theta)\|_{\mathcal{W}\cap C_0} \le C(1 + \|u_0\|_{\mathcal{W}}) , \tag{2.25}$$

for all $1/4 \le \theta \le 1$. Since $|u(t,x)| \le (\alpha t)^{-\frac{1}{\alpha}}$ (note that $(\alpha t)^{-\frac{1}{\alpha}}$ is a solution of (1.1)), we deduce that

$$\| \, |u(\theta)|^{\alpha+1} \|_{\mathcal{W}\cap C_0} \le C(1 + \|u_0\|_{\mathcal{W}}) , \tag{2.26}$$

for all $1/4 \le \theta \le 1$. Let now $1/2 \le s < t \le 1$ and $1/4 \le \tau < s$. We write

$$\begin{aligned} u(t) - u(s) = (e^{(t-\tau)\Delta} - e^{(s-\tau)\Delta})u(\tau) - \int_s^t e^{(t-\sigma)\Delta} |u|^\alpha u(\sigma) \\ - \int_\tau^s (e^{(t-\sigma)\Delta} - e^{(s-\sigma)\Delta}) |u|^\alpha u(\sigma) = a_1 - a_2 - a_3 , \end{aligned}$$

and we estimate separately the a_j's. It follows from (2.25) and formula (4.2) in [5] that

$$\|a_1\|_{\mathcal{W}\cap C_0} \le C(1 + \|u_0\|_{\mathcal{W}}) \frac{t-s}{s-\tau} . \tag{2.27}$$

Next, it follows from (2.26) and formula (4.1) in [5] that

$$\|a_2\|_{\mathcal{W}\cap C_0} \le C(1 + \|u_0\|_{\mathcal{W}})(t-s) , \tag{2.28}$$

and

$$\|a_3\|_{\mathcal{W}\cap C_0} \le C(1 + \|u_0\|_{\mathcal{W}})(s-\tau) , \tag{2.29}$$

The result follows from (2.27), (2.28) and (2.29), by letting for example $s - \tau = \sqrt{t-s}/4$. □

PROPOSITION 2.9. *Let $\alpha > 0$. Let $u_0 \in \mathcal{W}$ and $u \in \Sigma(u_0)$. It follows that for any $\tau \ge 0$,*

$$\|D_{\sqrt{t}} u(t+\tau) - D_{\sqrt{t}} u(t)\|_{\mathcal{W}\cap C_0} \underset{t\to\infty}{\longrightarrow} 0 .$$

In particular, $\omega(u_\tau) = \omega(u)$ and $\omega_1(u_\tau) = \omega_1(u)$, where $u_\tau(t) \equiv u(\tau + t)$ and ω and ω_1 are as in Definition 1.2.

PROOF. Let $\tau \ge 0$ and $t \ge 1$, $t \ge 2\tau$. Since

$$\begin{aligned} D_{\sqrt{t}} u(t+\tau) - D_{\sqrt{t}} u(t) &= D_{\sqrt{t}} \mathcal{S}\left(\frac{t}{2} + \tau\right) u\left(\frac{t}{2}\right) - D_{\sqrt{t}} \mathcal{S}\left(\frac{t}{2}\right) u\left(\frac{t}{2}\right) \\ &= \mathcal{S}\left(\frac{1}{2} + \frac{\tau}{t}\right) D_{\sqrt{t}} u\left(\frac{t}{2}\right) - \mathcal{S}\left(\frac{1}{2}\right) D_{\sqrt{t}} u\left(\frac{t}{2}\right) , \end{aligned}$$

by (1.7), we deduce from (2.24) that

$$\|D_{\sqrt{t}}u(t+\tau) - D_{\sqrt{t}}u(t)\|_{\mathcal{W}\cap C_0} \le C(1 + \|D_{\sqrt{t}}u(t/2)\|_{\mathcal{W}})\sqrt{\frac{\tau}{t}}\,.$$

The result follows, since $\|D_{\sqrt{t}}u(t/2)\|_{\mathcal{W}} = \|D_{\sqrt{\frac{t}{2}}}u(t/2)\|_{\mathcal{W}} \le C(1 + \|u_0\|_{\mathcal{W}})$ by (2.13). □

REMARK 2.10. Assume that $0 < \alpha < 4/N$. Let $u, v \in C((0,\infty), C_0(\mathbb{R}^N))$ be solutions of (1.1) such that $u(t), v(t) \in \mathcal{W}$ for all $t > 0$. It follows (since $\alpha < 4/N$) that u and v are L^2 solutions of (1.1). Therefore, there is backward uniqueness (see [1], [12]). In particular, if $u(t_0) = v(t_0)$ for some $t_0 > 0$, then $u(t) = v(t)$ for all $t \in (0,\infty)$.

We end this section with a useful property of self-similar solutions of (1.1).

PROPOSITION 2.11. *Let* $\alpha > 0$. *Given* $f \in C_0(\mathbb{R}^N)$*, the following properties are equivalent.*

(i) f *is the profile of a self-similar solution of* (1.1).
(ii) $\mathcal{S}(\tau)D_{\frac{1}{\sqrt{t}}}f = D_{\frac{1}{\sqrt{t+\tau}}}f$ *for all* $t, \tau > 0$.
(iii) $D_{\sqrt{1+s}}\mathcal{S}(s)f = f$, *for all* $s \ge 0$.

PROOF. (i) means that $u(t) = D_{\frac{1}{\sqrt{t}}}f$ is a solution of (1.1), which is clearly equivalent to (ii). Let now $s > 0$. Letting $t = 1/(1+s)$ and $\tau = s/(1+s)$ in (ii) and applying (1.7) to the left hand side, we obtain (iii). Conversely, let $t, \tau > 0$ and set $s = \tau/t$. Applying $D_{\frac{1}{\sqrt{t+\tau}}}$ to (iii) and using (1.7) in the left hand side, we obtain (ii). □

3. – The case $\alpha \ge 2/N$

Throughout this section, we assume $\alpha \ge 2/N$. We begin by showing that equation (1.1), which determines the semiflow $\mathcal{S}(t)$ on $C_0(\mathbb{R}^N)$, also determines a semiflow in a natural and unique way on $\mathcal{W}$ which agrees with $\mathcal{S}(t)$ on $\mathcal{W} \cap C_0(\mathbb{R}^N)$. We use the same notation $\mathcal{S}(t)$ to denote the semiflow on $\mathcal{W}$ (see Definition 3.2 below).

PROPOSITION 3.1. *Assume* $\alpha \ge 2/N$. *Given* $u_0 \in \mathcal{W}$*, there exists a unique solution* $u \in C((0,\infty), C_0(\mathbb{R}^N))$ *of* (1.1) *such that* $u(t) \to u_0$ *in* $L^1_{\text{loc}}(\mathbb{R}^N \setminus \{0\})$ *as* $t \downarrow 0$.

PROOF. Existence follows immediately from Theorem 2.2, by letting $u_0^n = \min\{n, \max\{u_0, -n\}\}$ and u^n the corresponding classical solution of (1.1).

We now prove uniqueness. In the case $\alpha > 2/N$ we observe that $u_0 \in L^1(\mathbb{R}^N) + L^p(\mathbb{R}^N)$ for all $p > N\alpha/2$. Consider a solution $u \in C((0,\infty), C_0(\mathbb{R}^N))$

of (1.1) such that $u(t) \to u_0$ in $L^1_{\mathrm{loc}}(\mathbb{R}^N \setminus \{0\})$ as $t \downarrow 0$. It follows from Corollary 2.4 that $|u(t,x)| \le A(t+|x|^2)^{-\frac{1}{\alpha}}$, and we deduce easily that $u(t) \to u_0$ in $L^1_{\mathrm{loc}}(\mathbb{R}^N)$, and in particular $|u(t) - u_0| \to 0$ in $\mathcal{S}'(\mathbb{R}^N)$. Therefore, u is the solution obtained in Theorem 8.8 in [4]. Hence uniqueness.

In the case $\alpha = 2/N$, uniqueness follows from an obvious modification of the argument of Brezis and Friedman [2]. For completeness, we give the details in the appendix to this paper. □

Definition 3.2. Assume $\alpha \ge 2/N$. Given $u_0 \in \mathcal{W}$, we set

$$\mathcal{S}(t)u_0 = u(t)\,, \tag{3.1}$$

for all $t > 0$, where $u \in C((0,\infty), C_0(\mathbb{R}^N))$ is the unique solution of (1.1) such that $u(t) \to u_0$ in $L^1_{\mathrm{loc}}(\mathbb{R}^N \setminus \{0\})$ as $t \downarrow 0$.

Remark 3.3.

(i) It follows from Corollary 2.4 that $\mathcal{S}(t)u_0 \in \mathcal{W} \cap C_0(\mathbb{R}^N)$ for all $t > 0$ and all $u_0 \in \mathcal{W}$.
(ii) By the scaling invariance of (1.1), i.e. formula (1.6), and uniqueness, we see that

$$D_\lambda \mathcal{S}(\lambda^2 \tau) = \mathcal{S}(\tau) D_\lambda\,, \tag{3.2}$$

on $\mathcal{W}$ for all $\lambda, \tau > 0$. In particular,

$$D_{\sqrt{t}} \mathcal{S}(t) = \mathcal{S}(1) D_{\sqrt{t}}\,, \tag{3.3}$$

on $\mathcal{W}$ for all $t > 0$.

We now give an application of Theorem 2.2

Proposition 3.4. *Assume $\alpha \ge 2/N$ and let $\mathcal{S}(t)$ be as in Definition 3.2. Fix $t > 0$ and $M > 0$.*

(i) *$\mathcal{S}(t)$ is continuous $(\mathcal{B}_M, \mathrm{d}^*_M) \to C_0(\mathbb{R}^N)$.*
(ii) *If $\alpha < 4/N$, then $\mathcal{S}(t)$ is a homeomorphism of $(\mathcal{B}_M, \mathrm{d}^*_M)$ onto $\mathcal{S}(t)\mathcal{B}_M \subset C_0(\mathbb{R}^N)$, this latter space considered with its norm topology.*
(iii) *$\mathcal{S}(t)$ is continuous $(\mathcal{B}_M, \mathrm{d}_M) \to \mathcal{W} \cap C_0(\mathbb{R}^N)$.*

Proof. By uniqueness in Proposition 3.1, the limit given by Theorem 2.2 is determined by u_0. In particular, it does not depend on the subsequence $(u^{n_k})_{k\ge 0}$, so that the whole sequence $(u^n)_{n\ge 0}$ converges. Therefore, (i) and (iii) follow from Theorem 2.2. Statement (ii) follows from (i), since a continuous, injective (by Remark 2.10), surjective map of a compact Hausdorff space onto a Hausdorff space is a homeomorphism. □

Proposition 3.5. *Assume $\alpha \ge 2/N$. A function $f \in \mathcal{W} \cap C_0(\mathbb{R}^N)$ is the profile of a self-similar solution of* (1.1) *if and only if there exists $\varphi \in \mathcal{W}$ homogeneous of degree $-2/\alpha$ such that $f = \mathcal{S}(1)\varphi$.*

PROOF. If φ is homogeneous, then $u(t) = \mathcal{S}(t)\varphi$ is self-similar by uniqueness. Conversely, we observe that by weak* compactness there exist $\varphi \in \mathcal{W}$ and $t_n \downarrow 0$ such that $D_{\frac{1}{\sqrt{t_n}}} f \to \varphi$ weak*. Therefore, by Propositions 2.11 (ii) and 3.4 (i), we see that

$$D_{\frac{1}{\sqrt{t}}} f = \lim_{n\to\infty} D_{\frac{1}{\sqrt{t+t_n}}} f = \lim_{n\to\infty} \mathcal{S}(t) D_{\frac{1}{\sqrt{t_n}}} f = \mathcal{S}(t)\varphi\,,$$

where the limits are in $C_0(\mathbb{R}^N)$. It follows that the solution $u(t) = \mathcal{S}(t)\varphi$ is self-similar. Moreover, φ must be homogeneous since $u(t) = D_\lambda(u(\lambda^2 t))$, for all $\lambda > 0$, and $u(t) \to \varphi$ in $L^1_{\mathrm{loc}}(\mathbb{R}^N \setminus \{0\})$ as $t \to 0$. □

REMARK 3.6. If $\varphi \in \mathcal{W} \cap C(\mathbb{R}^N \setminus \{0\})$ and $u(t) = \mathcal{S}(t)\varphi$, then $u(t) \to \varphi$ in $C(S^{N-1})$ as $t \downarrow 0$ by Remark 2.5. If, in addition, φ is homogeneous of degree $-2/\alpha$, so that $u(t)$ is self-similar, it follows that $|x|^{\frac{2}{\alpha}} u(t,x) - \varphi(x/|x|) \to 0$ as $x \to \infty$ for any fixed $t > 0$ (since $|x|^{\frac{2}{\alpha}} u(t,x) = u(t/|x|^2, x/|x|)$).

LEMMA 3.7. *If $\alpha > 2/N$, then the operators $D_{\sqrt{1+s}}\mathcal{S}(s)$ are Lipschitz continuous $\mathcal{W} \cap C_0(\mathbb{R}^N) \to \mathcal{W} \cap C_0(\mathbb{R}^N)$, uniformly in $s \geq 0$. If $\alpha = 2/N$, these operators are uniformly equicontinuous $\mathcal{W} \cap C_0(\mathbb{R}^N) \to \mathcal{W} \cap C_0(\mathbb{R}^N)$.*

PROOF. Let $f, g \in \mathcal{W} \cap C_0(\mathbb{R}^N)$, so that

$$|f - g| \leq (1 + |x|^2)^{-\frac{1}{\alpha}} \|f - g\|_{\mathcal{W}\cap C_0}\,. \tag{3.4}$$

We next observe that by (2.1) and (3.4),

$$\begin{aligned} |D_{\sqrt{1+s}}\mathcal{S}(s) f - D_{\sqrt{1+s}}\mathcal{S}(s) g| &\leq 2 D_{\sqrt{1+s}}\mathcal{S}(s)|f - g| \\ &\leq 2 D_{\sqrt{1+s}}\mathcal{S}(s)[(1 + |\cdot|^2)^{-\frac{1}{\alpha}} \|f - g\|_{\mathcal{W}\cap C_0}]\,. \end{aligned} \tag{3.5}$$

If $\alpha > 2/N$, it follows from Corollary 8.3 in [4] that there exists L such that $D_{\sqrt{1+s}} e^{s\Delta}(1 + |\cdot|^2)^{-\frac{1}{\alpha}} \leq L(1 + |x|^2)^{-\frac{1}{\alpha}}$, i.e. $\|D_{\sqrt{1+s}} e^{s\Delta}(1 + |\cdot|^2)^{-\frac{1}{\alpha}}\|_{\mathcal{W}\cap C_0} \leq L$. Since $\mathcal{S}(s)(1 + |\cdot|^2)^{-\frac{1}{\alpha}} \leq e^{s\Delta}(1 + |\cdot|^2)^{-\frac{1}{\alpha}}$, we then deduce from (3.5) that $\|D_{\sqrt{1+s}}\mathcal{S}(s) f - D_{\sqrt{1+s}}\mathcal{S}(s) g\|_{\mathcal{W}\cap C_0} \leq L\|f - g\|_{\mathcal{W}\cap C_0}$, which is the desired estimate. In the case $\alpha = 2/N$, we need only show in view of (3.5) that $\|D_{\sqrt{1+s}}\mathcal{S}(s)[\varepsilon(1 + |\cdot|^2)^{-\frac{1}{\alpha}}]\|_{\mathcal{W}\cap C_0} \to 0$ as $\varepsilon \downarrow 0$, uniformly in $s \geq 0$. Given $\delta > 0$, let

$$\mu(\delta) = \inf_{x\in\mathbb{R}^N} (1 + |x|^2)^{\frac{1}{\alpha}} \mathcal{S}(1)(\delta|\cdot|^{-\frac{2}{\alpha}})(x)\,.$$

Since, by Remark 3.6, $(1 + |x|^2)^{\frac{1}{\alpha}} \mathcal{S}(1)(\delta|\cdot|^{-\frac{2}{\alpha}})(x) \to \delta$ as $|x| \to \infty$, we see that $\mu(\delta) > 0$, $\mu(\delta) \to 0$ as $\delta \downarrow 0$ and that μ is increasing (for the last property we also use the maximum principle for x in a bounded region). Therefore, if $\varepsilon_0 > 0$ is small, we may consider the increasing function $\delta = \mu^{-1}$, defined $(0, \varepsilon_0) \to (0, \infty)$ and such that $\delta(\varepsilon) \to 0$ as $\varepsilon \downarrow 0$. By definition, we have $\varepsilon(1+|x|^2)^{-\frac{1}{\alpha}} \le \mathcal{S}(1)[\delta(\varepsilon)|\cdot|^{-\frac{2}{\alpha}}]$. Therefore, $D_{\sqrt{1+s}}\mathcal{S}(s)[\varepsilon(1+|\cdot|^2)^{-\frac{1}{\alpha}}] \le D_{\sqrt{1+s}}\mathcal{S}(1+s)[\delta(\varepsilon)|\cdot|^{-\frac{2}{\alpha}}] = \mathcal{S}(1)D_{\sqrt{1+s}}[\delta(\varepsilon)|\cdot|^{-\frac{2}{\alpha}}] = \mathcal{S}(1)[\delta(\varepsilon)|\cdot|^{-\frac{2}{\alpha}}]$. The result follows, since $\delta(\varepsilon) \to 0$ as $\varepsilon \downarrow 0$ and, by Lemma 3.3 in [4], $\|\mathcal{S}(1)(\delta|\cdot|^{-\frac{2}{\alpha}})\|_{\mathcal{W}\cap C_0} \underset{\delta\downarrow 0}{\longrightarrow} 0$. □

The following proposition is analogous to Proposition 3.6 in [5].

PROPOSITION 3.8. *Assume $\alpha \ge 2/N$ and let $u_0 \in \mathcal{W}$. Set $u(t) = \mathcal{S}(t)u_0$ where $\mathcal{S}(t)$ is as in Definition* 3.2 *and let $\omega(u)$ and $\omega_1(u)$ be defined by* (1.25) *and* (1.26), *respectively.*

(i) *If $f \in \omega(u)$, then $D_{\sqrt{1+s}}\mathcal{S}(s)f \in \omega(u)$ for all $s \ge 0$.*
(ii) *If $\omega(u) = \{f\}$, then $f = \mathcal{S}(1)\varphi$ with $\varphi \in \mathcal{W}$ homogeneous of degree $-2/\alpha$. Moreover, $D_{\sqrt{t}}u(t) = \mathcal{S}(1)D_{\sqrt{t}}u_0 \to f$ in $L^\infty(\mathbb{R}^N)$ as $t \to \infty$.*
(iii) *If $\alpha > 2/N$ and $u_0 \in L^p(\mathbb{R}^N)$, for some p with $1 \le p \le N\alpha/2$, then $\omega(u) = \{0\}$.*
(iv) *If $\omega_1(u) \ne \emptyset$, then $\omega_1(u) = \overline{\bigcup_{s\ge 0}\{D_{\sqrt{1+s}}\mathcal{S}(s)f\}} = \bigcap_{s_0\ge 0}\overline{\bigcup_{s\ge s_0}\{D_{\sqrt{1+s}}\mathcal{S}(s)f\}}$ for every $f \in \omega_1(u)$, where the closures are in $\mathcal{W}\cap C_0(\mathbb{R}^N)$. Moreover, $\inf_{w\in\omega_1(u)}\|D_{\sqrt{t}}u(t) - w\|_{\mathcal{W}\cap C_0} \underset{t\to\infty}{\longrightarrow} 0$.*
(v) *If $\omega_1(u) = \{f\}$, then $f = \mathcal{S}(1)\varphi$ with $\varphi \in \mathcal{W}$ homogeneous of degree $-2/\alpha$.*
(vi) *If $f \in \omega_1(u)$ and $f = \mathcal{S}(1)\varphi$ with $\varphi \in \mathcal{W}$ homogeneous of degree $-2/\alpha$, then $\|D_{\sqrt{t}}u(t) - f\|_{\mathcal{W}\cap C_0} \to 0$ as $t \to \infty$. In particular, $\omega_1(u) = \omega(u) = \{f\}$.*
(vii) *If $\omega_1(u) \ne \emptyset$, then $\omega(u) = \overline{\omega_1(u)}$, where the closure is in $C_0(\mathbb{R}^N)$.*

PROOF.

(i) This follows since, by (2.22),

$$D_{\sqrt{1+s}}\mathcal{S}(s)[D_{\sqrt{t_n}}u(t_n)] = D_{\sqrt{(1+s)t_n}}u((1+s)t_n)\,, \tag{3.6}$$

and, by (2.1), $D_{\sqrt{1+s}}\mathcal{S}(s)$ is continuous $C_0(\mathbb{R}^N) \to C_0(\mathbb{R}^N)$.

(ii) It follows from (i), setting $s = \tau/t$ and applying (1.) with t replaced by τ and $\lambda = 1/\sqrt{t}$, that $\mathcal{S}(\tau)D_{\frac{1}{\sqrt{t}}}f = D_{\frac{1}{\sqrt{t+\tau}}}f$ for all $t,\tau > 0$. Since $f \in \mathcal{W}$ by Proposition 2.7 (ii), the first statement follows from Propositions 3.5 and 2.11 (iii). The second statement follows by relative compactness (Proposition 2.7 (iii)).

(iii) This follows from the fact that $t^{\frac{1}{\alpha}}\|u(t)\|_{L^\infty} \le t^{\frac{1}{\alpha}}\|e^{t\Delta}|u_0|\,\|_{L^\infty} \underset{t\to\infty}{\longrightarrow} 0$.

(iv) Let $f \in \omega_1(u)$ and set $E(s_0) = \bigcup_{s\ge s_0}\{D_{\sqrt{1+s}}\mathcal{S}(s)f\}$. We first show that $\omega_1(u) = \overline{E(s_0)}$, for all $s_0 \ge 0$. We deduce from (1.26), (3.6) and the continuity of $D_{\sqrt{1+s}}\mathcal{S}(s) : \mathcal{W}\cap C_0(\mathbb{R}^N) \to \mathcal{W}\cap C_0(\mathbb{R}^N)$ (see Proposition 2.1) that $E(s_0) \subset E(0) \subset \omega_1(u)$ and, since $\omega_1(u)$ is closed in $\mathcal{W}\cap C_0(\mathbb{R}^N)$, $\overline{E(s_0)} \subset \omega_1(u)$. To show the reverse inclusion, let $s_0 \ge 0$ and consider a sequence $t_n \to \infty$ such that

$$\|D_{\sqrt{t_n}}u(t_n) - f\|_{\mathcal{W}\cap C_0} \underset{n\to\infty}{\longrightarrow} 0\,.$$

Applying (3.6) and Lemma 3.7, we deduce that

$$\sup_{s\geq 0}\|D_{\sqrt{(1+s)t_n}}u((1+s)t_n)-D_{\sqrt{1+s}}\mathcal{S}(s)f\|_{\mathcal{W}\cap C_0}$$
$$=\sup_{s\geq 0}\|D_{\sqrt{1+s}}\mathcal{S}(s)D_{\sqrt{t_n}}u(t_n)-D_{\sqrt{1+s}}\mathcal{S}(s)f\|_{\mathcal{W}\cap C_0}\underset{n\to\infty}{\longrightarrow}0\,.$$

In particular,

$$\sup_{s\geq s_0}\inf_{w\in E(s_0)}\|D_{\sqrt{(1+s)t_n}}u((1+s)t_n)-w\|_{\mathcal{W}\cap C_0}\underset{n\to\infty}{\longrightarrow}0\,,$$

so that $\omega_1(u)\subset\overline{E(s_0)}$. The last statement in (iv) easily follows from the previous estimate.

(v) This follows from exactly the same argument of the analogous property in (ii), except using (iv) instead of (i).

(vi) This follows from (iv) and Proposition 2.11 (iii).

(vii) It follows from the last property in (iv) that $\omega(u)\subset\overline{\omega_1(u)}$. On the other hand, $\omega_1(u)\subset\omega(u)$ and, since $\omega(u)$ is closed, $\overline{\omega_1(u)}\subset\omega(u)$. □

REMARK 3.9. It follows from part (iv) of Proposition 3.8 that if $f\in\omega_1(u)$, then there exist $s_n\to\infty$ such that $f=\lim_{n\to\infty}D_{\sqrt{1+s_n}}\mathcal{S}(s_n)f$ in $\mathcal{W}\cap C_0(\mathbb{R}^N)$.

Finally, we state and prove the main results of this section.

THEOREM 3.10. *Assume $\alpha\geq 2/N$ and let $\mathcal{S}(t)$ be as in Definition 3.2. Let $u_0\in\mathcal{W}$, set $u(t)=\mathcal{S}(t)u_0$ and let $\Omega(u_0)$ and $\omega(u)$ be defined by (1.20) and (1.25), respectively.*

(i) *$\omega(u)=\mathcal{S}(1)\Omega(u_0)$. In particular, if $u_0\in\mathcal{B}_M$ for some $M>0$, then $\omega(u)\subset\mathcal{S}(1)\mathcal{B}_M$.*

(ii) *If $\alpha<4/N$, then $\mathcal{S}(1)$ is a homeomorphism of $\Omega(u_0)\subset(\mathcal{B}_M,\mathrm{d}_M^*)$ onto $\omega(u)\subset C_0(\mathbb{R}^N)$, this latter space considered with its norm topology.*

PROOF. Let $z\in\Omega(u_0)$. It follows that there exists $\lambda_n\to\infty$ such that $D_{\lambda_n}u_0\to z$ weak*. Since $\mathcal{S}(1)$ is continuous $(\mathcal{B}_M,\mathrm{d}_M^*)\to C_0(\mathbb{R}^N)$ (by Proposition 3.4 (i)), we deduce from (3.3) that $D_{\lambda_n}\mathcal{S}(\lambda_n^2)u_0=\mathcal{S}(1)D_{\lambda_n}u_0\to\mathcal{S}(1)z$ in $L^\infty(\mathbb{R}^N)$, i.e. $\mathcal{S}(1)z\in\omega(u)$. Thus $\mathcal{S}(1)\Omega(u_0)\subset\omega(u)$. Conversely, let $f\in\omega(u)$. It follows from (1.25) and (3.6) that there exists $\lambda_n\to\infty$ such that $\mathcal{S}(1)D_{\lambda_n}u_0\to f$ uniformly. By weak* compactness, there exists a subsequence $(\lambda_{n_k})_{k\geq 0}$ and $z\in\mathcal{W}$ such that $D_{\lambda_{n_k}}u_0\to z$ weak*. It follows that $z\in\Omega(u_0)$ and, by the previous argument, we see that $\mathcal{S}(1)D_{\lambda_{n_k}}u_0\to\mathcal{S}(1)z$ uniformly. Therefore, $f=\mathcal{S}(1)z\in\mathcal{S}(1)\Omega(u_0)$. The second statement follows from (i) and Proposition 3.4 (ii). □

COROLLARY 3.11. *Assume $\alpha\geq 2/N$ and let $M>0$. There exists $U_0\in\mathcal{B}_M\cap C^\infty(\mathbb{R}^N)$ such that $\omega(U)=\mathcal{S}(1)\mathcal{B}_M$, where $U=\mathcal{S}(\cdot)U_0$. In particular, given any $u_0\in\mathcal{B}_M$, there exist $t_n\to\infty$ such that $D_{\sqrt{t_n}}U(t_n)\to\mathcal{S}(1)u_0$ in $C_0(\mathbb{R}^N)$, or equivalently*

$$\|U(t_n)-\mathcal{S}(t_n)D_{\frac{1}{\sqrt{t_n}}}u_0\|_{L^\infty}=\|U(t_n)-D_{\frac{1}{\sqrt{t_n}}}\mathcal{S}(1)u_0\|_{L^\infty}=o(t_n^{-\frac{1}{\alpha}})\,,$$

as $n\to\infty$.

PROOF BY THEOREM 1.2. in [5], there exists $U_0 \in \mathcal{B}_M \cap C^\infty(\mathbb{R}^N)$ such that $\Omega(U_0) = \mathcal{B}_M$, and the result follows from Theorem 3.10 □

REMARK 3.12. The solution $U(t) = \mathcal{S}(t)U_0$ given by Corollary 3.11 is "universal", in the sense that $\mathcal{S}(1)u_0 \in \omega(U)$ and $\omega(u) \subset \omega(U)$ for all $u_0 \in \mathcal{B}_M \cap C_0(\mathbb{R}^N)$, where $u(t) = \mathcal{S}(t)u_0$. Moreover, $\omega_1(U) \neq \omega(U)$. To see this, recall that if $\varphi \in \mathcal{B}_M$ is homogeneous of degree $-\sigma$ (for example $\varphi = 0$), then $\mathcal{S}(1)\varphi \notin \omega_1(U)$, for otherwise Proposition 3.8 (vi) would imply that $\omega(U) = \{\mathcal{S}(1)\varphi\}$. We do not know if $\omega_1(U) = \emptyset$.

COROLLARY 3.13. *Assume $\alpha \geq 2/N$. Let $M > 0$ and suppose that H is a nonempty, compact, connected subset of $(\mathcal{B}_M, \mathrm{d}_M^*)$ such that every $\varphi \in H$ is homogeneous of degree $-2/\alpha$. Then there exists $V_0 \in \mathcal{B}_M \cap C_0(\mathbb{R}^N)$ such that $\omega(V) = \mathcal{S}(1)H$, where $V(t) = \mathcal{S}(t)V_0$. In particular, given any $\varphi \in H$, there exist $t_n \to \infty$ such that $D_{\sqrt{t_n}}V(t_n) \to \mathcal{S}(1)\varphi$ in $C_0(\mathbb{R}^N)$, or equivalently*

$$\|V(t_n) - \mathcal{S}(t_n)\varphi\|_{L^\infty} = o(t_n^{-\frac{1}{\alpha}})\,.$$

In other words, $V(t)$ is asymptotic, along an appropriate subsequence, to every possible self-similar solution of (1.1) *with initial value in H. In addition, if $H \subset C(\mathbb{R}^N \setminus \{0\})$, then we may choose $V_0 \in \mathcal{B}_M \cap C_0(\mathbb{R}^N)$. Furthermore, if H contains at least two elements, then $\omega_1(V) = \emptyset$.*

PROOF. It follows from Proposition 2.9 in [5] that there exists $V_0 \in \mathcal{B}_M$ ($V_0 \in \mathcal{B}_M \cap C_0(\mathbb{R}^N)$ if $H \subset C(\mathbb{R}^N \setminus \{0\})$) such that $\Omega(V_0) = H$. The result is then a consequence of Theorem 3.10 (the last property follows from Proposition 3.8 (vi)). □

We recall that the discrete dynamical system given by the mapping $F = D_\lambda$ on $(\mathcal{B}_M, \mathrm{d}_M^*)$, for any fixed $\lambda \neq 1$, is an example of chaos, as defined in Devaney [8] (see Proposition 2.11 in [5]).

COROLLARY 3.14. *Assume $\alpha \geq 2/N$. Given $M > 0$ and $\lambda > 1$, the map $D_\lambda : \mathcal{B}_M \to \mathcal{B}_M$ becomes, under the action of $\mathcal{S}(1) : (\mathcal{B}_M, \mathrm{d}_M^*) \to \mathcal{S}(1)\mathcal{B}_M$,*

$$F_\lambda = D_\lambda \mathcal{S}(\lambda^2 - 1) = \mathcal{S}\left(1 - \frac{1}{\lambda^2}\right) D_\lambda\,.$$

Moreover, the mapping F_λ of $\mathcal{S}(1)\mathcal{B}_M$ is chaotic.

PROOF. By (3.3), $\mathcal{S}(1)D_\lambda = D_\lambda\mathcal{S}(\lambda^2) = F_\lambda\mathcal{S}(1)$, and the first statement follows. The second statement follows from Propositions 4.5 (ii) and 2.11 in [5]. (We note that if $z = m|x|^{-\frac{2}{\alpha}}$ and $\widetilde{z} = \widetilde{m}|x|^{-\frac{2}{\alpha}}$ with $m \neq \widetilde{m}$, then the corresponding solutions of (1.1) are self-similar with profiles $\mathcal{S}(1)z$ and $\mathcal{S}(1)\widetilde{z}$, so that $\mathcal{S}(1)z \neq \mathcal{S}(1)\widetilde{z}$.) □

COROLLARY 3.15. *Assume $2/N \leq \alpha < 4/N$. It follows that $\Omega(\mathcal{S}(t_0)u_0) = \Omega(u_0)$ for all $t_0 \geq 0$ and all $u_0 \in \mathcal{W}$.*

PROOF. It follows from Proposition 2.9 that $\omega(\mathcal{S}(\cdot + t_0)u_0) = \omega(\mathcal{S}(\cdot)u_0)$. The result now follows from Theorem 3.10. □

Next, we turn our attention to $\omega_1(u)$. Since $(\mathcal{B}_M, \mathrm{d}_M)$ is not compact, the relationship between $\Omega_1(u_0)$ and $\omega_1(u)$ is not nearly as strong as that between $\Omega(u_0)$ and $\omega(u)$, as the following theorem and subsequent remark show. Nonetheless, Proposition 3.19 below describes conditions under which $\mathcal{U}(1)\Omega_1(u_0) = \omega_1(u)$ and $\omega(u) = \omega_1(u)$.

THEOREM 3.16. *Assume $\alpha \geq 2/N$ and let $\mathcal{S}(t)$ be as in Definition 3.2. Let $u_0 \in \mathcal{W}$, set $u(t) = \mathcal{S}(t)u_0$ and let $\Omega_1(u_0)$ and $\omega_1(u)$ be defined by (1.21) and (1.26), respectively. If $\Omega_1(u_0) \neq \emptyset$, then $\omega_1(u) = \overline{\mathcal{S}(1)\Omega_1(u_0)}$, where the closure is in $\mathcal{W} \cap C_0(\mathbb{R}^N)$.*

PROOF. Given $z \in \Omega_1(u_0)$, there exists $\lambda_n \to \infty$ such that $\mathrm{d}_M(D_{\lambda_n}u_0, z) \to 0$ (where M is such that $u_0 \in \mathcal{B}_M$). It follows from Proposition 3.4 (iii) that $\mathcal{S}(1)D_{\lambda_n}u_0 \to \mathcal{S}(1)z$ in $\mathcal{W} \cap C_0(\mathbb{R}^N)$, i.e. $\mathcal{S}(1)z \in \omega_1(u)$. Since $\omega_1(u)$ is closed, $\overline{\mathcal{S}(1)\Omega_1(u_0)} \subset \omega_1(u)$. We now show the reverse inclusion. Given $\varphi \in \Omega_1(u_0)$, it follows from what precedes that $f = \mathcal{S}(1)\varphi \in \omega_1(u)$. Proposition 3.8 (iv) implies that $\omega_1(u) = \overline{\bigcup_{s\geq 0}\{D_{\sqrt{1+s}}\mathcal{S}(s)f\}}$, and thus

$$\omega_1(u) = \overline{\bigcup_{s\geq 0}\{D_{\sqrt{1+s}}\mathcal{S}(1+s)\varphi\}} = \overline{\bigcup_{s\geq 0}\{\mathcal{S}(1)D_{\sqrt{1+s}}\varphi\}} = \overline{\mathcal{S}(1)\bigcup_{s\geq 0}\{D_{\sqrt{1+s}}\varphi\}}.$$

By Proposition 2.5 (iv) in [5], $D_{\sqrt{1+s}}\varphi \in \Omega_1(u_0)$ for all $s \geq 0$. Thus $\overline{\mathcal{S}(1)\bigcup_{s\geq 0}\{D_{\sqrt{1+s}}\varphi\}} \subset \overline{\mathcal{S}(1)\Omega_1(u_0)}$ and the result follows. □

REMARK 3.17. At least in the case $\alpha > 2/N$, there exists $u_0 \in \mathcal{W} \cap C^\infty(\mathbb{R}^N)$ such that $\Omega_1(u_0) = \emptyset$ but $\omega_1(u) \neq \emptyset$. Indeed, it follows from Remark 3.17 in [5] that there exists $u_0 \in \mathcal{W} \cap C^\infty(\mathbb{R}^N)$ such that $\Omega_1(u_0) = \emptyset$ and $\sup_{x\in\mathbb{R}^N}(t + |x|^2)^{\frac{1}{\alpha}}|e^{t\Delta}u_0(x)| \to 0$ as $t \to \infty$. We deduce from (2.1) that $\sup_{x\in\mathbb{R}^N}(t + |x|^2)^{\frac{1}{\alpha}}|\mathcal{S}(t)u_0(x)| \to 0$ as $t \to \infty$, so that $\omega_1(u) = \{0\}$.

REMARK 3.18. There exists an initial value $u_0 \in \mathcal{W} \cap C_0(\mathbb{R}^N)$ such that $\omega_1(u) \neq \emptyset$ and $\omega_1(u) \neq \omega(u)$. Indeed, if u_0 is the initial value given in Proposition 2.13 in [5] with $\sigma = 2/\alpha$, then $c|\cdot|^{-\frac{2}{\alpha}} \in \Omega(u_0)$ and $v \in \Omega_1(u_0)$ for some $v \in \mathcal{W}$, $v \neq c|\cdot|^{-\frac{2}{\alpha}}$. It follows from Theorems 3.10 and 3.16 that $\mathcal{S}(1)(c|\cdot|^{-\frac{2}{\alpha}}) \in \omega(u)$ and $\mathcal{S}(1)v \in \omega_1(u)$. On the other hand, $\mathcal{S}(1)(c|\cdot|^{-\frac{2}{\alpha}}) \notin \omega_1(u)$, since if it were, then $\omega_1(u) = \{\mathcal{S}(1)(c|\cdot|^{-\frac{2}{\alpha}})\}$ by Proposition 3.8 (vi). Furthermore, by Proposition 3.8 (vii), we see that $\omega_1(u)$ is not closed in $C_0(\mathbb{R}^N)$, therefore not compact in either $C_0(\mathbb{R}^N)$ or $\mathcal{W} \cap C_0(\mathbb{R}^N)$.

We conclude this section by giving various conditions under which $\omega_1(u) = \omega(u)$. Our results depend on the theory of almost periodic functions, where we need to consider in particular metric space valued almost periodic functions. We recall the following two definitions. (See Haraux [14].)

An almost periodic function on $\mathbb{R}$ taking values in a metric space (X, d) is a continuous, bounded function p such that any sequence of translations $p(s_n + \cdot)$ has a uniformly (on $\mathbb{R}$) convergent subsequence. A function $g \in C_{\mathrm{b}}(\mathbb{R}, X)$ is asymptotically almost periodic if there exists an almost periodic function p such that $\mathrm{d}(g(s), p(s)) \to 0$ as $s \to \infty$.

Proposition 3.19. *Let $u_0(x) = |x|^{-\frac{2}{\alpha}} g(\log|x|)\zeta(x/|x|)$ with $\zeta \in C(S^{N-1})$ and $g \in C(\mathbb{R}, \mathbb{R})$ and set $u(t) = \mathcal{S}(t)u_0$ for $t \geq 0$ and $v(s) = \mathcal{S}(1)D_{e^s}u_0 = D_{\sqrt{t}}u(t)$ where $t = e^{2s}$, $s \in \mathbb{R}$.*

(i) *If g is periodic, then v is periodic.*
(ii) *If $g(t)$ is asymptotically periodic as $t \to \infty$, then $v(s)$ is asymptotically periodic as $s \to \infty$ as a function $\mathbb{R} \to \mathcal{W} \cap C_0(\mathbb{R}^N)$.*
(iii) *If g is almost periodic, then v is almost periodic as a function $\mathbb{R} \to \mathcal{W} \cap C_0(\mathbb{R}^N)$.*
(iv) *If $g(t)$ is asymptotically almost periodic as $t \to \infty$, then $v(s)$ is asymptotically almost periodic as $s \to \infty$, as a function $\mathbb{R} \to \mathcal{W} \cap C_0(\mathbb{R}^N)$.*

In particular, in all these cases, $\omega(u) = \omega_1(u)$. Moreover, if $f \in \omega(u) = \omega_1(u)$ and $s_n \to \infty$ are such that $\|v(s_n) - f\|_{L^\infty} \to 0$ as $n \to \infty$, then $\|v(s_n) - f\|_{\mathcal{W}\cap C_0} \to 0$ as $n \to \infty$.

Proof. We prove (iii) and (iv), the proof of (i) and (ii) being similar but simpler. Let $z(s) = D_{e^s}u_0$ and assume g is almost periodic. It follows from Proposition 2.12 (iii) in [5] that z is almost periodic $\mathbb{R} \to (\mathcal{B}_M, \mathrm{d}_M)$, where $M > 0$ is such that $u_0 \in \mathcal{B}_M$. In particular, the set $K = \overline{\bigcup_{s\in\mathbb{R}} \{z(s)\}}$ is a compact subset of $(\mathcal{B}_M, \mathrm{d}_M)$, so that by Proposition 3.4 (iii), $\mathcal{S}(1)$ is uniformly continuous $K \to \mathcal{W} \cap C_0(\mathbb{R}^N)$. Given $(s_n)_{n\geq 0}$, there exists a subsequence $(s_{n_k})_{k\geq 0}$ such that $z(s_{n_k} + \cdot)$ converges uniformly in $(\mathcal{B}_M, \mathrm{d}_M)$ to a function $\widetilde{z} : \mathbb{R} \to K$. By uniform continuity of $\mathcal{S}(1)$, $v(s_{n_k} + \cdot)$ converges uniformly in $\mathcal{W} \cap C_0(\mathbb{R}^N)$, which proves (iii). Assume now g is asymptotically almost periodic. It follows from Proposition 2.12 (iv) in [5] that z is asymptotically almost periodic $\mathbb{R} \to (\mathcal{B}_M, \mathrm{d}_M)$, where $M > 0$ is such that $u_0 \in \mathcal{B}_M$. Therefore, there exists an almost periodic function $w : \mathbb{R} \to (\mathcal{B}_M, \mathrm{d}_M)$ such that $\mathrm{d}_M(z(s), w(s)) \to 0$ as $s \to \infty$. Setting $h(s) = \mathcal{S}(1)w(s)$, we deduce from (iii) above that h is almost periodic $\mathbb{R} \to \mathcal{W} \cap C_0(\mathbb{R}^N)$. Since the set $\widetilde{K} = \overline{\bigcup_{s\geq 0} \{z(s)\}} \cup \overline{\bigcup_{s\geq 0} \{w(s)\}}$ is a compact subset of $(\mathcal{B}_M, \mathrm{d}_M)$, it follows from Proposition 3.4 (iii) that $\mathcal{S}(1)$ is uniformly continuous $\widetilde{K} \to \mathcal{W} \cap C_0(\mathbb{R}^N)$. We conclude that $\|v(s) - h(s)\|_{\mathcal{W}\cap C_0} \to 0$ as $s \to \infty$, which proves (iv). Finally, the last statement follows from the fact that in all cases (i)-(iv), the set $\bigcup_{s\geq 0} \{v(s)\}$ is relatively compact in $\mathcal{W} \cap C_0(\mathbb{R}^N)$.

Remark 3.20. Assume $u_0 \in C_0(\mathbb{R}^N)$ is asymptotically homogeneous as $|x| \to \infty$, i.e. there exists $\eta \in C(S^{N-1})$ such that $|x|^{\frac{2}{\alpha}} u_0(x) - \eta(x/|x|) \to 0$ as $|x| \to \infty$. Setting $\varphi(x) = |x|^{-\frac{2}{\alpha}} \eta(x/|x|)$, we see that $u_0, \varphi \in \mathcal{W}$ and that $\Omega_1(u_0) = \{\varphi\}$. We deduce from Theorem 3.16 that $\omega_1(u) = \{f\}$ with $f = \mathcal{S}(1)\varphi$. Remark 3.6 implies that $|x|^{\frac{2}{\alpha}} f(x) - \eta(x/|x|) \to 0$ as $x \to \infty$. By

Proposition 3.8 (vi), we see that $u(t)$ is asymptotically self-similar as $t \to \infty$ in the sense that $\|D_{\sqrt{t}} u(t) - f\|_{\mathcal{W}\cap C_0} \to 0$ as $t \to \infty$. Thus the results of this section contain as a very particular case Theorem 1.2 of [4] (case $\alpha \geq 2/N$).

4. – The case $\alpha < 2/N$

Throughout this section, we assume $\alpha < 2/N$. In particular, $\mathcal{W}\cap C_0(\mathbb{R}^N) \hookrightarrow L^1(\mathbb{R}^N)$. The major difficulty, in comparison with the case $\alpha \geq 2/N$, is that even for nonnegative initial values and solutions, there may be nonuniqueness. The best known example of this is the "very singular" solution of (1.1) (see [3], [9], [10], [11], [21]): there exists a unique $R_0 \in C_0(\mathbb{R}^N)$, $R_0 > 0$ with exponential decay, such that

$$r(t,x) = t^{-\frac{1}{\alpha}} R_0\left(\frac{x}{\sqrt{t}}\right), \tag{4.1}$$

for $t > 0$, $x \in \mathbb{R}^N$, is a self-similar solution of (1.1). Furthermore, it is clear that

$$r(t) \to 0 \quad \text{in} \quad L^1_{\text{loc}}(\mathbb{R}^N \setminus \{0\}) \quad \text{and} \quad \|r(t)\|_{L^1} = t^{-\frac{1}{\alpha}+\frac{N}{2}} \|R_0\|_{L^1} \to \infty\,, \tag{4.2}$$

as $t \downarrow 0$.

In fact, one can describe all nonnegative solutions with a given nonnegative initial value in $\mathcal{W}$. This is the object of Proposition 4.3, based on the results of Marcus and Véron [18].

Definition 4.1. Given $u_0 \in \mathcal{W}$, $u_0 \geq 0$, we denote by $\Sigma^+(u_0)$ the set of nonnegative solutions $u \in C((0,\infty), C_0(\mathbb{R}^N))$ of (1.1) such that $u(t) \to u_0$ in $L^1_{\text{loc}}(\mathbb{R}^N \setminus \{0\})$ as $t \downarrow 0$. Note that $\Sigma^+(u_0) \subset \Sigma(u_0)$.

Remark 4.2. Since $2/\alpha > N$, it follows easily from Corollary 2.4 that $u(t) \to u_0$ in $L^p(\{|x| > \varepsilon\})$ as $t \downarrow 0$, for all $\varepsilon > 0$, $1 \leq p < \infty$ and $u \in \Sigma^+(u_0)$.

Proposition 4.3. *Assume* $0 < \alpha < 2/N$, *and let* $u_0 \in \mathcal{W}$, $u_0 \geq 0$.

(i) *If* $u \in \Sigma^+(u_0)$, *then there exists* $\|u_0\|_{L^1} \leq \ell \leq \infty$ *such that* $\|u(t)\|_{L^1} \to \ell$ *as* $t \downarrow 0$. *Moreover, if* $\ell = \infty$, *then* u *has initial trace* $(\{0\}, u_0)$, *and if* $\ell < \infty$, *then* u *has initial trace* $(\emptyset, u_0 + (\ell - \|u_0\|_{L^1})\delta_0)$ *in the sense of* [18], *where* δ_0 *is the Dirac measure at* $x = 0$.
(ii) *Given* $\|u_0\|_{L^1} \leq \ell \leq \infty$, *there exists a unique* $u \in \Sigma^+(u_0)$ *such that* $\|u(t)\|_{L^1} \to \ell$ *as* $t \downarrow 0$. *Moreover, the solutions are increasing with respect to* ℓ.
(iii) *If* $u_0 \notin L^1(\mathbb{R}^N)$, *then* $\Sigma^+(u_0)$ *is a singleton.*
(iv) *If* $u^1, u^2 \in \Sigma^+(u_0)$, *then* $|u^1 - u^2| \leq 2r$ *with* r *given by* (4.1).

For the proof of Proposition 4.3, we will use the following lemma.

LEMMA 4.4. *Assume* $0 < \alpha < 2/N$. *Let* $M > 0$ *and* $(u_0^n)_{n\geq 0} \subset \mathcal{B}_M^+$. *For every* $n \geq 0$, *let* $u^n \in \Sigma^+(u_0^n)$. *If* $u^n \underset{n\to\infty}{\longrightarrow} u$ *in* $L^\infty_{\mathrm{loc}}((0,\infty), C_0(\mathbb{R}^N))$ *and if* $\limsup_{t\downarrow 0} \|u^n(t)\|_{L^1} \underset{n\to\infty}{\longrightarrow} \infty$, *then* $\|u(t)\|_{L^1} \underset{t\downarrow 0}{\longrightarrow} \infty$.

PROOF. Let $\tau_n > 0$, $\tau_n \downarrow 0$ be such that $\|u^n(\tau_n)\|_{L^1} \underset{n\to\infty}{\longrightarrow} \infty$. Since $u^n(\tau_n, x) \leq C|x|^{-\frac{2}{\alpha}}$ by Corollary 2.4, we deduce that there exist $\varepsilon_n > 0$, $\varepsilon_n \downarrow 0$ such that

$$\int_{\{|x|<\varepsilon_n\}} u^n(\tau_n, x)\, dx \underset{n\to\infty}{\longrightarrow} \infty\,.$$

We deduce that, given any $c > 0$, there exists a sequence $(v_0^n)_{n\geq 0} \subset C_0(\mathbb{R}^N)$ such that $0 \leq v_0^n \leq u^n(\tau_n) 1_{\{|x|<\varepsilon_n\}}$ and $\|v_0^n\|_{L^1} = c$ for n large enough. It follows that $v_0^n \to c\delta_0$ (the Dirac measure at 0) in the weak* topology of measures, so that by Theorem 3.10 in [18], $\mathcal{S}(\cdot)v_0^n$ converges in $L^\infty_{\mathrm{loc}}((0,\infty)\times\mathbb{R}^N)$ to v_c, the (unique) positive solution of (1.1) with the initial value $c\delta_0$. We note that, given $A > 0$, $u^{n,A} = \min\{u^n, A\}$ is a supersolution of (1.1). Also, since $u^{n,A}(t,x) \leq C|x|^{-\frac{2}{\alpha}}$ by Corollary 2.4 and $u^{n,A}(t) \to u^{n,A}(0)$ in $L^1_{\mathrm{loc}}(\mathbb{R}^N \setminus \{0\})$ as $t \downarrow 0$, we see that $u^{n,A}(t) \to u^{n,A}(0)$ in $L^p(\mathbb{R}^N)$ as $t \downarrow 0$, for all $1 \leq p < \infty$. Since $u^{n,A}(\tau_n, x) \geq v_0^n(x)$ for A sufficiently large, we deduce from the maximum principle that $u^n(\tau_n + t) \geq u^{n,A}(\tau_n + t) \geq \mathcal{S}(t)v_0^n$; and so, $u \geq v_c$. On the other hand, $\|v_c(t)\|_{L^1} \underset{t\downarrow 0}{\longrightarrow} c$, and we deduce that $\liminf_{t\downarrow 0}\|u(t)\|_{L^1} \geq c$. Since c is arbitrary, the result follows. □

PROOF OF PROPOSITION 4.3. We proceed in five steps.

STEP 1. Proof of (i). We use Lemma 2.2 and Definition 2.3 in [18]. Given $u \in \Sigma^+(u_0)$, the initial trace of u in the sense of [18] is either $(\{0\}, u_0)$ or else $(\emptyset, \nu)$, where ν is a positive measure on $\mathbb{R}^N$. In the first case, $\|u(t)\|_{L^1} \to \infty$ as $t \to 0$. In the second case, u_0 must be a positive measure on $\mathbb{R}^N$ (in particular, since $u_0 \in \mathcal{W}$, $u_0 \in L^1(\mathbb{R}^N)$), and there exists $0 \leq c < \infty$ such that $\nu = u_0 + c\delta_0$, where δ_0 is the Dirac measure at $x = 0$. Therefore, it follows that

$$\int_{\{|x|<1\}} u(t,x)\, dx \underset{t\downarrow 0}{\longrightarrow} c + \int_{\{|x|<1\}} u_0(x)\, dx\,.$$

Thus $\|u(t)\|_{L^1} \to c + \|u_0\|_{L^1}$ by Remark 4.2, which completes the proof.

STEP 2. Proof of uniqueness and monotonicity in (ii). Given u as in the statement, it follows from (i) that u has the initial trace $(\{0\}, u_0)$ (if $\ell = \infty$) or $(\emptyset, u_0 + (\ell - \|u_0\|_{L^1})\delta_0)$ (if $\ell < \infty$) in the sense of [18]. Hence uniqueness, by Theorem 3.5 in [18]. The solutions are increasing in ℓ by Theorem 3.4 in [18].

STEP 3. Proof of existence in (ii) for $\ell < \infty$. Let $u_0^n = u_0 + c|\{|x| < 1/n\}|^{-1} 1_{\{|x|<1/n\}}$ with $c = \ell - \|u_0\|_{L^1}$, so that $u_0^n \to u_0$ in $L^1_{\mathrm{loc}}(\mathbb{R}^N \setminus \{0\})$ and $u_0^n \to u_0 + c\delta_0$ in the weak* topology of measures as $n \to \infty$. Since $u_0^n \in L^1(\mathbb{R}^N)$, there exists a unique solution $u^n \in C((0,\infty), C_0(\mathbb{R}^N))$ of (1.1) such that $u^n(t) \to u_0^n$ in $L^1(\mathbb{R}^N)$ as $t \downarrow 0$. Note that $u_0^n \to u_0$ in $\mathcal{W}$. It follows from Theorem 2.2 that there exist a subsequence n_k and a solution $u \in \Sigma^+(u_0)$ such that $u^{n_k}(t) \to u(t)$ in $C_0(\mathbb{R}^N)$ as $k \to \infty$ for all $t > 0$. It remains to show that $\lim_{t\downarrow 0} \|u(t)\|_{L^1} = c + \|u_0\|_{L^1}$. Since $u_0^n \to u_0 + c\delta_0$ in the weak* topology of measures, u^n converges in $L^\infty_{\text{loc}}((0,\infty)\times\mathbb{R}^N)$ to the unique solution of (1.1) with the initial value $u_0 + c\delta_0$ (see e.g. Theorem 3.10 in [18]). Therefore, $u(t) \to u_0 + c\delta_0$ in the weak* topology of measures as $t \downarrow 0$. Applying Remark 4.2, we deduce easily that $\lim_{t\downarrow 0} \|u(t)\|_{L^1} = c + \|u_0\|_{L^1}$.

STEP 4. Proof of existence in (ii) for $\ell = \infty$. Setting $u_0^n = \min\{u_0, n\} + r(1/n)$, where $r(t)$ is the very singular solution (4.1), we deduce from (4.2) that $u_0^n \to u_0$ in $L^1_{\text{loc}}(\mathbb{R}^N \setminus \{0\})$ and $\|u_0^n\|_{L^1} \to \infty$ as $n \to \infty$. If u^n is the classical solution of (1.1) with the initial condition $u^n(0) = u_0^n \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$, we deduce from Theorem 2.2 that there exist a subsequence $(n_k)_{k\geq 0}$ and a solution $u \in \Sigma^+(u_0)$ such that $u^{n_k} \to u$ in $C([\varepsilon,\infty), C_0(\mathbb{R}^N))$ for all $\varepsilon > 0$. In addition, it follows from Lemma 4.4 that $\|u(t)\|_{L^1} \to \infty$ as $t \downarrow 0$.

STEP 5. Proof of (iv). We first note that by (4.2), r is the element of $\Sigma^+(0)$ corresponding to $\ell = \infty$, so that every $w \in \Sigma^+(0)$ satisfies $w \leq r$. Let now $u^1, u^2 \in \Sigma^+(u_0)$, let $\tau_n \downarrow 0$ and set $w^n(t) = \mathcal{S}(t)w_0^n$ with $w_0^n = |u^1(\tau_n) - u^2(\tau_n)|/2$. It follows from (2.1) that $|u^1(t) - u^2(t)| \leq 2w^n(t - \tau_n)$ for $t \geq \tau_n$. Since $w_0^n \to 0$ in $L^1_{\text{loc}}(\mathbb{R}^N \setminus \{0\})$, it follows from Theorem 2.2 that there exist a subsequence n_k and $w \in \Sigma^+(0)$ such that $w^{n_k} \to w$ in $C([\varepsilon, T], C_0(\mathbb{R}^N))$ for all $0 < \varepsilon < T < \infty$. We deduce that $|u^1(t) - u^2(t)| \leq 2w(t)$ for all $t > 0$, which completes the proof. □

Consider $u_0 \in \mathcal{W}$, $u_0 \geq 0$. It follows from Proposition 4.3 that if $u_0 \in L^1(\mathbb{R}^N)$, then the set $\Sigma^+(u_0)$ is infinite. This is the case in particular if $u_0 \in \mathcal{W} \cap C_0(\mathbb{R}^N)$. It turns out that the case $\ell = \infty$ in Proposition 4.3 plays a crucial role in the description of the asymptotic behavior of solutions. Thus we make the following definition.

DEFINITION 4.5. Assume $0 < \alpha < 2/N$. Given $u_0 \in \mathcal{W}$, $u_0 \geq 0$, we set

$$\mathcal{U}(t)u_0 = u(t)\,, \tag{4.3}$$

for all $t > 0$, where u is the unique element of $\Sigma^+(u_0)$ such that $\|u(t)\|_{L^1} \to \infty$ as $t \downarrow 0$.

REMARK 4.6.

(i) If $r(t)$ is defined by (4.1), then it follows from (4.2) that $r(t) = \mathcal{U}(t)0$.

(ii) It follows easily from the proof of existence in Proposition 4.3 that $\mathcal{U}(t)u_0 \geq \mathcal{U}(t)v_0$ for all $u_0, v_0 \in \mathcal{W}$ such that $u_0 \geq v_0 \geq 0$. In particular, we deduce from (i) above that if $u_0 \in \mathcal{W}$, $u_0 \geq 0$, then $\mathcal{U}(t)u_0 \geq r(t)$.

(iii) It follows from (ii) above that if $u_0 \in \mathcal{W} \cap C_0(\mathbb{R}^N)$, $u_0 \geq 0$, then $\mathcal{S}(t)u_0 \neq \mathcal{U}(t)u_0$. Note also that $(\mathcal{U}(t))_{t\geq 0}$ does not satisfy the semigroup property. Indeed, if $s, t > 0$, then $\mathcal{U}(t+s) = \mathcal{S}(t)\mathcal{U}(s) \neq \mathcal{U}(t)\mathcal{U}(s)$.

(iv) By the scaling invariance of (1.1), i.e. formula (1.6), and uniqueness,

$$D_\lambda \mathcal{U}(\lambda^2\tau) = \mathcal{U}(\tau) D_\lambda\,, \tag{4.4}$$

on $\{u \in \mathcal{W};\ u \geq 0\}$. In particular,

$$D_{\sqrt{t}}\mathcal{U}(t) = \mathcal{U}(1) D_{\sqrt{t}}\,, \tag{4.5}$$

for all $t > 0$.

We now establish some continuity properties.

PROPOSITION 4.7. *Assume* $0 < \alpha < 2/N$ *and* $M > 0$. *Let* $(u_0^n)_{n\geq 0} \subset \mathcal{B}_M^+$, $z_0 \in \mathcal{B}_M^+$. *For* $n \geq 0$, *let* $u^n \in \Sigma^+(u_0^n)$ *be such that* $\lim_{t\downarrow 0} \|u^n(t)\|_{L^1} \underset{n\to\infty}{\longrightarrow} \infty$.

(i) *If* $u_0^n \to z_0$ *in* $(\mathcal{B}_M, \mathrm{d}_M^*)$ *as* $n \to \infty$, *then* $u^n(t) \to \mathcal{U}(t) z_0$ *in* $C_0(\mathbb{R}^N)$ *for every* $t > 0$.
(ii) *If* $\mathrm{d}_M(u_0^n, z_0) \to 0$ *as* $n \to \infty$, *then* $u^n(t) \to \mathcal{U}(t) z_0$ *in* $\mathcal{W} \cap C_0(\mathbb{R}^N)$ *for every* $t > 0$.

PROOF. By Lemma 4.4 and Proposition 4.3, any limiting solution given by Theorem 2.2 must be $\mathcal{U}(t)u_0$. In particular, it does not depend on the subsequence $(n_k)_{k\geq 0}$, so that the whole sequence $(u^n(t))_{n\geq 0}$ converges. Thus the result follows from Theorem 2.2 □

PROPOSITION 4.8. *Assume* $0 < \alpha < 2/N$ *and* $M > 0$. *Let* $t > 0$ *be fixed.*

(i) $\mathcal{U}(t)$ *is continuous* $(\mathcal{B}_M^+, \mathrm{d}_M^*) \to C_0(\mathbb{R}^N)$.
(ii) $\mathcal{U}(t)$ *is a homeomorphism of* $(\mathcal{B}_M^+, \mathrm{d}_M^*)$ *onto* $\mathcal{U}(t)\mathcal{B}_M^+ \subset C_0(\mathbb{R}^N)$, *this latter space considered with its norm topology.*
(iii) $\mathcal{U}(t)$ *is continuous* $(\mathcal{B}_M^+, \mathrm{d}_M) \to \mathcal{W} \cap C_0(\mathbb{R}^N)$.

PROOF. Let $(u_0^n)_{n\geq 0} \subset \mathcal{B}_M^+$ and $u_0 \in \mathcal{B}_M^+$ such that $\mathrm{d}_M^*(u_0^n, u_0) \to 0$. By Remark 4.6 (ii), $\mathcal{U}(t)u_0^n \geq r(t)$ given by (4.1). If $u \in \Sigma^+(u_0)$ is obtained from the sequence $(\mathcal{U}(t)u_0^n)_{n\geq 0}$ by Theorem 2.2, then $u(t) \geq r(t)$; and so, by (4.2), $\|u(t)\|_{L^1} \to \infty$ as $t \downarrow 0$. Thus $u(t) = \mathcal{U}(t)u_0$ by Proposition 4.3. In particular, the limiting solution u in Theorem 2.2 is unique. This proves (i). Statement (iii) follows from the last part of Theorem 2.2. Finally, (ii) follows from (i), since a continuous, injective (by Remark 2.10), surjective map of a compact Hausdorff space onto a Hausdorff space is a homeomorphism. □

PROPOSITION 4.9. *Let* $u_0 \in \mathcal{B}_M^+$ *and* $u \in \Sigma(u_0)$, $u \not\equiv 0$. *It follows that* $\|D_{\sqrt{t}}[u(t) - \mathcal{U}(t)u_0]\|_{\mathcal{W}\cap C_0} \underset{t\to\infty}{\longrightarrow} 0$. *In particular,* $\omega(u) = \omega(\mathcal{U}(\cdot)u_0)$ *and* $\omega_1(u) = \omega_1(\mathcal{U}(\cdot)u_0)$.

PROOF. We first show

$$\|D_{\sqrt{t}}[u(t) - \mathcal{U}(t)u_0]\|_{L^\infty} \underset{t\to\infty}{\longrightarrow} 0\,. \tag{4.6}$$

Assume by contradiction that there exist $\varepsilon > 0$ and a sequence $t_n \to \infty$ such that

$$\|D_{\sqrt{t_n}}[u(t_n) - \mathcal{U}(t_n)u_0]\|_{L^\infty} \geq \varepsilon \,. \tag{4.7}$$

Note that

$$D_{\sqrt{t_n}}u(t_n) - D_{\sqrt{t_n}}\mathcal{U}(t_n)u_0 = \Gamma_{\sqrt{t_n}}u(1) - \mathcal{U}(1)D_{\sqrt{t_n}}u_0 \,,$$

with Γ_λ defined by (1.4). By weak* compactness, passing to a subsequence if necessary, we may assume that $D_{\sqrt{t_n}}u_0 \to z$ weak* for some $z \in \mathcal{B}_M^+$. Since $u \not\equiv 0$, it follows from Proposition 4.3 that there exists $\ell > 0$ such that $\|u(t)\|_{L^1} \to \ell$ as $t \downarrow 0$. Since

$$\|\Gamma_{\sqrt{t_n}}u(t)\|_{L^1} = t_n^{\frac{1}{\alpha}-\frac{N}{2}}\|u(tt_n)\|_{L^1} \underset{t\downarrow 0}{\longrightarrow} t_n^{\frac{1}{\alpha}-\frac{N}{2}}\ell \,,$$

we deduce from Proposition 4.7 that $\Gamma_{\sqrt{t_n}}u(1)$ converges in $C_0(\mathbb{R}^N)$ to $\mathcal{U}(1)z$. On the other hand, Proposition 4.8 (i) implies that $\mathcal{U}(1)D_{\sqrt{t_n}}u_0$ also converges to $\mathcal{U}(1)z$, thus contradicting (4.7).

Next, it follows from Proposition 4.3 (iv) that $|D_{\sqrt{t}}[u(t) - \mathcal{U}(t)u_0]| \leq 2D_{\sqrt{t}}r(t) = 2R_0$. Fix $\varepsilon > 0$ and let R be large enough so that $2(1 + |x|^2)^{\frac{1}{\alpha}}R_0(x) \leq \varepsilon$ for $|x| \geq R$. We deduce that $(1+|x|^2)^{\frac{1}{\alpha}}|D_{\sqrt{t}}[u(t)-\mathcal{U}(t)u_0]| \leq \varepsilon$ for $|x| \geq R$. On the other hand, it follows from (4.6) that $(1+|x|^2)^{\frac{1}{\alpha}}|D_{\sqrt{t}}[u(t)-\mathcal{U}(t)u_0]| \leq \varepsilon$ for $|x| \leq R$ and t sufficiently large. Hence the result, since $\varepsilon > 0$ is arbitrary. □

PROPOSITION 4.10. *Assume $0 < \alpha < 2/N$. A function $f \in \mathcal{W} \cap C_0(\mathbb{R}^N)$, $f \geq 0$, $f \not\equiv 0$ is the profile of a self-similar solution u of* (1.1) *if and only if there exists $\varphi \in \mathcal{W}$, $\varphi \geq 0$ homogeneous of degree $-2/\alpha$ such that $f = \mathcal{U}(1)\varphi$.*

PROOF. The proof is analogous to the proof of Proposition 3.5, using Proposition 4.7 (i) instead of Proposition 3.4 (i). □

REMARK 4.11. If $\varphi \in \mathcal{W}\cap C(\mathbb{R}^N\setminus\{0\})$, $\varphi \geq 0$ and $u \in \Sigma^+(\varphi)$, then $u(t) \to \varphi$ in $C(S^{N-1})$ as $t \downarrow 0$ by Remark 2.5. If, in addition, φ is homogeneous of degree $-2/\alpha$, so that $u(t)$ is self-similar, it follows that $|x|^{\frac{2}{\alpha}}u(t,x) - \varphi(x/|x|) \to 0$ as $x \to \infty$ for any fixed $t > 0$.

The following lemma is analogous to Lemma 3.7. It plays a fundamental role in the analysis of $\omega_1(u)$ in the subsequent proposition and is thus crucial to the proof of Theorem 4.21 below. Note that while the statement concerns only the classical flow $\mathcal{S}(s)$, its proof makes essential use of the "singular flow" $\mathcal{U}(s)$.

LEMMA 4.12. *Assume $0 < \alpha < 2/N$. Let $f \in \mathcal{W}\cap C_0(\mathbb{R}^N)$ and $(f_n)_{n\geq 0} \subset \mathcal{W}\cap C_0(\mathbb{R}^N)$. If $f_n, f \geq 0$, $f \not\equiv 0$, and if $\|f - f_n\|_{\mathcal{W}\cap C_0} \underset{n\to\infty}{\longrightarrow} 0$, then $\sup_{s\geq 0}\|D_{\sqrt{1+s}}\mathcal{S}(s)f - D_{\sqrt{1+s}}\mathcal{S}(s)f_n\|_{\mathcal{W}\cap C_0} \underset{n\to\infty}{\longrightarrow} 0$.*

Proof. We first show that

$$\sup_{s\geq 0} \|D_{\sqrt{1+s}}\mathcal{S}(s)f - D_{\sqrt{1+s}}\mathcal{S}(s)f_n\|_{L^\infty} \underset{n\to\infty}{\longrightarrow} 0 . \tag{4.8}$$

Assume by contradiction that there exist $\delta > 0$, $n_k \to \infty$ and $s_k \geq 0$ such that $\|D_{\sqrt{1+s_k}}\mathcal{S}(s_k)f - D_{\sqrt{1+s_k}}\mathcal{S}(s_k)f_{n_k}\|_{L^\infty} \geq \delta$. By continuous dependence (Proposition 2.1) and the fact that $D_{\sqrt{1+s}}$ is uniformly bounded on $L^\infty(\mathbb{R}^N)$ for s in a bounded set, it follows that $s_k \to \infty$ as $k \to \infty$. Thus we may assume $s_k \geq 1$. Since $D_{\sqrt{1+s}}\mathcal{S}(s) = D_{\frac{\sqrt{1+s}}{\sqrt{s}}}\mathcal{S}(1)D_{\sqrt{s}}$ by (1.7) and since $D_{\frac{\sqrt{1+s}}{\sqrt{s}}}$ is uniformly bounded on $L^\infty(\mathbb{R}^N)$ for $s \geq 1$, we deduce that $\|\mathcal{S}(1)D_{\sqrt{s_k}}f - \mathcal{S}(1)D_{\sqrt{s_k}}f_{n_k}\|_{L^\infty} \geq \delta' > 0$. Consider a subsequence, which we still denote by $(n_k)_{k\geq 0}$, such that $D_{\sqrt{s_k}}f \to z$ in $\mathcal{W}$ weak* with $z \in \mathcal{W}$. Since $\|D_{\sqrt{s_k}}f\|_{L^1} = s_k^{\frac{1}{\alpha}-\frac{N}{2}}\|f\|_{L^1} \to \infty$ as $k \to \infty$, we deduce from Proposition 4.7 that $\|\mathcal{S}(1)D_{\sqrt{s_k}}f - \mathcal{U}(1)z\|_{L^\infty} \to 0$. On the other hand, $|f_{n_k} - f| \leq \varepsilon_k(1+|x|^2)^{-\frac{1}{\alpha}}$ with $\varepsilon_k \to 0$, so that $|D_{\sqrt{s_k}}f_{n_k} - D_{\sqrt{s_k}}f| \leq \varepsilon_k|x|^{-\frac{2}{\alpha}} \to 0$ weak*. Therefore, $D_{\sqrt{s_k}}f_{n_k} \to z$ in $\mathcal{W}$ weak*. Since $\|f_{n_k} - f\|_{L^1} \to 0$, we see that $\|D_{\sqrt{s_k}}f_{n_k}\|_{L^1} = s_k^{\frac{1}{\alpha}-\frac{N}{2}}\|f_{n_k}\|_{L^1} \to \infty$ as $k \to \infty$ and deduce from Proposition 4.7 that $\|\mathcal{S}(1)D_{\sqrt{s_k}}f_{n_k} - \mathcal{U}(1)z\|_{L^\infty} \to 0$. This yields a contradiction.

Next, given $\varepsilon > 0$, let $\theta_\varepsilon = \mathcal{U}(1)(\varepsilon|\cdot|^{-\frac{2}{\alpha}})$, so that $(1+|x|^2)^{\frac{1}{\alpha}}\theta_\varepsilon(x) \geq c_\varepsilon > 0$ by Remark 4.11. We note that, by Proposition 4.8 (iii) and Remark 4.6 (i), $\theta_\varepsilon \to R_0$ in $\mathcal{W} \cap C_0(\mathbb{R}^N)$ as $\varepsilon \downarrow 0$, with R_0 given by (4.1). We now observe that $|f - f_n| \leq 2\theta_{\varepsilon_n}$ with $\varepsilon_n \to 0$, so that by (2.1) and Proposition 2.11 (iii), $|D_{\sqrt{1+s}}\mathcal{S}(s)f - D_{\sqrt{1+s}}\mathcal{S}(s)f_n| \leq 2D_{\sqrt{1+s}}\mathcal{S}(s)\theta_{\varepsilon_n} = 2\theta_{\varepsilon_n}$. Therefore, given $\mu > 0$, we see that there exist $n_0 > 0$ and $R > 0$ such that $|D_{\sqrt{1+s}}\mathcal{S}(s)f - D_{\sqrt{1+s}}\mathcal{S}(s)f_n| \leq \mu(1+|x|^2)^{-\frac{2}{\alpha}}$ for $x \geq R$ and $n \geq n_0$. We deduce from (4.8) that, by possibly choosing n_0 larger, $|D_{\sqrt{1+s}}\mathcal{S}(s)f - D_{\sqrt{1+s}}\mathcal{S}(s)f_n| \leq \mu(1+|x|^2)^{-\frac{2}{\alpha}}$ for $x \in \mathbb{R}^N$ and $n \geq n_0$. Since $\mu > 0$ is arbitrary, the result follows. □

Proposition 4.13. *Assume* $0 < \alpha < 2/N$. *Let* $u_0 \in \mathcal{W}$, $u \in \Sigma(u_0)$ *and let* $\omega(u)$ *and* $\omega_1(u)$ *be defined by* (1.25) *and* (1.26).

(i) *If* $f \in \omega(u)$, *then* $D_{\sqrt{1+s}}\mathcal{S}(s)f \in \omega(u)$ *for all* $s \geq 0$.
(ii) *If* $\omega(u) = \{f\}$ *and* $f \geq 0$, $f \not\equiv 0$, *then* $f = \mathcal{U}(1)\varphi$ *with* $\varphi \in \mathcal{W}$ *homogeneous of degree* $-2/\alpha$, $\varphi \geq 0$. *Moreover,* $D_{\sqrt{t}}u(t) \to f$ *in* $L^\infty(\mathbb{R}^N)$ *as* $t \to \infty$.
(iii) *If* $f \in \omega_1(u)$ *and* $f \geq 0$, $f \not\equiv 0$, *then* $\omega_1(u) = \overline{\bigcup_{s\geq 0}\{D_{\sqrt{1+s}}\mathcal{S}(s)f\}} = \bigcap_{s_0\geq 0}\overline{\bigcup_{s\geq s_0}\{D_{\sqrt{1+s}}\mathcal{S}(s)f\}}$, *where the closures are in* $\mathcal{W} \cap C_0(\mathbb{R}^N)$. *Moreover,* $\inf_{w\in\omega_1(u)}\|D_{\sqrt{t}}u(t) - w\|_{\mathcal{W}\cap C_0} \underset{t\to\infty}{\longrightarrow} 0$.
(iv) *If* $\omega_1(u) = \{f\}$ *and* $f \geq 0$, $f \not\equiv 0$, *then* $f = \mathcal{U}(1)\varphi$ *with* $\varphi \in \mathcal{W}$ *homogeneous of degree* $-2/\alpha$, $\varphi \geq 0$.

(v) *If* $f \in \omega_1(u)$ *and* $f = \mathcal{U}(1)\varphi$ *with* $\varphi \in \mathcal{W}$ *homogeneous of degree* $-2/\alpha$, $\varphi \geq 0$, *then* $\|D_{\sqrt{t}}u(t) - f\|_{\mathcal{W}\cap C_0} \to 0$ *as* $t \to \infty$. *In particular,* $\omega_1(u) = \omega(u) = \{f\}$.

(vi) *If* $\omega_1(u) \neq \emptyset$, *then* $\omega(u) = \overline{\omega_1(u)}$, *where the closure is in* $C_0(\mathbb{R}^N)$.

PROOF. The proofs of (i), (ii), (iii), (iv), (v) and (vi) are analogous to the proofs of (i), (ii), (iv), (v), (vi) and (vii), respectively, in Proposition 3.8. One uses Proposition 4.10 instead of Proposition 3.5 and Lemma 4.12 instead of Lemma 3.7. □

REMARK 4.14. It follows from part (iii) of Proposition 4.13 that if $f \in \omega_1(u)$, then there exist $s_n \to \infty$ such that $f = \lim_{n\to\infty} D_{\sqrt{1+s_n}}\mathcal{S}(s_n)f$ in $\mathcal{W} \cap C_0(\mathbb{R}^N)$.

We now give the principal results of this section, i.e. the characterizations of $\omega(u)$ and $\omega_1(u)$.

THEOREM 4.15. *Assume* $0 < \alpha < 2/N$. *Let* $u_0 \in \mathcal{W}$, $u_0 \geq 0$ *and* $u \in \Sigma^+(u_0)$, $u \not\equiv 0$. *Let* $\Omega(u_0)$ *and* $\omega(u)$ *be defined by* (1.20) *and* (1.25), *respectively.*

(i) $\omega(u) = \mathcal{U}(1)\Omega(u_0)$. *In particular, if* $u_0 \in \mathcal{B}_M^+$ *for some* $M > 0$, *then* $\omega(u) \subset \mathcal{U}(1)\mathcal{B}_M^+$.

(ii) $\mathcal{U}(1)$ *is a homeomorphism of* $\Omega(u_0) \subset (\mathcal{B}_M^+, \mathrm{d}_M^*)$ *onto* $\omega(u) \subset C_0(\mathbb{R}^N)$, *this latter space considered with its norm topology.*

PROOF. (i) By Proposition 4.9 it suffices to consider the solution $u(t) = \mathcal{U}(t)u_0$. In this case, the desired relation is an immediate consequence of (4.5) and Proposition 4.8 (ii).

(ii) This follows from (i) and Proposition 4.8 (ii). □

REMARK 4.16. Suppose $u_0 \in \mathcal{W}$, $u_0 \geq 0$ and $u \in \Sigma^+(u_0)$, $u \not\equiv 0$.

(i) $\Omega(u(t)) = \Omega(u_0)$ for all $t > 0$. Indeed, Proposition 2.9 and Theorem 4.15 (i) imply that $\mathcal{U}(1)\Omega(u(t_0)) = \mathcal{U}(1)\Omega(u_0)$, and the result follows by backward uniqueness (see Remark 2.10).

(ii) Let $f \in \omega(u)$. Since $f \in \mathcal{U}(1)\Omega(u_0)$, it follows from Remark 4.6 (ii) that $f \geq R_0$, where R_0 is given by (4.1). In particular, $f > 0$.

COROLLARY 4.17. *Assume* $0 < \alpha < 2/N$ *and let* $M > 0$. *There exists* $U_0 \in \mathcal{B}_M^+ \cap C^\infty(\mathbb{R}^N)$ *such that* $\omega(U) = \mathcal{U}(1)\mathcal{B}_M^+$ *for all* $U \in \Sigma^+(U_0)$. *In particular, given any* $u_0 \in \mathcal{B}_M^+$ *and* $U \in \Sigma^+(U_0)$, *there exist* $t_n \to \infty$ *such that* $D_{\sqrt{t_n}}U(t_n) \to \mathcal{U}(1)u_0$ *in* $C_0(\mathbb{R}^N)$, *or equivalently*

$$\|U(t_n) - \mathcal{U}(t_n)D_{\frac{1}{\sqrt{t_n}}}u_0\|_{L^\infty} = \|U(t_n) - D_{\frac{1}{\sqrt{t_n}}}\mathcal{U}(1)u_0\|_{L^\infty} = o(t_n^{-\frac{1}{\alpha}}),$$

as $n \to \infty$.

PROOF. By Lemma 2.7 in [5] (see the proof of Theorem 1.2 in [5]), there exists $U_0 \in \mathcal{B}_M^+ \cap C^\infty(\mathbb{R}^N)$ such that $\Omega(U_0) = \mathcal{B}_M^+$, and the result follows from Theorem 4.15. □

REMARK 4.18. Any solution $U \in \Sigma^+(U_0)$ with U_0 given by Corollary 4.17 is "universal", in the sense that $\mathcal{U}(1)u_0 \in \omega(U)$ and $\omega(u) \subset \omega(U)$ for all $u_0 \in \mathcal{B}_M^+$ and $u \in \Sigma^+(u_0)$. Moreover, $\omega_1(U) \neq \omega(U)$ for all $U \in \Sigma^+(U_0)$. To see this, recall that if $\varphi \in \mathcal{B}_M^+$ is homogeneous of degree $-\sigma$ (for example $\varphi = 0$), then $\mathcal{U}(1)\varphi \notin \omega_1(U)$, for otherwise Proposition 4.13 (v) would imply that $\omega(U) = \{\mathcal{U}(1)\varphi\}$. We do not know if $\omega_1(U) = \emptyset$.

COROLLARY 4.19. *Assume* $0 < \alpha < 2/N$. *Let* $M > 0$ *and suppose that* $H \subset C(\mathbb{R}^N \setminus \{0\})$ *is a nonempty, compact, connected subset of* $(\mathcal{B}_M^+, \mathrm{d}_M^*)$ *such that every* $\varphi \in H$ *is homogeneous of degree* $-2/\alpha$. *Then there exists* $V_0 \in \mathcal{B}_M^+ \cap C_0(\mathbb{R}^N)$ *such that* $\omega(V) = \mathcal{U}(1)H$ *for all* $V \in \Sigma^+(V_0)$. *In particular, given any* $\varphi \in H$ *and* $V \in \Sigma^+(V_0)$, *there exist* $t_n \to \infty$ *such that* $D_{\sqrt{t_n}}V(t_n) \to \mathcal{U}(1)\varphi$ *in* $C_0(\mathbb{R}^N)$, *or equivalently*

$$\|V(t_n) - \mathcal{U}(t_n)\varphi\|_{L^\infty} = o(t_n^{-\frac{1}{\alpha}}) .$$

In other words, $V(t)$ *is asymptotic, along an appropriate subsequence, to every possible self-similar solution of* (1.1) *with initial value in* H. *Furthermore, if* H *contains at least two elements, then* $\omega_1(V) = \emptyset$.

PROOF. It follows from Proposition 2.9 in [5] that there exists $V_0 \in \mathcal{B}_M^+ \cap C_0(\mathbb{R}^N)$ such that $\Omega(V_0) = H$. The first statement is then a consequence of Theorem 4.15 and the second statement follows from Proposition 4.13 (v). □

The following result is an analogue of Corollary 3.14.

COROLLARY 4.20. *Assume* $0 < \alpha < 2/N$. *Given* $M > 0$ *and* $\lambda > 1$, *the map* $D_\lambda : \mathcal{B}_M^+ \to \mathcal{B}_M^+$ *becomes, under the homeomorphism* $\mathcal{U}(1) : (\mathcal{B}_M^+, \mathrm{d}_M^*) \to \mathcal{U}(1)\mathcal{B}_M^+$,

$$F_\lambda = D_\lambda \mathcal{S}(\lambda^2 - 1) = \mathcal{S}\left(1 - \frac{1}{\lambda^2}\right) D_\lambda .$$

Moreover, the mapping F_λ *of* $\mathcal{U}(1)\mathcal{B}_M^+$ *is chaotic.*

PROOF. We note that

$$\mathcal{U}(1)D_\lambda = D_\lambda \mathcal{U}(\lambda^2) = D_\lambda \mathcal{S}(\lambda^2 - 1)\mathcal{U}(1) = \mathcal{S}\left(1 - \frac{1}{\lambda^2}\right) D_\lambda \mathcal{U}(1) = F_\lambda \mathcal{U}(1) ,$$

by (4.4), Remark 4.6 (iii) and (1.7). This proves the first statement.

The second statement follows from Proposition 4.5 (ii) and (the proof of) Proposition 2.11 in [5]. (We note that if $z = m|x|^{-\frac{2}{\alpha}}$ and $\widetilde{z} = \widetilde{m}|x|^{-\frac{2}{\alpha}}$ with $m \neq \widetilde{m}$, then the corresponding solutions of (1.1) are self-similar with profiles $\mathcal{U}(1)z$ and $\mathcal{U}(1)\widetilde{z}$, so that $\mathcal{U}(1)z \neq \mathcal{U}(1)\widetilde{z}$.) □

THEOREM 4.21. *Assume* $0 < \alpha < 2/N$. *Let* $u_0 \in \mathcal{W}$, $u_0 \geq 0$ *and let* $u \in \Sigma^+(u_0)$. *Let* $\Omega_1(u_0)$ *and* $\omega_1(u)$ *be defined by* (1.21) *and* (1.26), *respectively. If* $u \not\equiv 0$ *and* $\Omega_1(u_0) \neq \emptyset$, *then* $\omega_1(u) = \overline{\mathcal{U}(1)\Omega_1(u_0)}$.

PROOF. Let $z \in \Omega_1(u_0)$ and $u \in \Sigma^+(u_0)$, $u \not\equiv 0$. There exists $t_n \to \infty$ such that $\mathrm{d}_M(D_{\sqrt{t_n}}u_0, z) \to 0$ (where M is such that $u_0 \in \mathcal{B}_M$). Note that, with Γ_λ defined by (1.4), $D_{\sqrt{t_n}}u(t_n) = [\Gamma_{\sqrt{t_n}}u](1)$. Since

$$\|\Gamma_{\sqrt{t_n}}u(t)\|_{L^1} = t_n^{\frac{1}{\alpha}-\frac{N}{2}}\|u(tt_n)\|_{L^1} \underset{t\downarrow 0}{\longrightarrow} t_n^{\frac{1}{\alpha}-\frac{N}{2}}\ell\,,$$

where $0 < \ell \leq \infty$ since $u \not\equiv 0$, we deduce from Proposition 4.7 (ii) that $D_{\sqrt{t_n}}u(t_n) \to \mathcal{U}(1)z$ in $\mathcal{W} \cap C_0(\mathbb{R}^N)$, i.e. $\mathcal{U}(1)z \in \omega_1(u)$. Since $\omega_1(u)$ is closed, $\overline{\mathcal{U}(1)\Omega_1(u_0)} \subset \omega_1(u)$. We now show the reverse inclusion. Given $\varphi \in \Omega_1(u_0)$, it follows from what precedes that $f = \mathcal{U}(1)\varphi \in \omega_1(u)$. Proposition 4.13 (iii) implies that $\omega_1(u) = \overline{\underset{s\geq 0}{\cup}\{D_{\sqrt{1+s}}\mathcal{S}(s)f\}}$, and thus

$$\omega_1(u) = \overline{\underset{s\geq 0}{\cup}\{D_{\sqrt{1+s}}\mathcal{U}(1+s)\varphi\}} = \overline{\underset{s\geq 0}{\cup}\{\mathcal{U}(1)D_{\sqrt{1+s}}\varphi\}} = \overline{\mathcal{U}(1)\underset{s\geq 0}{\cup}\{D_{\sqrt{1+s}}\varphi\}}\,.$$

By Proposition 2.5 (iv) in [5], $D_{\sqrt{1+s}}\varphi \in \Omega_1(u_0)$ for all $s \geq 0$. Thus $\overline{\mathcal{U}(1)\underset{s\geq 0}{\cup}\{D_{\sqrt{1+s}}\varphi\}} \subset \overline{\mathcal{U}(1)\Omega_1(u_0)}$ and the result follows. □

REMARK 4.22. There exists an initial value $u_0 \in \mathcal{W} \cap C_0(\mathbb{R}^N)$, $u_0 > 0$ such that $\omega_1(u) \neq \emptyset$ and $\omega_1(u) \neq \omega(u)$ for all $u \in \Sigma^+(u_0)$. Indeed, if u_0 is the initial value given in Proposition 2.13 in [5] with $\sigma = 2/\alpha$, then $c|\cdot|^{-\frac{2}{\alpha}} \in \Omega(u_0)$ for some $c > 0$ and $v \in \Omega_1(u_0)$ for some $v \in \mathcal{W}$, $v \neq c|\cdot|^{-\frac{2}{\alpha}}$. It follows from Theorems 4.15 and 4.21 that $\mathcal{U}(1)(c|\cdot|^{-\frac{2}{\alpha}}) \in \omega(u)$ and $\mathcal{U}(1)v \in \omega_1(u)$. On the other hand, $\mathcal{U}(1)(c|\cdot|^{-\frac{2}{\alpha}}) \notin \omega_1(u)$, since if it were, then $\omega_1(u) = \{\mathcal{U}(1)(c|\cdot|^{-\frac{2}{\alpha}})\}$ by Proposition 4.13 (v). Furthermore, by Proposition 4.13 (vi), we see that $\omega_1(u)$ is not closed in $C_0(\mathbb{R}^N)$, therefore not compact in either $C_0(\mathbb{R}^N)$ or $\mathcal{W}\cap C_0(\mathbb{R}^N)$.

PROPOSITION 4.23. *Let* $u_0(x) = |x|^{-\frac{2}{\alpha}}g(\log|x|)\zeta(x/|x|)$ *with* $\zeta \in C(S^{N-1})$, $\zeta \geq 0$ *and* $g \in C(\mathbb{R}, \mathbb{R})$, $g \geq 0$. *Set* $u(t) = \mathcal{U}(t)u_0$ *for* $t > 0$ *and* $v(s) = \mathcal{U}(1)D_{e^s}u_0$ *for* $s \in \mathbb{R}$.

(i) *If* g *is periodic, then* v *is periodic.*
(ii) *If* $g(t)$ *is asymptotically periodic as* $t \to \infty$, *then* $v(s)$ *is asymptotically periodic as* $s \to \infty$ *as a function* $\mathbb{R} \to \mathcal{W} \cap C_0(\mathbb{R}^N)$.
(iii) *If* g *is almost periodic, then* v *is almost periodic as a function* $\mathbb{R} \to \mathcal{W}\cap C_0(\mathbb{R}^N)$.
(iv) *If* $g(t)$ *is asymptotically almost periodic as* $t \to \infty$, *then* $v(s)$ *is asymptotically almost periodic as* $s \to \infty$, *as a function* $\mathbb{R} \to \mathcal{W} \cap C_0(\mathbb{R}^N)$.

In particular, in all these cases, $\omega(u) = \omega_1(u)$ *Moreover, if* $f \in \omega(u) = \omega_1(u)$ *and* $s_n \to \infty$ *are such that* $\|v(s_n) - f\|_{L^\infty} \to 0$ *as* $n \to \infty$, *then* $\|v(s_n) - f\|_{\mathcal{W}\cap C_0} \to 0$ *as* $n \to \infty$.

PROOF. The proof is analogous to the proof of Proposition 3.19, using Proposition 4.8 instead of Proposition 3.4. □

REMARK 4.24. Assume $u_0 \in C_0(\mathbb{R}^N)$, $u_0 \geq 0$, $u_0 \not\equiv 0$ is asymptotically homogeneous as $|x| \to \infty$, i.e. there exists $\eta \in C(S^{N-1})$ such that $|x|^{\frac{2}{\alpha}} u_0(x) - \eta(x/|x|) \to 0$ as $|x| \to \infty$. Setting $z(x) = |x|^{-\frac{2}{\alpha}} \eta(x/|x|)$, we see that $u_0, z \in \mathcal{W}$ and that $\Omega_1(u_0) = \{z\}$. We deduce from Theorem 4.21 that $\omega_1(u) = \{f\}$ with $f = \mathcal{U}(1)z$, for every $u \in \Sigma^+(u_0)$. Remark 4.11 implies that $|x|^{\frac{2}{\alpha}} f(x) - \eta(x/|x|) \to 0$ as $x \to \infty$. By Proposition 4.13 (v), we see that $u(t)$ is asymptotically self-similar as $t \to \infty$ in the sense that $\|D_{\sqrt{t}} u(t) - f\|_{\mathcal{W} \cap C_0} \to 0$ as $t \to \infty$, for every $u \in \Sigma^+(u_0)$. Thus the results of this section contain as a very particular case Theorem 1.2 of [4] (case $\alpha < 2/N$).

5. – The asymptotically linear case

In this section, we return to the case $\alpha > 2/N$, and so $\mathcal{S}(t)$ acts on $\mathcal{W}$ as in Definition 3.2. Our goal is to give a more complete description of the asymptotic behavior of a class of solutions for which $\omega(u) = \omega_1(u) = \{0\}$. More precisely, let $u_0 \in \mathcal{W}$ and set $u(t) = \mathcal{S}(t)u_0$. Suppose that for some $2/\alpha < \sigma < N$, $|\cdot|^\sigma u_0(\cdot) \in L^\infty(\mathbb{R}^N)$. It follows that $\Omega(u_0) = \Omega_1(u_0) = \{0\}$; and so, $\omega(u) = \omega_1(u) = \{0\}$ by Theorems 3.10 and 3.16.

LEMMA 5.1. *Suppose $\alpha > 2/N$ and let $\mathcal{S}(t)$ be as in Definition 3.2. Let $u_0 \in \mathcal{W}$ and let $u(t) = \mathcal{S}(t)u_0$. If $|\cdot|^\sigma u_0(\cdot) \in L^\infty(\mathbb{R}^N)$ for some $2/\alpha < \sigma < N$, then*

$$\sup_{x \in \mathbb{R}^N} (t + |x|^2)^{\frac{\sigma}{2}} |u(t,x) - e^{t\Delta} u_0| \to 0\,, \tag{5.1}$$

as $t \to \infty$.

PROOF. It follows easily from (2.1) and Proposition 3.4 (i) that $|u(t)| \leq e^{t\Delta}|u_0|$. We deduce from Corollary 8.3 in [4] that $|u(t,x)| \leq C(t+|x|^2)^{-\frac{1}{\alpha}} \|u_0\|_{\mathcal{W}}$ and $|u(t,x)| \leq C(t + |x|^2)^{-\frac{\sigma}{2}} \| |\cdot|^\sigma u_0\|_{L^\infty}$. Therefore, there exists C such that for all $1/\alpha \leq \mu \leq \sigma/2$,

$$|u(t,x)| \leq C(t + |x|^2)^{-\mu}\,. \tag{5.2}$$

Given $1/\alpha \leq \mu \leq \sigma/2$, we write $\mu(\alpha+1) = a + b$ with $a, b > 0$ and $b < N/2$. We deduce from (5.2) that

$$|u|^{\alpha+1}(s,x) \leq C(s + |x|^2)^{-\mu(\alpha+1)} \leq Cs^{-a}(s + |x|^2)^{-b}\,;$$

and so, by using again Corollary 8.3 in [4],

$$|e^{(t-s)\Delta} |u(s)|^\alpha u(s)| \leq Cs^{-a}(t + |x|^2)^{-b}\,. \tag{5.3}$$

We now consider $1/\alpha \le \mu_1, \mu_2 \le \sigma/2$ such that $\mu_i(\alpha+1) = a_i + b_i$ with $\sigma/2 < b_i < N/2$, $0 < a_1 < 1$ and $a_2 > 1$. For $t > 1$, we write

$$u(t) - e^{t\Delta}u_0 = -\left(\int_0^1 + \int_1^t\right) e^{(t-s)\Delta}|u(s)|^\alpha u(s)\,ds\,,$$

and we use (5.3) with a_1, b_1 for $s < 1$ and (5.3) with a_2, b_2 for $s > 1$. It follows that

$$\begin{aligned}|u(t,x) - e^{t\Delta}u_0(x)| &\le C(t+|x|^2)^{-b_1}\int_0^1 s^{-a_1}\,ds + C(t+|x|^2)^{-b_2}\int_1^\infty s^{-a_2}\,ds\\ &\le C(t+|x|^2)^{-b_1} + C(t+|x|^2)^{-b_2}\end{aligned}$$

which shows (5.1). □

In view of (5.1), the asymptotic behavior of $u(t)$ with the scale $(t+|x|^2)^{\frac{\sigma}{2}}$ is the same as the asymptotic behavior of $e^{t\Delta}u_0$. This last behavior is described in Section 3 of [5] in terms of sets like $\Omega(u_0)$ and $\Omega_1(u_0)$, but corresponding to the dilations $\lambda^\sigma u_0(\lambda x)$. See in particular Theorems 3.9 and 3.16 in [5].

6. – A Ginzburg-Landau equation

In this section, we consider the equation

$$u_t - \xi\Delta u = \zeta|u|^\alpha u\,, \tag{6.1}$$

in $(0,\infty)\times\mathbb{R}^N$, where $\alpha > 0$, $\xi, \zeta \in \mathbb{C}$ and $\operatorname{Re}\xi > 0$. Of course, the solutions of (6.1) are complex valued, so all the function spaces that we consider in this section are complex valued. In particular, we set $C_0(\mathbb{R}^N) = C_0(\mathbb{R}^N, \mathbb{C})$ and we consider the space $\mathcal{W}$, the sets $\mathcal{B}_M$ and the metric spaces $(\mathcal{B}_M, \mathrm{d}_M^*)$ and $(\mathcal{B}_M, \mathrm{d}_M)$ as in the introduction, but where the functions are allowed to be complex-valued. Given $u_0 \in \mathcal{W}$, the sets $\Omega(u_0)$ and $\Omega_1(u_0)$ are defined by (1.20) and (1.21).

As is well known, given any $u_0 \in C_0(\mathbb{R}^N)$, there exists a unique solution $u \in C([0,T_{\max}), C_0(\mathbb{R}^N))$ of (1.1) with the initial condition $u(0,x) = u_0(x)$, which is defined on the maximal interval $[0,T_{\max})$ with $T_{\max} = T_{\max}(u_0)$. We denote the solution by

$$u(t) = \mathcal{S}(t)u_0\,. \tag{6.2}$$

We will show that for small initial data in $\mathcal{W}$, the solutions are global and the description of their asymptotic behavior is similar to the description given in Section 3 for the heat equation.

We note that if $\alpha \le 2/N$ and $\xi = \zeta = 1$, then all positive solutions of (6.1) blow up in finite time, so we assume $\alpha > 2/N$. We begin with a simple existence result.

LEMMA 6.1. *Assume $\alpha > 2/N$. There exist $\delta > 0$ and $M > 0$ such that for all $u_0 \in \mathcal{B}_\delta$, there exists a unique solution $u \in C((0,\infty), C_0(\mathbb{R}^N))$ of* (6.1) *such that $u(t) \to u_0$ in $L^1_{\mathrm{loc}}(\mathbb{R}^N)$ as $t \downarrow 0$ and $\sup_{t>0} \|D_{\sqrt{t}} u(t)\|_{\mathcal{W}\cap C_0} \le M$.*

PROOF. We note that $e^{t\xi\Delta}$ is the convolution with the kernel

$$G^\xi_t = (4\pi\xi t)^{-\frac{N}{2}} e^{-\frac{|x|^2}{4\xi t}} .$$

Since $|G^\xi_t| \le (4\pi|\xi|t)^{-\frac{N}{2}} e^{-\frac{|x|^2 \mathrm{Re}\,\xi}{4|\xi|^2 t}}$, we deduce that

$$|e^{t\xi\Delta}\varphi| \le a e^{bt\Delta}|\varphi| , \tag{6.3}$$

for all $\varphi \in C_0(\mathbb{R}^N)$, where $a = (|\xi|/\mathrm{Re}\,\xi)^{\frac{N}{2}}$ and $b = |\xi|^2/\mathrm{Re}\,\xi$.
Let $u \in C((0,\infty), C_0(\mathbb{R}^N))$ satisfy $\sup_{t>0} \|D_{\sqrt{t}} u(t)\|_{\mathcal{W}\cap C_0} \le M$ for some $M>0$. Since $|u(s,x)|^{\alpha+1} \le M^{\alpha+1}(s+|x|^2)^{-\frac{\alpha+1}{\alpha}} \le M^{\alpha+1}s^{-1+\varepsilon}(s+|x|^2)^{-\frac{1}{\alpha}-\varepsilon}$ for $0 < \varepsilon < 1$, we deduce from (6.3) and Corollary 8.3 in [4] that $|e^{(t-s)\xi\Delta}|u(s)|^\alpha u(s)| \le CM^{\alpha+1}s^{-1+\varepsilon}(t+|x|^2)^{-\frac{1}{\alpha}-\varepsilon}$. Therefore, for $0 \le \tau \le t$,

$$\begin{aligned} \left| \int_0^\tau e^{(t-s)\xi\Delta}|u(s)|^\alpha u(s)\,ds \right| &\le CM^{\alpha+1}\tau^\varepsilon (t+|x|^2)^{-\frac{1}{\alpha}-\varepsilon} \\ &\le CM^{\alpha+1}(t+|x|^2)^{-\frac{1}{\alpha}} . \end{aligned} \tag{6.4}$$

On the other hand, it follows from (6.3) and Corollary 8.3 in [4] that $|e^{t\xi\Delta}u_0| \le C(t+|x|^2)^{-\frac{1}{\alpha}}\|u_0\|_{\mathcal{W}}$. By a standard contraction mapping argument, if M and δ are sufficiently small, there exists a unique solution $u \in C((0,\infty), C_0(\mathbb{R}^N))$ of

$$u(t) = e^{t\xi\Delta}u_0 + \zeta \int_0^t e^{(t-s)\xi\Delta}|u(s)|^\alpha u(s)\,ds , \tag{6.5}$$

such that $\sup_{t>0} \|D_{\sqrt{t}} u(t)\|_{\mathcal{W}\cap C_0} \le M$. Using the first inequality in (6.4) with $\tau = t$, we see that $u(t) \to u_0$ in $L^1_{\mathrm{loc}}(\mathbb{R}^N)$. That u is a solution of (6.1) follows from standard arguments. □

PROPOSITION 6.2. *Suppose $\alpha > 2/N$ and let δ be as in Lemma* 6.1. *Given $u_0 \in \mathcal{B}_\delta$, let $\mathcal{S}(t)u_0$ be the solution of* (6.1) *given by Lemma* 6.1. *Given any $t > 0$, the following properties hold.*

(i) *$\mathcal{S}(t)$ is continuous $(\mathcal{B}_\delta, \mathrm{d}^*_\delta) \to C_0(\mathbb{R}^N)$.*
(ii) *$\mathcal{S}(t)$ is continuous $(\mathcal{B}_\delta, \mathrm{d}_\delta) \to \mathcal{W} \cap C_0(\mathbb{R}^N)$.*

PROOF.

(i) By uniqueness, we only need an analogue of the first part of Theorem 2.2 (see the proof of Proposition 3.4). This is proved as in Theorem 2.2, because of the a priori estimate $|u(t,x)| \le M(t+|x|^2)^{-\frac{1}{\alpha}}$ of Lemma 6.1, except for one technical point. More precisely, in the notation of the proof of Theorem 2.2, we must show that the limit u of the solutions u^{n_k} satisfies (6.5). It clearly suffices to show that

$$\int_0^t e^{(t-s)\xi\Delta}|u^{n_k}(s)|^\alpha u^{n_k}(s)\,ds \underset{k\to\infty}{\longrightarrow} \int_0^t e^{(t-s)\xi\Delta}|u^{n_k}(s)|^\alpha u^{n_k}(s)\,ds\,,$$

in $L^1_{\mathrm{loc}}(\mathbb{R}^N)$. This is an immediate consequence of the fact that $u^{n_k} \to u$ in $C([\tau,t],C_0(\mathbb{R}^N))$ for $0<\tau<t$ and the first inequality in (6.4) (applied to u^{n_k}).

(ii) Suppose $\mathrm{d}_\delta(u_0^n,u_0)\to 0$ as $n\to\infty$, and let $u^n(t)=\mathcal{S}(t)u_0^n$ and $u(t)=\mathcal{S}(t)u_0$. Given $t>0$, it follows from (6.3) and (6.4) that

$$|u^n(t)-u(t)| \le |e^{t\xi\Delta}(u_0^n-u_0)| + CM^{\alpha+1}t^{\frac{\theta-1}{\alpha}}(t+|x|^2)^{-\frac{\theta}{\alpha}}\,. \tag{6.6}$$

Fix $\varepsilon>0$. Since $\|e^{t\xi\Delta}(u_0^n-u_0)\|_{\mathcal{W}\cap C_0}\to 0$ by Proposition 3.8 (iii) in [5], and $\theta>1$, we deduce from (6.6) that there exists $R>0$ such that $(1+|x|^2)^{\frac{1}{\alpha}}|u^n(t,x)-u(t,x)|\le\varepsilon$ for $|x|\ge R$ and n large. Since there is local convergence by (i), we see that if n is large enough, then $(1+|x|^2)^{\frac{1}{\alpha}}|u^n(t,x)-u(t,x)|\le\varepsilon$ for all $x\in\mathbb{R}^N$. Since $\varepsilon>0$ is arbitrary, the result follows. □

In view of Lemma 6.1 and and Proposition 6.2, we see that we have results formally similar to those of Section 3, but limited to small initial values (and excluding those related to backward uniqueness). In particular, we have the following analogue of Theorems 1.3 and 1.4, whose proof is similar.

THEOREM 6.3. *Suppose $\alpha>2/N$ and let δ be sufficiently small. Given $u_0\in\mathcal{B}_\delta$, let $u(t)=\mathcal{S}(t)u_0$ be the solution of* (6.1) *given by Lemma* 6.1. *Also, let $\omega(u)$ and $\omega_1(u)$ be defined by formulas* (1.25) *and* (1.26), *respectively. It follows that $\omega(u)=\mathcal{S}(1)\Omega(u_0)$. Moreover, if $\Omega_1(u_0)\neq\emptyset$, then $\omega_1(u)=\overline{\mathcal{S}(1)\Omega_1(u_0)}$, where the closure is in $\mathcal{W}\cap C_0(\mathbb{R}^N)$.*

We leave it to the reader to formulate the analogues of the other results.

7. – Appendix. Proof of uniqueness in Proposition 3.1 for $\alpha = 2/N$

We use the argument of Brezis and Friedman [2]. Let $u,v\in C((0,\infty),C_0(\mathbb{R}^N))$ be two solutions of (1.1) such that $u(t),v(t)\to u_0$ in $L^1_{\mathrm{loc}}(\mathbb{R}^N\setminus\{0\})$

as $t \downarrow 0$. Set $w = |u - v|/2$. We observe that $w \in C((0,\infty), C_0(\mathbb{R}^N)) \cap H^1_{\mathrm{loc}}((0,\infty)\times\mathbb{R}^N)$. Moreover, it follows from Kato's parabolic inequality that

$$w_t - \Delta w + \frac{1}{2}\big|\,|u|^\alpha u - |v|^\alpha v\big| \le 0\,.$$

Since $\big|\,|u|^\alpha u - |v|^\alpha v\big| \ge 2^{-\alpha}|u-v|^{\alpha+1}$, we deduce that

$$w_t - \Delta w + w^{\alpha+1} \le 0\,. \tag{7.1}$$

Moreover,

$$w(t) \underset{t\downarrow 0}{\longrightarrow} 0 \quad \text{in} \quad L^1_{\mathrm{loc}}(\mathbb{R}^N \setminus \{0\})\,, \tag{7.2}$$

and, by (2.2),

$$0 \le w(t,x) \le C(t + |x|^2)^{-\frac{1}{\alpha}}\,. \tag{7.3}$$

We let

$$\widetilde{w}(t,x) = \begin{cases} w(t,x) \text{ for } t > 0\,, \\ 0 \text{ for } t \le 0\,, \end{cases}$$

so that $\widetilde{w} \in L^1_{\mathrm{loc}}(\mathbb{R}\times\mathbb{R}^N \setminus \{0,0\})$ by (7.3). We also consider a function $\eta \in C^\infty(\mathbb{R})$, $\eta \ge 0$ such that $\eta(t) = 0$ for $t \le 1$, $\eta(t) = 1$ for $t \ge 2$ and $\eta' \ge 0$, and we set $\eta_n(t) = \eta(nt)$. We now proceed in four steps.

STEP 1. $\widetilde{w}_t - \Delta\widetilde{w} + \widetilde{w}^{\alpha+1} \le 0$ in $\mathcal{D}'(\mathbb{R}\times\mathbb{R}^N \setminus \{0,0\})$. Let $\varphi \in C^\infty_{\mathrm{c}}(\mathbb{R}\times\mathbb{R}^N \setminus \{0,0\})$, $\varphi \ge 0$. We need to show that

$$\int_0^\infty \int_{\mathbb{R}^N} w(-\varphi_t - \Delta\varphi + w^\alpha\varphi) \le 0\,. \tag{7.4}$$

Since $\eta_n\varphi \in C^\infty_{\mathrm{c}}((0,\infty)\times\mathbb{R}^N)$, $\eta_n\varphi \ge 0$, it follows from (7.1) that

$$\int_0^\infty \int_{\mathbb{R}^N} w(-\eta_{nt}\varphi - \eta_n\varphi_t - \eta_n\Delta\varphi + w^\alpha\eta_n\varphi) \le 0\,. \tag{7.5}$$

Note that, by (7.3) and dominated convergence,

$$\int_0^\infty \int_{\mathbb{R}^N} \eta_n w(-\varphi_t - \Delta\varphi + w^\alpha\varphi) \underset{n\to\infty}{\longrightarrow} \int_0^\infty \int_{\mathbb{R}^N} w(-\varphi_t - \Delta\varphi + w^\alpha\varphi)\,. \tag{7.6}$$

Also, by (7.2),

$$g(t) = \int_{\mathbb{R}^N} w(t,x)\varphi(t,x)\,dx \underset{t\downarrow 0}{\longrightarrow} 0\,,$$

so that

$$\int_0^\infty \int_{\mathbb{R}^N} \eta_{nt} w\varphi = \int_0^\infty \eta_{nt} g \underset{n\to\infty}{\longrightarrow} 0\,. \tag{7.7}$$

(7.4) now follows from (7.5)-(7.7).

STEP 2. $\int_0^1 \int_{\{|x|<1\}} w^{\alpha+1} < \infty$. To see this, let $\varphi \in C_c^\infty(\mathbb{R}^N)$, $\varphi \geq 0$ and $\varphi(x) = 1$ for $|x| \leq 2$ and set $\phi_n(t,x) = \eta_n(t+|x|^2)\varphi(x)$. Since ϕ_n vanishes in a neighborhood of $x = 0$, we deduce from Step 1 that

$$\int_0^1 \int_{\mathbb{R}^N} w(-\partial_t \phi_n - \Delta\phi_n + w^\alpha \phi_n) \leq -\int_{\mathbb{R}^N} w(1,x)\phi_n(1,x)\,dx \leq 0\,. \tag{7.8}$$

Setting $\Sigma_n = \{(t,x);\ t > 0,\ 1/n < t + |x|^2 < 2/n\}$, an easy calculation shows that $|\partial_t \phi_n| + |\Delta\phi_n| \leq C$ outside Σ_n and $|\partial_t \phi_n| + |\Delta\phi_n| \leq Cn$ on Σ_n. Therefore, it follows from (7.8) that

$$\int_0^1 \int_{\mathbb{R}^N} w^{\alpha+1}\phi_n \leq C + nC \int_{\Sigma_n} w\,. \tag{7.9}$$

On the other hand, we deduce from (7.3) that

$$\int_{\Sigma_n} w \leq C \int_{\Sigma_n} (t+|x|^2)^{-\frac{N}{2}} \leq C n^{\frac{N}{2}} |\Sigma_n| = \frac{C}{n}|\Sigma_1|\,. \tag{7.10}$$

The result follows by letting $n \to \infty$ in (7.9).

STEP 3. $\widetilde{w}_t - \Delta\widetilde{w} \leq 0$ in $\mathcal{D}'(\mathbb{R}\times\mathbb{R}^N)$. We need to show that

$$\int_0^\infty \int_{\mathbb{R}^N} w(-\varphi_t - \Delta\varphi) \leq 0\,, \tag{7.11}$$

for all $\varphi \in C_c^\infty(\mathbb{R}\times\mathbb{R}^N)$, $\varphi \geq 0$. Consider such a φ, and set $\phi_n(t,x) = \eta_n(t+|x|^2)\varphi(t,x)$. It follows from Step 1 that

$$\int_0^\infty \int_{\mathbb{R}^N} w(\eta_{nt}\varphi - \eta_n\varphi_t - \eta_n\Delta\varphi - \varphi\Delta\eta_n - 2\nabla\eta_n\cdot\nabla\varphi) \leq 0\,. \tag{7.12}$$

Note that, by (7.3) and dominated convergence,

$$\int_0^\infty \int_{\mathbb{R}^N} \eta_n w(-\varphi_t - \Delta\varphi) \underset{n\to\infty}{\longrightarrow} \int_0^\infty \int_{\mathbb{R}^N} w(-\varphi_t - \Delta\varphi)\,. \tag{7.13}$$

Also, it follows from an easy calculation that

$$\begin{aligned}&\int_0^\infty \int_{\mathbb{R}^N} w(|\eta_{nt}|\varphi + \varphi|\Delta\eta_n| + 2|\nabla\eta_n|\,|\nabla\varphi|)\\ &\leq nC\int_{\Sigma_n} w \leq nC|\Sigma_n|^{\frac{\alpha}{\alpha+1}}\Big(\int_{\Sigma_n} w^{\alpha+1}\Big)^{\frac{1}{\alpha+1}}\,.\end{aligned} \tag{7.14}$$

Note that, by (7.10) and Step 2,

$$nC|\Sigma_n|^{\frac{\alpha}{\alpha+1}} \left(\int_{\Sigma_n} w^{\alpha+1}\right)^{\frac{1}{\alpha+1}} \le C \left(\int_{\Sigma_n} w^{\alpha+1}\right)^{\frac{1}{\alpha+1}} \underset{n\to\infty}{\longrightarrow} 0 .$$

(7.11) follows from (7.12)-(7.14).

STEP 4. Conclusion. It follows easily from Step 3 that $\widetilde{w} \in L^\infty_{\mathrm{loc}}(\mathbb{R}\times\mathbb{R}^N)$. Therefore, by (7.3), $w \in L^\infty((0,\infty)\times\mathbb{R}^N)$. We then deduce from (7.2)-(7.3) that $w(t) \to 0$ in $L^p(\mathbb{R}^N)$ for all, $1 < p < \infty$ as $t \downarrow 0$, and we conclude that $w(t) \equiv 0$. □

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