

## Failure of Analytic Hypoellipticity in a Class of Differential Operators

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**Abstract.** For the hypoelliptic differential operators  $P = \partial_x^2 + (x^k \partial_y - x^l \partial_t)^2$  introduced by T. Hoshiro, generalizing a class of M. Christ, in the cases of  $k$  and  $l$  left open in the analysis, the operators  $P$  also fail to be *analytic* hypoelliptic (except for  $(k, l) = (0, 1)$ ), in accordance with Treves' conjecture. The proof is constructive, suitable for generalization, and relies on evaluating a family of eigenvalues of a non-self-adjoint operator.

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### 1. – Introduction and result

A differential operator  $P$  is said to be hypoelliptic (respectively, analytic hypoelliptic) on  $\Omega$  if for any  $C^\infty$  (respectively,  $C^\omega$ ) function  $f$  on some open set  $U \subset \Omega$  all the solutions  $u$  of  $Pu = f$  belong to  $C^\infty(U)$  (respectively,  $C^\omega(U)$ ).

The basic result about the hypoellipticity of operators of the type “sum of squares”,  $P = X_1^2 + \dots + X_n^2$ , where  $X_1, \dots, X_n$  are real vector fields of class  $C^\omega(\Omega)$ , is Hörmander's theorem [1] which gives necessary and sufficient conditions for hypoellipticity. But these assumptions are not sufficient for analytic hypoellipticity as was first proved by Baouendi and Goulaouic [2]. Other classes of hypoelliptic operators which fail to be analytic hypoelliptic have been found and there are important results on analytic regularity (see Christ [3]-[8], Christ and Geller [9], Derridj and Tartakoff [10]-[13], Derridj and Zuily [14], Francsics and Hanges [15]-[17], Grigis and Sjöstrand [19], Hanges and Himonas [20]-[22], Helffer [23], Hoshiro [24], Metivier [25]-[26], Pham The Lai and Robert [27], Sjöstrand [28], Tartakoff [29], Treves [30]-[31], see also the survey [18] for more references).

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However, the question of finding a general characterisation of analytic hypoellipticity for sum of squares operators is still open. Christ gave a criterion for analytic hypoellipticity in the two-dimensional case [8]. Treves conjectured a general criterion for analytic hypoellipticity [31]: “For a sum of squares of analytic vector fields to be analytic hypoelliptic it is necessary and sufficient that every Poisson stratum (defined by the symbols of the vector fields) of its characteristic variety be symplectic.”

Extending a result of Christ [3], Hoshiro gave new examples of hypoelliptic operators in  $\mathbb{R}^3$  which fail to be analytic hypoelliptic [24]. They have the form

$$(1.1) \quad P = \frac{\partial^2}{\partial x^2} + \left( x^k \frac{\partial}{\partial y} - x^l \frac{\partial}{\partial t} \right)^2$$

where  $k < l$  are non-negative integers. It is shown, through an elegant proof, that  $P$  is not analytic hypoelliptic if either of the following assumptions are satisfied:

- (i)  $(l + 1)/(l - k)$  is not a positive integer.
- (ii) Both  $l - k$  and  $(l + 1)/(l - k)$  are odd integers.

The case  $k = 0$  was studied by Christ [7] and further refined by Yu [35]. In view of Treves’ conjecture, it is interesting to investigate the cases remained open.

The purpose of the present paper is to show that in the remaining cases, except for  $(k, l) = (0, 1)$ , the operator  $P$  fails to be analytic hypoelliptic as well.

This result is in agreement with Treves’ conjecture. Indeed, the Poisson strata of the characteristic variety  $\Sigma = \{\xi = x^k \eta - x^l \tau = 0\}$  of the operators (1.1) with  $k < l$  are as follows. For  $k \geq 1$  then  $\Sigma_0 = \{\xi = x^k \eta - x^l \tau = 0, x \neq 0\}$ ,  $\Sigma_1 = \{x = \xi = 0, \eta \neq 0\}$ , and  $\Sigma_2 = \{x = \xi = \eta = 0, \tau \neq 0\}$  which is not symplectic since its codimension is odd. For  $k = 0$  and  $l \geq 2$  the strata are  $\Sigma_0 = \{\xi = \eta - x^l \tau = 0, x \neq 0\}$  and  $\Sigma_1 = \{x = \xi = \eta = 0, \tau \neq 0\}$  which is not symplectic due to its odd codimension. By contrast, for  $k = 0$  and  $l = 1$  the characteristic variety is symplectic, and the operator is analytic hypoelliptic (see also [3]).

## 2. – Outline of proof

The proof uses a standard reduction to ordinary differential equations:

LEMMA 1 ([24]). *Let, for  $\zeta \in \mathbb{C}$ ,*

$$(2.2) \quad P_\zeta = -\frac{d^2}{dx^2} + (x^l - x^k \zeta)^2.$$

If there exist  $\zeta \in \mathbb{C}$  and  $f \in L^\infty(\mathbb{R})$ ,  $f \neq 0$ , satisfying

$$(2.3) \quad P_\zeta f = 0$$

then the operator  $P$  is not analytic hypoelliptic.

The proof of Lemma 1 and further references are found in [24].

The results of [24] and of the present paper give:

**THEOREM 2.** *For all non-negative  $k, l$  with  $k < l$  and  $(k, l) \neq (0, 1)$  there are values of  $\zeta \in \mathbb{C}$  such that equation (2.3) has a solution  $f \in L^\infty(\mathbb{R})$ ,  $f \neq 0$ .*

*As a consequence (1.1) is not analytic hypoelliptic.*

To complete the proof of Theorem 2 we only need to consider  $k$  and  $l$  which fail both conditions (i) and (ii), and we prove:

**PROPOSITION 3.** *For all integer values of  $k, l$  with  $1 \leq k < l$  which fail both conditions (i) and (ii) there are values of  $\zeta \in \mathbb{C}$  such that equation (2.3) has a solution  $f \in L^\infty(\mathbb{R})$ ,  $f \neq 0$ .*

*An array of such values of  $\zeta$  satisfy the estimate*

$$(2.4) \quad \zeta = \zeta_M = e^{\frac{i\pi(l-k)}{2(l+1)}} \left[ M \frac{\pi(l+1)(k+1)}{l-k} \right]^{\frac{l-k}{l+1}} \left[ 1 + O\left(M^{-1}\right) \right] \quad (M \rightarrow +\infty)$$

for  $M \in \mathbb{Z}_+$  large enough.

Note that  $k = 0$  is not considered in Proposition 3. In fact for  $k = 0$  and  $l \geq 2$  condition (i) is satisfied so failure of analytic hypoellipticity was proved in [24]. The case  $k = 0$  was studied earlier in [3] (see also the references therein) where it was shown that such operators are not analytic hypoelliptic except for the case  $l = 1$ . The case  $(k, l) = (0, 1)$  is fundamentally distinct as in this case the operator (1.1) is analytic hypoelliptic [3]; this distinction appears naturally in our proof (see Remark 7).

Our proof of existence of bounded solutions of (3) differs from the approaches in [3] and [24] and the method is constructive and general; we restrict the proof to the cases not covered in [24] for simplicity. The values of  $\zeta$  of Theorem 2 are related to the eigenvalues of a non-selfadjoint operator associated to (2.2). Allowing  $\zeta$  to be large is a crucial element in our proof, as it leads to substantial asymptotic simplification of the equations involved. There are other arrays of eigenvalues  $\zeta$  besides (2.4), but we will not pursue this issue here.

### 3. – Main Proofs

#### 3.1. – Notations

We denote by  $\text{const}$  positive constants independent of the parameter  $\xi$ .

$O(\rho^{-c})$  will denote functions (of the variable  $v$ , or of  $s$ ), depending on the parameter  $\rho$ , which are less, in absolute value, than  $\text{const } \rho^{-c}$  uniformly on specified paths.

$o(1)$  will stand here only for terms decaying like powers of  $\rho$ , i.e. for  $O(\rho^{-c})$  for some  $c > 0$ .

#### 3.2. – General setting

Classical methods in the theory of ordinary differential equations were used to show:

LEMMA 4 ([24]). *Equation (2.3) has solutions which decrease exponentially for  $x \rightarrow +\infty$ , respectively  $x \rightarrow -\infty$  (see [24]):*

$$(3.5) \quad \begin{aligned} f_{-}^{\{+\infty\}}(x) &\sim x^{-\frac{l}{2}} \exp(-W(x)) & (x \rightarrow +\infty) \\ f_{-}^{\{-\infty\}}(x) &\sim x^{-\frac{l}{2}} \exp\left((-1)^l W(x)\right) & (x \rightarrow -\infty) \end{aligned}$$

where  $W(x) = x^{l+1}/(l+1) - \zeta x^{k+1}/(k+1)$ , and solutions with exponential growth

$$(3.6) \quad \begin{aligned} f_{+}^{\{+\infty\}}(x) &\sim x^{-\frac{l}{2}} \exp(W(x)) & (x \rightarrow +\infty) \\ f_{+}^{\{-\infty\}}(x) &\sim x^{-\frac{l}{2}} \exp\left((-1)^{l+1} W(x)\right) & (x \rightarrow -\infty). \end{aligned}$$

We will show that there are values of  $\zeta$  for which  $f_{-}^{\{+\infty\}}$  is a multiple of  $f_{-}^{\{-\infty\}}$  by finding the analytic continuation of  $f_{-}^{\{+\infty\}}$  from  $+\infty$  to  $-\infty$  along  $\mathbb{R}$ .

NOTATION. To simplify the notations we will omit from now on the superscript  $+\infty$  from the notations of the small, or a large solution for  $x \rightarrow +\infty$ ; we however need to keep the superscript  $-\infty$  to make the distinction for the small, or a large solution as  $x \rightarrow -\infty$ .

Denote  $m = l - k \geq 1$  and  $q = (l + 1)/(l - k) - 2$ . Since we assume that  $(l + 1)/(l - k)$  is a positive integer, then  $q \geq 0$ ,  $q \in \mathbb{Z}$ . If  $m$  is odd we assume that  $q$  is even (since the case with  $m$  and  $q$  odd falls under the already studied case (ii) [24]).

The substitution

$$(3.7) \quad v = \zeta^{-1} x^{l-k}$$

$g(v) = f(x)$ , brings (2.3) to the form

$$(3.8) \quad g''(v) + \frac{m-1}{m} \frac{1}{v} g'(v) = -\xi^2 v^{2q} (v-1)^2 g(v)$$

where

$$(3.9) \quad \xi = -\frac{i}{m}\zeta^{q+2}.$$

We will only consider  $\xi$  large enough in a strip containing  $\mathbb{R}_+$  (cf. (2.4)):

$$(3.10) \quad \begin{aligned} \xi &= \rho + iZ \text{ with } \rho \in \mathbb{R}_+ \text{ large enough,} \\ &\text{and } Z \in \mathbb{R}, \quad Z = O(1) \quad (\rho \rightarrow +\infty) \end{aligned}$$

(see Section 3.1 for notations).

Correspondingly,

$$(3.11) \quad \zeta = e^{i\frac{\pi}{2(q+2)}}(\rho m)^{\frac{1}{q+2}} [1 + o(1)] \quad (\rho \rightarrow +\infty).$$

The path  $-\mathbb{R}$  from  $+\infty$  to  $-\infty$  in the  $x$ -plane corresponds to the path  $v(-\mathbb{R})$  on the Riemann surface above  $\mathbb{C} \setminus \{0\}$  of the  $v$ -plane. If  $m$  is even then  $v(-\mathbb{R})$  comes from  $\infty$  along the half-line

$$d = e^{-i\frac{\pi}{2(q+2)}}\mathbb{R}_+$$

then turns by an angle of  $m\pi$  around  $v = 0$  and returns towards  $\infty$  along  $d$ . If  $m$  is odd then  $v(-\mathbb{R})$  comes from  $\infty$  along  $d$ , turns by an angle  $m\pi$  around  $v = 0$  then follows  $-d$  towards  $\infty$ .

To prove Proposition 3 we show that there are values of  $\xi$  satisfying (3.10) for which equation (3.8) has a bounded solution on  $v(-\mathbb{R})$  (in fact, exponentially small for  $v \rightarrow \infty$  on the path).

The proof is done as follows. For not too small  $v \in d$  we find precise estimates of the unique (up to a multiplicative factor) small solution  $g_-(v)$  of (3.8) (Lemma 5). On the other hand, for small  $v$ , we find precise estimates for two independent solutions of (3.8),  $g_a(v)$  and  $g_s(v) = v^{1/m}g_{a;s}(v)$  with  $g_a, g_{a;s}$  analytic (Lemma 8). The two regions where the said estimates hold overlap (for  $v$  small, but not too close to 0), so the two representations can be matched in this common region ((3.47), Lemma 10).

If  $m$  is even, we find values of  $\xi$  for which  $g_-$  is a multiple of  $g_a$ ; then the continuation of  $g_-$  along  $v(-\mathbb{R})$  yields a small solution (Lemma 11) and Proposition 3 is proved for  $m$  even.

If  $m$  is odd, and  $q$  is even, we also find precise estimates of the small solution  $g_-^{\{-\infty\}}(v)$  of (3.8) for  $v \in -d$ , from  $\infty$  down to not too small  $v$  (Lemma 6). We decompose  $g_-$  along  $g_a, g_s$  ((3.47), Lemma 10), perform analytic continuation upon a rotation by angle of  $m\pi$  then determine  $\xi$  so that this result is a multiple of  $g_-^{\{-\infty\}}(v)$  ((3.55), Lemma 13). This ends the proof of Proposition 3.

### 3.3. – Small solutions of (3.8) for $|v| \rightarrow \infty, v \in d$

In Section 3.3 we find precise estimates for  $g_-$ , the exponentially small solution of (3.8) on  $d$ ; the main result is Lemma 5 (ii) and (3.30).

#### 3.3.1. – Setting

We make suitable substitutions in (3.8) to obtain uniform estimates for a relatively large domain of variation of  $v$ .

Let  $a$  be a number satisfying

$$(3.12) \quad \frac{1}{q+1} \left[ 1 - \frac{2m}{3m(q+1) + 2m - 1} \right] < a < \frac{1}{q+1}.$$

Note that  $m(q+1) = k+1 > 1$  under our assumptions on  $k, l$ . Also, a simple calculation shows that (3.12) implies  $a > 1/(q+2)$ .

Denote  $s(v) = i\xi \int_1^v t^q (t-1) dt$ ; then

$$(3.13) \quad s(v) = i\xi \left( \frac{1}{q+2} v^{q+2} - \frac{1}{q+1} v^{q+1} + P_0 \right) \equiv i\xi (v-1)^2 P(v).$$

We note that  $P(v) = (q+1)^{-1}(q+2)^{-1} \sum_{j=0}^q (j+1)v^j$  hence  $P(1) \neq 0$  and

$$P_0 = P(0) = (q+1)^{-1}(q+2)^{-1}.$$

The inverse function  $v(s)$  of (3.13) is algebraic, ramified only when  $v = 1$  (at  $s = 0$ ) and  $v = 0$  (at  $s = s_0 \equiv i\xi P_0$ , if  $q \neq 0$ ).

The substitution

$$(3.14) \quad g(v) = P(v)^{1/4} v^{q_1} g_1(v) \quad \text{where} \quad q_1 = -\frac{q+1}{2} + \frac{1}{2m}$$

followed by

$$(3.15) \quad s = s(v) \quad , \quad g_1(v) = h(s(v))$$

transform (3.8) into

$$(3.16) \quad h''(s) + \frac{1}{2s} h'(s) - h(s) = R(s)h(s)$$

where

$$(3.17) \quad R(s) = \frac{\tilde{R}(v)}{\xi^2 v^{2q+2} (v-1)^2} \Big|_{v=v(s)}$$

where

$$(3.18) \quad \tilde{R}(v) = -\frac{3}{16} \left( \frac{vP'(v)}{P(v)} \right)^2 + \frac{1}{4} \frac{v^2 P''(v)}{P(v)} - \frac{q}{4} \frac{vP'(v)}{P(v)} + \left( \frac{q+1}{2} \right)^2 - \frac{1}{4m^2}.$$

The branch of  $v(s)$  in (3.17) is  $v \in d$  if  $s \in s(d)$ , respectively  $v \in -d$  if  $s \in s(-d)$ ; we denote the latter by  $v^-(s)$ .

### 3.3.2. – Shifting the path $s(d)$

We will use some known facts on the behavior of a solution of a differential equation towards a rank one irregular singular point to calculate solutions of (3.8) on a path different from  $s(d)$ .

By Lemma 5, equation (3.8) has an exponentially small solution  $g_-$  along  $d$ , unique up to a multiplicative factor (since  $g_-(v) = f_-^{(+\infty)}(x)$ , see (3.5), (3.7)).

After the substitution (3.14), (3.15) equation (3.8) becomes (3.16). The coefficients of the linear equation (3.16) are analytic on the Riemann surface above  $\mathbb{C} \setminus \{0, s_0\}$ . In addition,  $s = 0$  is an algebraic singularity for the solutions of (3.16). For large values of  $s$  on one sheet of this surface standard theory of linear equations applies.<sup>(1)</sup>

We need the small solution  $h_-$  of (3.16) on  $s(d)$  as we can then define  $g_-(v) = P(v)^{1/4} v^{q_1} h_-(s(v))$  to be the small solution of (3.8) on  $d$ .

The path  $s(d)$  comes from  $\infty$  in the right half-plane, crosses the left half-plane and ends at  $s_0$ — see the Appendix 5.1 for details. By standard theory of linear equations (see [32], [34]) there is a solution  $h_-$  of (3.16) which is exponentially small for  $|s| \rightarrow +\infty$ ,  $\arg(s) \in (-\frac{\pi}{2}, \frac{\pi}{2})$ , and its asymptotic representation has the form

$$(3.19) \quad h_-(s) = s^{-1/4} e^{-s} (1 + O(s^{-1}))$$

which is valid for  $\arg(s) \in (-\frac{\pi}{2}, \pi)$  ( $|s| \rightarrow \infty$ ). Similarly, there exists a solution satisfying

$$(3.20) \quad h_+(s) = s^{-1/4} e^s (1 + O(s^{-1}))$$

for  $|s| \rightarrow +\infty$ ,  $\arg(s) \in (-\pi/2, \pi)$ .<sup>(2)</sup>

We will follow the values of  $g_-(v)$  along  $d$  from  $\infty$  down to small values of  $v$ :

$$(3.21) \quad v_P = e^{-i \frac{\pi}{2(q+2)}} \rho^{-a}$$

on

$$(3.22) \quad d_a \equiv v_P [1, +\infty) \subset d.$$

In variable  $s$  this amounts to following  $h_-(s)$  along  $s(d)$  from  $\infty$  down to

$$(3.23) \quad s_P = s(v_P) = i\rho P_0 - e^{i \frac{\pi}{2(q+2)}} \frac{\rho^{1-a(q+1)}}{q+1} - ZP_0 + o(1) \quad (\rho \rightarrow +\infty).$$

<sup>(1)</sup>The study of solutions of differential equations (linear or nonlinear) having a prescribed asymptotic behavior is done on shifted sectors, for large values of the independent variables (for linear equations see [32], [33], and for nonlinear equations see [34]).

<sup>(2)</sup>The intervals stated for the validity of asymptotic representations are not optimal, but we do not need more.

The behavior (3.19) of  $h_-$  (respectively, (3.20) of  $h_+$ ) holds for  $s \rightarrow \infty$ ,  $s \in s(d)$  and also for  $s \rightarrow \infty$ ,  $s \in i\mathbb{R}_+$ . It is more convenient to use the latter path to continue  $h_\pm$  from  $\infty$  to the finite plane.

We define a path  $\ell$  coming from infinity along  $i\mathbb{R}_+$  and ending at  $s_P$ , homotopically equivalent to  $s(d_a)$  in a subregion of  $\mathbb{C} \setminus \{0, s_0\}$  in which  $\arg s \in (-\frac{\pi}{2}, \frac{\pi}{2}]$  for  $s$  large. Then the value of  $h_-$  at  $s_P$  can be calculated along  $\ell$ .

Let  $t_0 > 0$  small enough, independent of  $\rho$ . Let

$$(3.24) \quad s_M = i\rho P_0 - i \frac{\rho^{1-a(q+1)}}{q+1} + it \quad , \quad t \in [-t_0, 0] \subset \mathbb{R}_-$$

with  $s_M = s(v_M)$  where

$$(3.25) \quad v_M = \rho^{-a} \left( 1 + (iZP_0 - t)\rho^{-1+a(q+1)}(1 + \Delta) \right) ,$$

with  $\Delta = O(\rho^{-a})$  ( $\rho \rightarrow +\infty$ ).

Let  $\ell$  be the path in the  $s$ -plane consisting of the following five parts. (Figure 1 sketches the first three parts of  $\ell$  in the case  $\Re s_0 \geq 0$ ; the vertical lines are in fact included in  $i\mathbb{R}_+$ .) Let  $\ell_1$  be the half-line coming from infinity along  $i\mathbb{R}_+$  down to  $i\rho P_0$ . If  $\Re s_0 \geq 0$  let also  $\ell_{12}$  be the segment from  $i\rho P_0$  to  $s_0$ , avoiding  $s_0$  clockwise on a small circle, then returning to  $i\rho P_0$ . (If  $\Re s_0 < 0$  then  $\ell_1$  continues with  $\ell_2$ .) Let  $\ell_2$  come from  $i\rho P_0$  along  $i\mathbb{R}_+$  towards 0, avoiding 0 clockwise on a small circle, then go up to  $i\rho P_0/2$ . Let  $\ell_3$  be the part from  $i\rho P_0/2$  to  $s_M$ . Finally,  $\ell_4$  is the segment  $[s_M, s_P]$ .

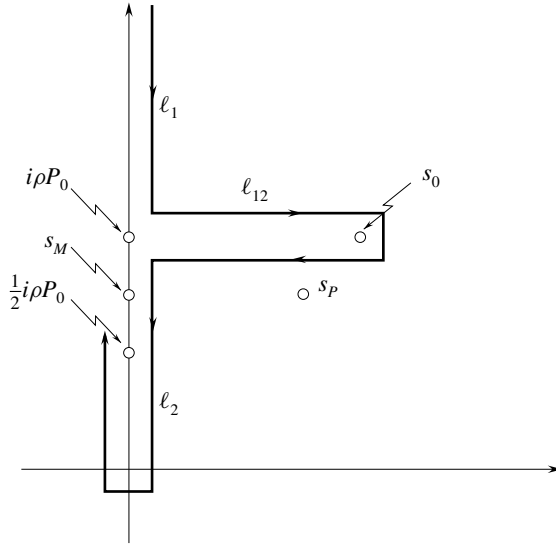


Fig. 1. The beginning of the path  $\ell$ :  $\ell_1$ ,  $\ell_{12}$  and  $\ell_2$ .



The branch of  $v(s)$  in (3.17) on  $\ell$  is obtained by analytic continuation of the branch on  $s(d_a)$  along the path that gives the homotopy between  $s(d_a)$  and  $\ell$  (see Section 4.1 for details).

### 3.3.3. – The solution $g_-$

The solutions of the left-hand side of (3.16), i.e. which satisfy

$$(3.26) \quad B''(s) + \frac{1}{2s}B'(s) - B(s) = 0$$

will turn out to be very close to solutions of (3.16) on  $\ell$ .

The solutions of (3.26) are Bessel functions, with two independent solutions given by

$$(3.27) \quad B_{\pm}(s) = \int_1^{+\infty} e^{\pm ps} (p^2 - 1)^{-3/4} dp \quad \text{for } s \in i\mathbb{R}_+.$$

Equation (3.16) can be written in integral form as

$$h(s) = C_1 B_+(s) + C_2 B_-(s) + \mathcal{J}(h)(s)$$

where

$$(3.28) \quad \mathcal{J}(h)(s) = W_0^{-1} \int_{+i\infty}^s [B_+(s)B_-(\sigma) - B_-(s)B_+(\sigma)] \sigma^{1/2} R(\sigma) h(\sigma) d\sigma$$

$W_0$  is a constant independent of  $\xi$  (the Wronskian of  $B_+$ ,  $B_-$ ) and the path of integration is along  $\ell$ .

For  $C_1 = 0$  and  $C_2 = 1$  we get the small solution (while for  $C_2 = 0$  and  $C_1 = 1$  we get an independent, large solution):

LEMMA 5. (i) *The integral equation*

$$(3.29) \quad h_-(s) = B_-(s) + \mathcal{J}(h_-)(s)$$

*has a unique solution on  $\ell$  and*

$$(3.30) \quad \sup_{s \in \ell} |h_-(s) - B_-(s)| < \text{const } \rho^{-\frac{5}{4} + a(q+1)}$$

(ii) *The solution  $g_-(v) = P(v)^{1/4} v^{q_1} h_-(s(v))$  of (3.8) tends to zero for  $v \rightarrow \infty$ ,  $v \in d$ .*

(iii)  *$g_-$  depends analytically on  $\xi$  for  $\xi$  large in a strip containing  $\mathbb{R}_+$ .*

The proof is given in Section 4.2.

### 3.4. – Solutions of (3.8) for $|v| \rightarrow \infty$ , $v \in -d$ when $m$ is odd and $q$ is even

In this section we find precise estimates for  $g_-^{\{-\infty\}}$ , the exponentially small solution of (3.8) on  $-d$ , and for the large solution  $g_+^{\{-\infty\}}$  and the main result is Lemma 6 (ii) and (3.35). The construction is similar to the one for  $v \in d$ ; we provide the details for completeness.

The proof uses constructions similar to those of Section 3.3.

#### 3.4.1. – Shifting the path $s(-d)$

For  $q$  even  $s(-d)$  starts at  $s_0$ , then goes to  $\infty$  in the right-half plane (see Appendix 5.1 for details).

We consider  $g_-^{\{-\infty\}}$  along  $-d_a$  (see (3.22), (3.23)). Denote

$$(3.31) \quad s_P^- = s(-v_P) = i\rho P_0 + e^{i\frac{\pi}{2(q+2)}} \frac{\rho^{1-a(q+1)}}{q+1} - ZP_0 + o(1) \quad (\rho \rightarrow +\infty).$$

The behavior (3.19), respectively (3.20) of the solution  $h_-^{\{-\infty\}}$ , respectively  $h_+^{\{-\infty\}}$ , holds for  $s \rightarrow \infty$ ,  $s \in s(-d)$  and also for  $s \rightarrow \infty$ ,  $s \in i\mathbb{R}_+$ . It is more convenient to use the latter path to continue  $h_{\pm}^{\{-\infty\}}$  in the finite plane.

We define a path  $\ell^-$  coming from  $\infty$  along  $i\mathbb{R}_+$  and ending at  $s_P^-$ , homotopically equivalent to  $s(-d_a)$  in a subset of  $\mathbb{C} \setminus \{0, s_0\}$  for which  $\arg s \in (-\frac{\pi}{2}, \frac{\pi}{2}]$  for  $s$  large. Then the value of  $h_{\pm}^{\{-\infty\}}$  at  $s_P^-$  can be calculated along  $\ell^-$ .

Let

$$(3.32) \quad s_M^- = i\rho P_0 + i \frac{\rho^{1-a(q+1)}}{q+1} + it \quad , \quad t \in [0, t_0] \subset \mathbb{R}_+$$

(with  $t_0$  small). We have  $s_M^- = s(v_M^-)$  for

$$(3.33) \quad v_M^- = v^-(s_M^-) = -\rho^{-a} \left( 1 - (iZP_0 - t)\rho^{-1+a(q+1)}(1 + \Delta_-) \right) \quad , \\ \Delta_- = O(\rho^{-a}) \quad (\rho \rightarrow +\infty).$$

Let  $\ell^-$  be the path in the  $s$ -plane consisting of  $(+i\infty, s_M^-] \subset i\mathbb{R}_+$  followed by the segment  $[s_M^-, s_P^-]$ .

The branch of  $v(s)$  in (3.17) on  $\ell^-$  is obtained by analytic continuation of the branch  $v^-(s)$  on  $s(-d_a)$  along the path that gives the homotopy between  $s(-d_a)$  and  $\ell^-$ .

#### 3.4.2. – The solutions $g_{\pm}^{\{-\infty\}}$

LEMMA 6. (i) *The integral equations*

$$(3.34) \quad h_{\pm}^{\{-\infty\}}(s) = B_{\pm}(s) + \mathcal{J}(h_{\pm}^{\{-\infty\}})(s)$$

have unique solutions on  $\ell^-$  and

$$(3.35) \quad \sup_{s \in \ell_-} |h_{\pm}^{\{-\infty\}}(s) - B_{\pm}(s)| < \text{const} \max\{\rho^{-3/4}, \rho^{-1+a(q+1)}\}.$$

(ii) The solution  $g_-^{\{-\infty\}}(v) = P(v)^{1/4} v^{q_1} h_-^{\{-\infty\}}(s(v))$  of (3.8) tends to zero for  $v \rightarrow \infty$ ,  $v \in -d$  and  $g_+^{\{-\infty\}}(v) = P(v)^{1/4} v^{q_1} h_+^{\{-\infty\}}(s(v))$  tends to infinity for  $v \rightarrow \infty$ ,  $v \in -d$ .

(iii) The solutions  $g_{\pm}^{\{-\infty\}}$  depend analytically on  $\xi$  for  $\xi$  large, in a strip containing  $\mathbb{R}_+$ .

The proof is given in Section 4.3.

### 3.5. – Solutions for small $v$

The linear equation (3.8) has a regular singular point at  $v = 0$ ; it has an analytic solution  $g_a$  and a ramified solution  $g_s = v^{1/m} g_{a;s}$  (with  $g_{a;s}$  analytic). For a proof and estimates we separate the dominant terms (for small  $v$ ), writing (3.8) in the form

$$(3.36) \quad g''(v) + \frac{m-1}{m} \frac{1}{v} g'(v) + \xi^2 v^{2q} g(v) = -\xi^2 v^{2q} (v^2 - 2v) g(v).$$

The the LHS (dominant part) of (3.36) has two independent solutions (see Section 4.4.1 for details)

$$(3.37) \quad \begin{aligned} \tilde{g}_a(v) &= \int_{-1}^1 e^{-ip\xi \frac{v^{q+1}}{q+1}} (1-p^2)^A dp, & \text{with } A &= -\frac{1}{2} \left(1 + \frac{1}{m(q+1)}\right) \\ \tilde{g}_s(v) &= v^{1/m} \int_{-1}^1 e^{-ip\xi \frac{v^{q+1}}{q+1}} (1-p^2)^B dp, & \text{with } B &= -\frac{1}{2} \left(1 - \frac{1}{m(q+1)}\right). \end{aligned}$$

We have  $A, B \in (-1, 0)$  since  $m(q+1) > 1$ .

REMARK 7. The condition  $A, B \in (-1, 0)$  is fundamental to our proof of Proposition 3, as the behavior of solutions  $g(v)$  for small  $v$  is different in other cases (cf. Remark 14). The only pair of values of the nonnegative integers  $k < l$  for which this condition fails is  $(k, l) = (0, 1)$ . It is known [8] that in this case  $P$  is in fact analytic hypoelliptic.

Then equation (3.36) can be written in integral form (see Section 4.4.1 for details)

$$(3.38) \quad g(v) = C_1 \tilde{g}_a(v) + C_2 \tilde{g}_s(v) + \mathcal{G}(g)(v)$$

where

$$(3.39) \quad \mathcal{G}(g)(v) = \text{const} \xi^{r+1} \int_0^v [\tilde{g}_a(v) \tilde{g}_s(t) - \tilde{g}_s(v) \tilde{g}_a(t)] t^{2(q+1) - \frac{1}{m}} (t-2) g(t) dt$$

where

$$(3.40) \quad r = 1 - \frac{1}{m(q+1)}.$$

Let  $\mathcal{S}$  be the path in the  $v$ -plane composed of two segments:  $\mathcal{S}_1 = [0, v_F]$  with  $v_F = \kappa_0 \rho^{-a}$  (where  $\kappa_0^{q+1} = (1+z)^{-1}$ ,  $\Re \kappa_0 > 0$ ) followed by the segment  $\mathcal{S}_2 = [v_F, v_M]$ .

LEMMA 8. *The integral equation (3.38) has a unique solution for  $v \in \mathcal{S}$ . For  $C_2 = 0$ ,  $C_1 = 1$ , its solution  $g_a(v)$  is analytic at  $v = 0$ . For  $C_1 = 0$ ,  $C_2 = 1$ , its solution has the form  $g_s(v) = v^{1/m} g_{a;s}(v)$  with  $g_{a;s}(v)$  analytic.*

*The following estimates hold*

$$(3.41) \quad |g_{a;s}(v) - \tilde{g}_{a;s}(v)| < \text{const } \rho^{1-a(q+2)} \text{ for } v \in \mathcal{S}.$$

*Also,  $g_a, g_s$  are analytic in  $\xi$  for  $\xi$  large in a strip containing  $\mathbb{R}_+$ .*

The proof is given in Section 4.4.

### 3.6. – Matching between $g_-$ and $g_a, g_s$ .

The calculations are done at  $v = v_M$ , see (3.25). Note that  $v_M$  depends on the real parameter  $t$  (independent of  $\xi$ ).

We first need asymptotic estimates for the solutions  $g_a, g_s, g_-$ :

LEMMA 9. *For  $v$  satisfying (3.25) we have for  $\xi \rightarrow \infty$*

$$(3.42) \quad g_a(v) = K_a \rho^{-\beta} \left[ e^{i(R+\lambda)} + e^{i\pi(1+A)} e^{-i(R+\lambda)} \right] + O(\rho^{-d})$$

$$(3.43) \quad g_s(v) = K_s \rho^{-\beta} \left[ e^{i(R+\lambda)} + e^{i\pi(1+B)} e^{-i(R+\lambda)} \right] + O(\rho^{-d})$$

and

$$(3.44) \quad g_-(v) = K \rho^\gamma \left[ e^{iP_0\xi} e^{-i(R+\lambda)} + \frac{i}{\sqrt{2}} e^{-iP_0\xi} e^{i(R+\lambda)} + O(\rho^{-1+a(q+1)}) \right]$$

where  $K, K_a, K_s$  are constants independent of  $\xi$  and

$$(3.45) \quad R = \frac{\rho^{1-a(q+1)}}{q+1}, \quad \lambda = iZP_0 - t$$

$$-d = \max\{[1 - a(q+1)](-2 - A), 1 - a(q+2)\}$$

$$(3.46) \quad \gamma = a \left( \frac{q+1}{2} - \frac{1}{2m} \right) - \frac{1}{4}, \quad -\beta = [1 - a(q+1)][-1 - A]$$

The proof is given in Section 4.5.

The solution  $g_-$  is a linear combination of the solutions  $g_a, g_s$ :

$$(3.47) \quad g_-(v) = C_a g_a(v) + C_s g_s(v)$$

where the coefficients  $C_a, C_s$  depend on  $\xi$ .

The representations (3.42), (3.43), (3.44) are used to find sharp estimates of  $C_a, C_s$  by identification of the coefficients of the independent functions whose leading behavior is  $e^{it}$ , respectively  $e^{-it}$ :

LEMMA 10. *The constants  $C_a, C_s$  of (3.47) are analytic in  $\xi$  for large  $\xi$  in a strip containing  $\mathbb{R}_+$  and satisfy*

$$(3.48) \quad C_a = \tilde{K}_a \xi^{\gamma+\beta} \left[ -e^{iP_0\xi} + \frac{i}{\sqrt{2}} e^{i\pi(1+B)} e^{-iP_0\xi} + O\left(\xi^{-d+\beta}\right) \right] \quad (\xi \rightarrow \infty)$$

$$(3.49) \quad C_s = \tilde{K}_s \xi^{\gamma+\beta} \left[ e^{iP_0\xi} - \frac{i}{\sqrt{2}} e^{i\pi(1+A)} e^{-iP_0\xi} + O\left(\xi^{-d+\beta}\right) \right] \quad (\xi \rightarrow \infty)$$

where

$$\tilde{K}_{a,s} = \frac{K}{(e^{i\pi A} - e^{i\pi B}) K_{a,s}}.$$

The proof is given in Section 4.6.

### 3.7. – Existence of a bounded solution of (2.3) for $m$ even

It is enough to find values of the parameter  $\xi$  for which that  $g_-$  is a multiple of  $g_a$ . Such values exist indeed:

PROPOSITION 11. *There exist infinitely many values of  $\xi$  for which  $C_s = 0$*

PROOF. For large  $\rho$  equation  $C_s(\xi)=0$  gives, up to lower order terms, (see (3.49))

$$(3.50) \quad e^{i\rho P_0} - i2^{-1/2} e^{-i\rho P_0 + i\pi(1+A)} = 0$$

having the solutions

$$(3.51) \quad \xi_M = \frac{\pi}{P_0} \left( M + \frac{1}{2m(q+1)} \right) - \frac{i}{4P_0} \ln 2, \quad M \in \mathbb{Z}.$$

We will show that for  $M$  large enough, near each  $\xi_M$  there are values of  $\xi$  for which  $C_s = 0$ . Note that the assumption (3.10) is satisfied for  $\xi = \xi_M$  if  $M$  is large enough.

The essence of the rest of the proof is to show that an “approximate” solution implies the existence of an actual solution using the principle of variation of the argument.

Let  $j \in \{1; 2\}$  and  $\alpha > 0$  with  $\alpha < d - \beta$ . Consider the circles  $\mathcal{C}_j$  centered at  $\xi_M$ , of radius  $jM^{-\alpha}$ :

$$(3.52) \quad \xi \in \mathcal{C}_j \quad \text{if} \quad \xi = \xi_M + jM^{-\alpha}\theta \quad \text{where} \quad |\theta| = 1.$$

Note that for  $\xi$  in the interior of the circle  $\mathcal{C}_j$  we have  $\xi = O(M)$  and  $M = O(\xi)$ .

Assume, to get a contradiction, that  $C_s(\xi) \neq 0$  for  $\xi$  in the interior of the circle  $\mathcal{C}_2$ .

Using (3.49), (3.50), (3.52) for  $\xi \in \mathcal{C}_j$  we get

$$(3.53) \quad C_s(\xi) = K_0 e^{iM\pi} j M^{\gamma+\beta-\alpha} (\theta + o(1)) \quad (M \rightarrow +\infty)$$

where  $K_0$  does not depend on  $M$ .

So  $|1/C_s(\xi)| \leq 1/(2|K_0|M^{\gamma+\beta-\alpha})(1 + o(1))$  for  $\xi$  on  $\mathcal{C}_2$ , hence also for  $\xi$  in the interior of this circle. Then  $|C_s(\xi)| \geq 2|K_0|M^{\gamma+\beta-\alpha}(1 + o(1))$  in the interior of  $\mathcal{C}_2$ .

However, using (3.53) for  $j = 1$  we get that  $|C_s(\xi)| \leq |K_0|M^{\gamma+\beta-\alpha}(1 + o(1))$  on the circle  $\mathcal{C}_1$ , which is a contradiction.

### 3.8. – Matching in the case $m$ odd, $q$ even

Denote by  $\phi(v)$  the solution of (3.8) which equals  $g_-$  on  $d$ . For  $v \in d$  we have  $\phi(v) = C_a g_a(v) + C_s g_s(v)$  with  $C_a, C_s$  given by (3.48), (3.49). After analytic continuation of  $\phi(v)$  on a small path rotating around  $v = 0$  by an angle  $m\pi$  the solution  $\phi$  becomes  $\phi(e^{im\pi}v) = C_a g_a(e^{im\pi}v) + C_s g_s(e^{im\pi}v)$  which decomposes along the small solution, and the large solution on  $s(-d)$ :  $\phi(e^{im\pi}v) \equiv D_+ g_{+;-\infty}(-v) + D_- g_{-;-\infty}(-v)$  for some constants  $D_+, D_-; g_{\pm;-\infty}$  are given in Lemma 6 (ii). Hence

$$(3.54) \quad C_a g_a(e^{im\pi}v) + C_s g_s(e^{im\pi}v) \equiv D_+ g_{+;-\infty}(-v) + D_- g_{-;-\infty}(-v).$$

Note that  $g_a(e^{im\pi}v) = g_a(-v) = g_a(v)$  since  $g_a$  is analytic and even, and that  $g_s(e^{im\pi}v) = -v^{1/m} g_{a;s}(v)$  since  $g_{a;s}$  is analytic and even. Relation (3.54) becomes

$$(3.55) \quad C_a g_a(v) - C_s g_s(v) \equiv D_+ g_{+;-\infty}(-v) + D_- g_{-;-\infty}(-v)$$

for  $v \in d$ , hence for all  $v$ .

We first need asymptotic estimates for the solutions  $g_a, g_s, g_{-;-\infty}$  for  $v = -v_M^-$ :

LEMMA 12. For  $v = -v_M^-$  where  $v_M^-$  is defined by (3.32), (3.33) we have

$$(3.56) \quad g_a(v) = K_a \xi^{-\beta} \left[ e^{i(R-\lambda)} + e^{i\pi(1+A)} e^{-i(R-\lambda)} \right] + O(\xi^{-d})$$

$$(3.57) \quad g_s(v) = K_s \xi^{-\beta} \left[ e^{i(R-\lambda)} + e^{i\pi(1+B)} e^{-i(R-\lambda)} \right] + O(\xi^{-d})$$

and

$$(3.58) \quad g_{\pm;-\infty}(v) = K^{-\xi^\gamma} \left[ e^{\pm i\pi/8} e^{\pm i P_0 \xi} e^{\pm i(R-\lambda)} + O(\xi^{-1+a(q+1)}) \right]$$

where the notations are those of (3.45), (3.46).

The proof is given in Section 4.7.

Denote

$$-c = -d + \beta \quad , \quad -c_1 = -1 + a(q + 1).$$

For  $v$  satisfying (3.33) the RHS of (3.55) becomes, in view of (3.56), (3.57), (3.48), (3.49)

$$\begin{aligned} C_a g_a(v) - C_s g_s(v) &= \\ &= K \xi^\gamma e^{i(R-\lambda)} \left[ -2e^{i\xi P_0} + \frac{i}{\sqrt{2}} \left( e^{i\pi(1+A)} + e^{i\pi(1+B)} \right) e^{-i\xi P_0} + O(\xi^{-c}) \right] \\ &\quad + K \xi^\gamma e^{-i(R-\lambda)} \left[ -e^{i\xi P_0} \left( e^{i\pi(1+A)} + e^{i\pi(1+B)} \right) + i\sqrt{2} e^{i\pi(1+A)} e^{i\pi(1+B)} e^{-i\xi P_0} \right. \\ &\quad \left. + O(\xi^{-c}) \right] \\ &\quad + \frac{K}{K^-} g_{+;-\infty} \left[ -2 + \frac{i}{\sqrt{2}} \left( e^{i\pi(1+A)} + e^{i\pi(1+B)} \right) e^{-2i\xi P_0} + O(\xi^{-c}) \right] (1 + O(\xi^{-c_1})) \\ &\quad + \frac{K}{K^-} g_{-;-\infty} \left[ -e^{2i\xi P_0} \left( e^{i\pi(1+A)} + e^{i\pi(1+B)} \right) + i\sqrt{2} e^{i\pi(1+A)} e^{i\pi(1+B)} + O(\xi^{-c}) \right] \\ &\quad \times (1 + O(\xi^{-c_1})) \end{aligned}$$

which in view of (3.55) gives

$$(3.59) \quad D_+ = K/K^- \left[ -2 + \frac{i}{\sqrt{2}} \left( e^{i\pi(1+A)} + e^{i\pi(1+B)} \right) e^{-2i\xi P_0} + O(\xi^{-c_1}) \right. \\ \left. + O(\xi^{-c}) \right] (1 + O(\xi^{-c_1})).$$

The existence of a bounded solution of (3.8) on  $v(-\mathbb{R})$ , hence of (2.3) on  $\mathbb{R}$  is assured if, for some value of the parameter  $\xi$  we have  $D_+ = 0$ . Indeed, there are such values:

**PROPOSITION 13.** *There are infinitely many values of  $\xi$  for which  $D_+ = 0$ , hence for which equation (2.3) has a bounded solution on  $\mathbb{R}$ .*

**PROOF.** We need to find valued for  $\xi$  satisfying (3.10) such that  $D_+ = 0$ . Up to smaller terms,  $\xi$  satisfies

$$-2 + \frac{i}{\sqrt{2}} \left( e^{i\pi(1+A)} + e^{i\pi(1+B)} \right) e^{-2i\xi P_0} = 0$$

which holds for

$$(3.60) \quad \xi_M = -\frac{i}{2P_0} \ln \left( 2^{-1/2} \cos \left( \frac{\pi}{2m(q+1)} \right) \right) + \frac{\pi}{2P_0} (2M + 1).$$

In a neighborhood of each  $\xi_M$  there is a zero of the function  $D_+$  if  $M$  is large enough, the details being as in the proof of Proposition 11.

## 4. – Proof of auxiliary Lemmas

### 4.1. – Some estimates

I. For  $v$  such that  $s(v)$  is on  $\ell$ , or on  $\ell^-$  we have  $|P(v)| > c_0 > 0$ .

Indeed, note first that if  $Z = 0$  (so  $\xi = \rho \in \mathbb{R}_+$ ) the part  $\ell_1$  of  $\ell$  corresponds in the  $v$ -plane to  $[1, +\infty)$ , the part  $\ell_{12}$  corresponds to a small half-circle avoiding  $v = 1$ , and the rest of  $\ell$  corresponds to a subsegment of  $(0, 1)$ . Since  $P(v) \neq 0$  for  $v \in \mathbb{R}_+$ , statement I follows.

In the general case when  $Z \neq 0$  the picture is similar. A continuous deformation of  $s(d)$  to  $\ell$  through the right half-plane corresponds in the  $v$ -plane to a counterclockwise rotation of  $d$ , of angle close to  $\pi/(2(q+2))$ . For large  $v$  we have  $s \sim i(\rho + iZ)v^{q+2}/(q+2)$  so if initially  $v \in d$ , after the deformation  $\arg v \sim -(q+2)^{-1} \arg(\rho + iZ)$  so  $\arg v$  is small. For  $v = O(1)$  ( $\rho \rightarrow +\infty$ ),  $Z$  is negligible in (3.13), (3.10) so  $v$  is close to  $\mathbb{R}_+$  and  $s = 0$  corresponds to  $v = 1$ . At this point the branch of  $v(s)$  (on  $\ell$ ) changes; denote this branch by  $v_+(s)$ .

II. We have  $|v_+(s)| < 1$ . Indeed, for  $s = i\rho\lambda P_0$  with  $\lambda \in [0, 1]$  substituting  $v = 1 + \Delta$  in (3.13) we get  $\Delta \sim \pm \sqrt{2\lambda P_0}(1 - iZ\rho^{-1}/2)$  (for large  $\rho$ ). Then  $v_+(s) \sim 1 - \sqrt{2\lambda P_0}(1 - iZ\rho^{-1}/2)$  and the estimate follows.

REMARK. From I and II it follows that  $s = s_0$  is not a ramification point of  $v(s)$  on the Riemann sheet of  $\mathbb{C} \setminus \{0; s_0\}$  where the homotopic deformation of  $s(d)$  to  $\ell$  was done; hence the presence of the path  $\ell_{12}$  is not necessary and can be omitted in the proofs.

III. Using I it follows that  $|vP'(v)/P(v)| < \text{const}$ ,  $|v^2P''(v)/P(v)| < \text{const}$  so that  $|\tilde{R}(v)| < \text{const}$ .

### 4.2. – Proof of Lemma 5

(i) Consider the operator  $\mathcal{J}$  of (3.28) defined on the Banach space of functions  $h$  which are continuous and bounded on  $\ell$ , in the sup norm, and analytic in  $\xi$ . Write  $\mathcal{J} = \mathcal{J}_+ - \mathcal{J}_-$  where

$$\mathcal{J}_\pm(h)(s) = B_\pm(s) \int_{+i\infty}^s B_\mp(\sigma) \frac{\sigma^{1/2}}{W_0} R(\sigma) h(\sigma) d\sigma.$$

Using (3.17) and (3.13) we have

$$\mathcal{J}_\pm(h)(s) = \text{const } B_\pm(s) \int_{+i\infty}^s B_\mp(\sigma) \frac{\sigma^{-1/2} P(v(\sigma))}{\xi v(\sigma)^{2q+2}} \tilde{R}(\sigma) h(\sigma) d\sigma.$$

Note that (see Section 4.1)

$$(4.61) \quad \left| \tilde{R}(s) \right| \leq \text{const} \quad \text{for } s \in \ell.$$



Let  $\|h\|_{\ell_k} = \sup_{s \in \ell_k} |h(s)|$ . Since  $B_{\pm}(s) \sim s^{-1/4} e^{\pm s}$  for  $s \in i\mathbb{R}_+$  and the solutions of (3.26) are bounded at  $s = 0$  then  $|B_{\pm}(s)| < \text{const}$  for  $s \in \ell_2$  and  $|B_{\pm}(s)| < \text{const} |s|^{-1/4}$  for  $s$  in  $\ell_1, \ell_{12}, \ell_3$  or  $\ell_4$ .

For  $s \in \ell_1$  we have  $|v(s)| \geq 1/2$ . Also, since  $|(v-1)^2 P(v)/|v|^{q+2}| < \text{const}$  on  $v(\ell_1)$  then  $|v|^{-q-2} < \text{const} \rho |s|^{-1}$ .

Hence (using also  $|P(v)/v^q| < \text{const}$ )

$$\left| \int_{i\rho P_0/2}^{+i\infty} B_{\pm}(\sigma) \frac{\sigma^{-1/2} P(v(\sigma))}{\xi v(\sigma)^{2q+2}} \tilde{R}(\sigma) h(\sigma) d\sigma \right| < \text{const} \|h\|_{\ell_1} \int_{\rho P_0/2}^{+\infty} \rho^{-1} \tau^{-3/4} d\tau \\ = \text{const} \rho^{-3/4} \|h\|_{\ell_1}.$$

For  $s \in \ell_2$  we have  $\text{const}_1 > |v| > \text{const}_2 > 0$  hence

$$\left| \int_{i\rho P_0/2}^{i\rho P_0/2} B_{\pm}(\sigma) \frac{\sigma^{-1/2} P(v(\sigma))}{\xi v(\sigma)^{2q+2}} \tilde{R}(\sigma) h(\sigma) d\sigma \right| < \text{const} \|h\|_{\ell_2} \int_0^{\rho P_0/2} \frac{\tau^{-3/4}}{\rho} d\tau \\ = \text{const} \rho^{-3/4} \|h\|_{\ell_2}.$$

If  $\ell_{12}$  is part of  $\ell$  the possible integral on the segment  $[i\rho P_0, i\xi P_0]$  does not modify these estimates since the length of this segment is  $O(1)$  as  $\rho \rightarrow +\infty$ .

For  $s \in \ell_3$  the corresponding  $v$  satisfies  $\text{const} \rho^{-a} < |v| < \text{const} < 1$  (since the branch of  $v(s)$  is  $v_+(s)$ , see Section 4.1.II). Then substituting  $\sigma = i\rho\tau$  in

$$T := \left| B_{\pm}(s) \int_{i\rho P_0/2}^{sM} B_{\mp}(\sigma) \frac{\sigma^{-1/2} P(v(\sigma))}{\xi v(\sigma)^{2q+2}} \tilde{R}(\sigma) h(\sigma) d\sigma \right|$$

and (since  $|P(v)| < \text{const}$ ) we get

$$(4.62) \quad T < \text{const} \rho^{-1} \|h\|_{\ell_3} \int_{1/4}^{1/2(1-\rho^{-a})} \frac{\tau^{-3/4}}{|v(i\rho\tau)|^{2q+2}} d\tau$$

where now  $v$  is given implicitly by  $\tau = (1+z)(v-1)^2 P(v)$ , with the notation  $z = iZ\rho^{-1}$ .

Let  $\tilde{v}(s)$  denote  $v(s)$  for  $z = 0$ . We have  $\tilde{v}_+ \in [c_0\rho^{-a}, v_0] \subset (0, 1)$ . Then  $|v_+(s) - \tilde{v}_+(s)| \leq |z| \sup_{|z| < \text{const} \rho^{-1}} |v'_+(z)| < \text{const} |z| \rho^{-aq}$ . It follows that  $|v|^{-1} < \text{const} \tilde{v}_+^{-1}$  on  $\ell_3$ .

Then (4.62) is less than

$$\text{const} \rho^{-1} \|h\|_{\ell_3} \int_{1/4}^{1/2(1-\rho^{-a})} \frac{1}{\tilde{v}(\tau)^{2q+2}} d\tau = \text{const} \rho^{-1} \|h\|_{\ell_3} \int_{K_0\rho^{-a}}^{v_0} \frac{1-\tilde{v}}{v_1^{q+2}} d\tilde{v} \\ \leq \text{const} \rho^{-1+a(q+1)} \|h\|_{\ell_3}.$$

Finally, for  $s \in \ell_4$  we have  $s = O(\rho)$ ,  $|s_M - s(v_P)| < \text{const } \rho^{1-a(q+1)}$  and  $|v| > \text{const } \rho^{-a}$  so that

$$\left| B_{\pm}(s) \int_{s_M}^s B_{\mp}(\sigma) \frac{\sigma^{-1/2} P(v(\sigma))}{\xi v(\sigma)^{2q+2}} \tilde{R}(\sigma) h(\sigma) d\sigma \right| < \text{const } \rho^{1-a(q+1)} \|h\|_{\ell_4}.$$

Therefore

$$|\mathcal{J}(h)(s)| < \text{const } \max\{\rho^{-3/4}, \rho^{-1+a(q+1)}\} \|h\|$$

and  $\mathcal{J}$  is a contraction for  $a$  satisfying (3.10). Equation (3.29) has a unique solution  $h_- = (1 - \mathcal{J})^{-1} \mathcal{J}(B_-)$ .

To show (3.30) we write  $h_- = B_- + (1 - \mathcal{J})^{-1} \mathcal{J}(B_-)$ . We showed that

$$|\mathcal{J}B_-(s)| \leq \text{const } \rho^{-3/4} \|B_-\|_{\ell_1 \cup \ell_{12} \cup \ell_2} + \text{const } \rho^{-1+a(q+1)} \|B_-\|_{\ell_3}$$

and since  $\|B_-\|_{\ell_3} < \text{const } \rho^{-1/4}$  and  $\|(1 - \mathcal{J})^{-1}\| < \text{const}$ , the estimate (3.30) follows.

(ii) is immediate since  $B_-$ , therefore  $h_-$  are exponentially small on  $s(d)$ .

Finally, (iii) holds since  $B_{\pm}(s)$  do not depend on  $\xi$  and the operator  $\mathcal{J}$  is analytic in  $\xi$ .

### 4.3. – Proof of Lemma 6

Denote by  $\ell_1^-$  the segment of  $\ell^-$  from  $+i\infty$  to  $2i\rho P_0$ , by  $\ell_3^-$  the segment from  $2i\rho P_0$  to  $s_M^-$ , and by  $\ell_4^-$  the segment from  $s_M^-$  to  $s_F^-$ .

The estimates on  $\ell_1^-$  (respectively  $\ell_3^-$ ) are the same as in the proof of Lemma 5(i) on  $\ell_1$  (respectively  $\ell_3$ ). For the integral on  $\ell_4^-$  the integrand has the same bounds as on  $\ell_3^-$ , but since the path of integration has an  $O(1)$  length the contribution of this term does not modify the estimate.

### 4.4. – Proof of Lemma 8

#### 4.4.1. – The integral equation

The substitution  $y = i\xi v^{q+1}/(q+1)$  transforms (3.36) to

$$(4.63) \quad h''(y) + \frac{r}{y} h'(y) - h(y) = \left( v(y)^2 - 2v(y) \right) h(y)$$

where  $r$  is given by (3.40).

The LHS of (4.63) is a Bessel equation, with an analytic solution

$$(4.64) \quad \tilde{h}_a(y) = \int_{-1}^1 e^{-py} (1-p^2)^A dp$$

and the branched solution

$$(4.65) \quad \tilde{h}_s(y) = y^{1-r} \tilde{h}_{a,s}(y) \quad , \quad \text{where } \tilde{h}_{a,s}(y) = \int_{-1}^1 e^{-py} (1-p^2)^B dp .$$

Their Wronskian is  $W[\tilde{h}_a, \tilde{h}_s] = \text{const } y^r$ .

Going back to the variable  $v$ , let  $\tilde{g}_{a,s}(v) = \tilde{h}_{a,s}(y(v))$ . Then

$$(4.66) \quad W[\tilde{g}_a, \tilde{g}_s] = \text{const } \xi^{1-r} v^{-1+1/m} .$$

The integral form of equation (3.36) is then (3.38) where

$$\mathcal{G}(g)(v) = \int_0^v [\tilde{g}_a(v) \tilde{g}_s(t) - \tilde{g}_s(v) \tilde{g}_a(t)] W[\tilde{g}_a, \tilde{g}_s]^{-1} \xi^2 t^{2q+1} (2-t) g(t) dt$$

which using (4.66) yields (3.39).

#### 4.4.2. – Estimates

We show that the operator  $\mathcal{G}$  is contractive in the Banach space of continuous and bounded functions on  $\mathcal{S}$  (with the sup norm).

We need the following estimates:

Consider first  $v \in \mathcal{S}_1$ . We have  $y \equiv y(v) \in i[0, \rho^{1-a(q+1)}/(q+1)] \subset i\mathbb{R}_+$ . Note that  $\tilde{h}_a, \tilde{h}_s$  are bounded for small  $y$ ; also, for large  $y$  on the segment  $y(v)$  we have  $|\tilde{h}_{a,s}(y)| < \text{const } |y|^{-1-A}$  (from (5.77) used for  $\alpha = -A$  and  $\alpha = -B$ ).

Denote  $v_1 = \kappa_0(q+1)^{1/(q+1)} \rho^{-1/(q+1)}$ . Then  $y(v_1) = i$ .

For  $v \in [0, v_1]$  we have  $|\tilde{g}_{a,s}(v)| \leq \sup_{y \in [0, i]} |\tilde{h}_{a,s}(y)| = \text{const}$  so that

$$\begin{aligned} |\mathcal{G}(g)(v)| &< \text{const } \rho^{1+r} \sup_{v \in [0, v_1]} |g(v)| \int_0^{\rho^{-1/(q+1)}} t^{2q+2-\frac{1}{m}} dt \\ &< \text{const } \rho^{-1/(q+1)} \sup_{v \in [0, v_1]} |g(v)| . \end{aligned}$$

For  $v \in [v_1, v_F]$  we have  $|\tilde{g}_{a,s}(v)| = |\tilde{h}_{a,s}(y(v))| < \text{const } |\rho v^{q+1}|^{-1-A}$  so that

$$\begin{aligned} |\mathcal{G}(g)(v)| &< \text{const } \rho^{-1/(q+1)} \sup_{v \in [0, v_1]} |g(v)| \\ (4.67) \quad &+ \text{const } \rho^{-2A-\frac{1}{m(q+1)}} \sup_{v \in [v_1, v_F]} |g(v)| |v_F|^{(q+1)(-1-A)} \int_{\rho^{-1/(q+1)}}^{\rho^{-a}} t^{(1-A)(q+1)-\frac{1}{m}} dt \\ &< \text{const } \rho^{1-a(q+2)} \sup_{v \in \mathcal{S}_1} |g(v)| . \end{aligned}$$

Therefore

$$(4.68) \quad \sup_{v \in \mathcal{S}_1} |\mathcal{G}(g)(v)| \leq \text{const } \sup_{v \in \mathcal{S}_1} |g(v)| .$$

For  $v \in \mathcal{S}_2$  we have

$$\xi v^{q+1} = \xi v_F^{q+1} \left[ 1 + O\left(\rho^{-a(q+1)}\right) \right] \quad (\rho \rightarrow \infty)$$

hence this path yields a negligible contribution to (2.68) and  $\|\mathcal{G}(g)\| \leq \text{const } \rho^{1-a(q+2)} \|g\|$ .

Then  $\mathcal{G}$  is contractive for  $\rho$  large enough. Taking  $C_1 = 1, C_2 = 0$  in (3.38) (respectively  $C_1 = 0, C_2 = 1$ ) we get the analytic (respectively branched) solution of (3.38)

$$(4.69) \quad g_{a,s} = (I - \mathcal{G})^{-1}(\tilde{g}_{a,s}) = \tilde{g}_{a,s} + (I - \mathcal{G})^{-1} \mathcal{G} \tilde{g}_{a,s}$$

and the estimate (3.41) follows.

#### 4.5. – Proof of Lemma 9

We first estimate the leading order  $B_-$  of  $h_-$ .

Formula (3.27) defines  $B_-(s)$  for  $s$  between  $+i\infty$  and 0. To get the values of  $B_-(s)$  in the matching region, where

$$s(v) = i \left\{ \rho P_0 - t^{q+1} R \right\} + o\left(\rho^{1-a(q+2)}\right) + O\left(\rho^{1-a(2q+3)}\right) \quad \text{with } R = \frac{\rho^{1-a(q+1)}}{q+1}$$

we need to find the analytic continuation of (3.27) along  $\ell$ .

We remark that in fact  $B_-$  also satisfies

$$(4.70) \quad B_-(e^{i\theta} s) = \int_{e^{-i\gamma}[1, +\infty)} e^{-ps} (p^2 - 1)^{-3/4} dp$$

for  $s \in i\mathbb{R}_+$  if  $\sin(\theta - \gamma) \leq 0$

and formula (4.70) can be used to determine the analytic continuation of  $B_-$  along  $\ell$ : when  $s$  goes on a small circle, clockwise around the origin, the path of integration in (3.27) must be rotated by an angle of  $2\pi$  (around the singular points  $p = \pm 1$ ). Then for  $s \in \ell_2$

$$(4.71) \quad B_-(s) = \int_1^{-1} e^{-ps} e^{-3\pi i/4} (1 - p^2)^{-3/4} dp$$

$$+ \int_1^1 e^{-ps} e^{-9\pi i/4} (1 - p^2)^{-3/4} dp + \int_1^{+\infty} e^{-ps} e^{-3\pi i} (p^2 - 1)^{-3/4} dp$$

$$(4.72) \quad = 2^{1/2} \int_{-1}^1 e^{-ps} (1 - p^2)^{-3/4} dp - \int_1^{+\infty} e^{-ps} (p^2 - 1)^{-3/4} dp.$$

Using now (5.75) and (5.77) we get

$$(4.73) \quad B_-(s) = \text{const } \rho^{-1/4} \left( i e^{-i\rho P_0} e^{it^{q+1}R} + 2^{1/2} i e^{i\rho P_0} e^{-it^{q+1}R} \right)$$

$$\times \left[ 1 + o\left(\rho^{1-a(q+2)}\right) + O\left(\rho^{-a(q+1)}\right) \right].$$

Finally, in view of Lemma 5(ii), and noting that in the matching region (3.25) we have  $P(v) = P_0 + O(\rho^{-a})$  and  $v^{-(q+1)/2+1/(2m)} = \rho^{a[(q+1)/2-1/(2m)]}(1+o(1))$  Lemma 9 follows.

Analyticity in  $\xi$  follows as in Section 4.2.

#### 4.6. – Proof of Lemma 10

I. Analytic dependence on  $\xi$  of  $C_a, C_s$ .

The solutions  $g_{\pm}(v), g_{a,s}(v)$ , depend analytically on  $\xi$  (Lemmas 5, 8). We have  $g_-(v(s)) \sim \text{const } s^{-1/4} e^{-s}$  for  $s$  in the initial part of  $\ell$ . After analytic continuation along  $\ell$ , rotating  $2\pi$  about  $s = 0$ , the solution  $g_-$  becomes  $ACg_-(v) = A(\xi)g_-(v) + B(\xi)g_+(v)$ . Evaluating this relation for two values of  $v$  it follows that  $A(\xi), B(\xi)$  are analytic. The same argument shows that the decomposition of  $g_a, g_s$  along  $g_+, g_-$  has analytic coefficients.

II. Estimates.

Introducing (3.42),(3.43),(3.44) in (3.47) and identifying the coefficients of  $e^{i(R+\lambda)} + O(\rho^{-c})$ , respectively  $e^{-i(R+\lambda)} + O(\rho^{-c})$  we get a system of two equations for the unknowns  $C_a, C_s$ , whose solutions have the form (3.48), (3.49).

#### 4.7. – Proof of Lemma 12

I. We first note that the result in Lemma 8 can be improved: the integral equation (3.38) has a unique solution on the longer path  $[0, v_M] \cup [v_M, -v_M^-]$  and the estimates (3.41) hold. Indeed, on the segment  $[v_M, -v_M^-]$  we have  $y = i\rho^{1-a(q+1)}/(q+1) + \mu + O(\rho^{-a})$  where  $\mu$  is bounded, therefore the estimates of  $\tilde{h}_{a,s}$  are the same as on the segment  $[0, v_M]$  (possibly after increasing const). Since the segment  $[v_M, -v_M^-]$  has an  $o(1)$  length, its contribution to the estimates in the proof of Lemma 8 is negligible.

II. Note that

$$s(-v_M^-) = i\rho P_0 - i \frac{\rho^{1-a(q+1)}}{q+1} + i(2iZP_0 - t) + o(1) \quad (\rho \rightarrow +\infty).$$

The rest follows in a straightforward way from (3.41), (3.35) and Remark 14.(iv).

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### 5. – Appendix

#### 5.1. – The paths $s(d), s(-d)$

Denote  $\theta = \frac{\pi}{2(q+2)}$ .

Let  $v \in d$  (respectively  $v \in -d$ ), so  $v = e^{-i\theta}t$  with  $t \in \mathbb{R}_+$ , (respectively  $t \in \mathbb{R}_-$ ). Then

$$\Re[s(v)] = \rho \left( \frac{1}{q+2} t^{q+2} - \frac{\cos \theta}{q+1} t^{q+1} \right) + Z \left( \frac{\sin \theta}{q+1} t^{q+1} + P_0 \right)$$

and

$$\Im[s(v)] = Z \left( \frac{1}{q+2} t^{q+2} - \frac{\cos \theta}{q+1} t^{q+1} \right) + \rho \left( -\frac{\sin \theta}{q+1} t^{q+1} + P_0 \right).$$

For large  $|t|$  we have  $\Re s(v) \sim \rho t^{q+2}/(q+2)$  and  $\Im s(v) \sim Z/(q+2)t^{q+2}$ . Hence  $s(d)$  (and  $s(-d)$  if  $q$  is even) comes from infinity in right half-plane, asymptotic to a line of small slope ( $Z\rho^{-1}$ ).

To obtain the form of  $s(d)$  consider first the case  $Z < 0$ . Then  $s(d)$  goes to infinity in the fourth quadrant. Furthermore, we have  $\Im s(v) = 0$  only for  $t = [(q+2)\sin\theta]^{-1/(q+1)}(1+o(1))$  (and at this point  $\Re s(v) < 0$ ); also,  $\Re s(v) = 0$  only for  $t = (q+2)/(q+1)(\cos\theta)(1+o(1))$  (at this point  $\Im s(v) < 0$ ) and for  $t = [-Z/(q+2)/\cos\theta\rho^{-1}]^{1/(q+1)}(1+o(1))$ . Therefore  $s(d)$  comes from  $\infty$  in the fourth quadrant, crosses  $i\mathbb{R}_-$ , then  $\mathbb{R}_-$ ,  $i\mathbb{R}_+$  and ends at  $s_0$  in the first quadrant.

The form of  $s(d)$  in the case  $Z > 0$  is obtained similarly. The curve comes from infinity in the first quadrant, crosses  $\mathbb{R}_+$ , then  $i\mathbb{R}_-$ , and  $\mathbb{R}_-$  and ends in the second quadrant at  $s_0$ .

For  $q$  even,  $s(-d)$  starts at  $s_0$  (in the second quadrant if  $Z > 0$ , respectively in the first quadrant if  $Z < 0$ ) crosses  $i\mathbb{R}_+$  (respectively  $\mathbb{R}_+$ ), then goes to infinity in the first (respectively fourth) quadrant.

## 5.2. – Asymptotic formulas

REMARK 14. Let  $T > 0$  and  $\alpha \in (0, 1)$ .

(i) We have

$$(5.74) \quad \int_0^{+\infty} e^{-Tq} q^{-\alpha} (2 \pm iq)^{-\alpha} dq = 2^{-\alpha} \Gamma(1-\alpha) T^{-1+\alpha} \left(1 + O(T^{-1})\right).$$

(ii) Also

$$(5.75) \quad \begin{aligned} \Phi_{\pm}(T) &\equiv \int_1^{+\infty} e^{\pm ipT} (p^2 - 1)^{-\alpha} dp \\ &= e^{\pm i\pi(1-\alpha)/2} 2^{-\alpha} \Gamma(1-\alpha) e^{\pm iT} T^{-1+\alpha} \left(1 + O(T^{-1})\right). \end{aligned}$$

(iii) Denote

$$(5.76) \quad I(T) = \int_{-1}^1 e^{-ipT} (1-p^2)^{-\alpha} dp.$$

We have

$$(5.77) \quad I(T) = \left[ e^{iT-i\pi(1-\alpha)/2} + e^{-iT+i\pi(1-\alpha)/2} \right] 2^{-\alpha} \Gamma(1-\alpha) T^{-1+\alpha} \left(1 + O(T^{-1})\right).$$

(iv) Let also  $K \in \mathbb{C}$  independent of  $T$  and  $\tilde{T} = O(T^{-\gamma})$  ( $T \rightarrow +\infty$ ) where  $\gamma > 0$ . Then the integral (5.76) satisfies

$$(5.78) \quad \begin{aligned} I(T + K + \tilde{T}) &= \left[ e^{i(T+K)-i\pi(1-\alpha)/2} + e^{-i(T+K)+i\pi(1-\alpha)/2} \right] \\ &\times 2^{-\alpha} \Gamma(1-\alpha) T^{-1+\alpha} \left(1 + O(T^{-1}) + O(T^{-\gamma})\right). \end{aligned}$$

PROOF.

- (i) Formula (5.74) is a direct consequence of Watson's Lemma [34].  
(ii) To show (5.75) we substitute  $r = p - 1$  then rotate the path of integration in the formula defining  $\Phi_{\pm}$  by an angle  $\pm\pi/2$ ; it follows that

$$(5.79) \quad \begin{aligned} \Phi_{\pm}(T) &= e^{\pm iT} \int_0^{-i\infty} e^{\pm irT} r^{-\alpha} (r+2)^{-\alpha} dr \\ &= \pm i e^{\mp i\pi\alpha/2} e^{\pm iT} \int_0^{+\infty} e^{-\tau T} \tau^{-\alpha} (2-i\tau)^{-\alpha} d\tau \end{aligned}$$

and (5.74) yields (5.75).

- (iii) The path of integration in (5.77) can be deformed to two vertical half-lines in the half-plane  $\Im p < 0$  (where the branches of the power are determined by analytic continuation of the usual branch on  $(-1, 1)$ ):

$$\begin{aligned} I(T) &= \left( \int_{-1}^{-1-i\infty} + \int_{1-i\infty}^1 \right) e^{-ipT} (1-p^2)^{\alpha} dp \\ &= -i e^{i\pi\alpha/2} e^{iT} \int_0^{+\infty} e^{-sT} s^{-\alpha} (2+is)^{-\alpha} ds \\ &\quad + i e^{-i\pi\alpha/2} e^{-iT} \int_0^{+\infty} e^{-sT} s^{-\alpha} (2-is)^{-\alpha} ds \end{aligned}$$

and using (5.74) the estimate (5.77) follows.

- (iv) For large  $T > 0$  and  $p$  with  $\Re p \in [-1, 1]$ ,  $\Im p < 0$ , and  $-\Im p$  large enough we have  $\Re[-ip(T+K+\tilde{T})] = [T+\Re(K+\tilde{T})]\Im p + [\Im(K+\tilde{T})]\Re p < 0$ . Then the path of integration in (5.76) can be deformed as in the proof of (iii), yielding

$$(5.80) \quad \begin{aligned} I(T+K+\tilde{T}) &= -i e^{i\pi\alpha/2} e^{i(T+K+\tilde{T})} \int_0^{+\infty} e^{-s(T+K+\tilde{T})} s^{-\alpha} (2+is)^{-\alpha} ds \\ &\quad + i e^{-i\pi\alpha/2} e^{-i(T+K+\tilde{T})} \int_0^{+\infty} e^{-s(T+K+\tilde{T})} s^{-\alpha} (2-is)^{-\alpha} ds \end{aligned}$$

and using again Watson's Lemma (5.78) follows.

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