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Well-Posedness of the Cauchy Problem for a Hyperbolic Equation with non-Lipschitz Coefficients

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Abstract. We prove that the Cauchy problem for a class of hyperbolic equations with non-Lipschitz coefficients is well-posed in $\mathcal{C}^\infty$ and in Gevrey spaces. Some counter examples are given showing the sharpness of these results.

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1. – Introduction

This work is devoted to the study of the Cauchy problem for the equation

$$(1.1)\qquad u_{tt} - \sum_{i,j=1}^{n} a_{ij}(t) u_{x_i x_j} + \sum_{i=1}^{n} b_i(t) u_{x_i} + c(t)u = 0 \quad \text{in} \quad [0,T]\times\mathbb{R}^n\,,$$

with initial data

$$(1.2)\qquad u(0,x) = u_0(x),\;\; u_t(0,x) = u_1(x) \quad \text{in} \quad \mathbb{R}^n\,,$$

under the hypothesis of strict hyperbolicity for (1.1). Supposing that $\mathcal{B}$ is a class of infinitely differentiable functions on $\mathbb{R}^n$ we will say that the Cauchy problem (1.1), (1.2) is $\mathcal{B}$-well-posed if for all given initial data u_0, $u_1 \in \mathcal{B}$ there exists a unique solution u to (1.1), (1.2) such that $u \in \mathcal{C}^1([0,T];\mathcal{B})$.

It is well known that if the coefficients of the principal part of the equation are Lipschitz-continuous then (1.1), (1.2) is $\mathcal{C}^\infty$-well-posed: more precisely in this case the Cauchy problem is well posed in Sobolev spaces and one can prove that for all $u_0 \in \mathcal{H}^s(\mathbb{R}^n)$, $u_1 \in \mathcal{H}^{s-1}(\mathbb{R}^n)$ there is a unique solution in $\mathcal{C}([0,T];\mathcal{H}^s(\mathbb{R}^n)) \cap \mathcal{C}^1([0,T];\mathcal{H}^{s-1}(\mathbb{R}^n))$ (see e.g. [5, Ch. 9]).

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Interesting results have been obtained in weakening the regularity assumption on the coefficients a_{ij}. In particular it as been proved that if the a_{ij}'s are Log-Lipschitz-continuous (we recall that a function f is a Log-Lipschitz-continuous if

$$\sup_{0<|t-s|<1/2} \frac{|f(t)-f(s)|}{|t-s\|\log|t-s\|} < +\infty$$

then the Cauchy problem (1.1), (1.2) is still $\mathcal{C}^\infty$-well-posed but in this case there exists $\delta > 0$ (depending on the Log-Lipschitz-norm of the coefficients a_{ij}) such that for all $u_0 \in \mathcal{H}^s(\mathbb{R}^n)$, $u_1 \in \mathcal{H}^{s-1}(\mathbb{R}^n)$ there is a unique solution in $\mathcal{C}([0,T];\mathcal{H}^{s-\delta}(\mathbb{R}^n)) \cap \mathcal{C}^1([0,T];\mathcal{H}^{s-1-\delta}(\mathbb{R}^n))$ (this behavior goes under the name of loss of derivatives).

Finally if the coefficients a_{ij} are only Hölder-continuous of exponent $\alpha < 1$ then (1.1), (1.2) is $\gamma^{(s)}$-well-posed for all $s < 1/(1-\alpha)$, where $\gamma^{(s)}$ the Gevrey space of order s (see [1] and [3], [6], [7] for related results for operators with coefficients depending also on x).

One may think to weaken the Lipschitz-continuity assumption on the coefficients of the principal part of the operator in a different way, supposing the a_{ij}'s to be $\mathcal{C}^1$ functions on $[0,T]\setminus\{\bar{t}\}$ and imposing a bound on the derivative of the a_{ij}'s of the type

$$|a'_{ij}(t)| \le C|t-\bar{t}|^{-q}\,, \tag{1.3}$$

for all $t \in [0,T]\setminus\{\bar{t}\}$ (remark that $\bar{t}$ may also be 0).

Elementary examples show that if $q \ge 1$ then condition (1.3) is independent from the regularity of the coefficients on the whole $[0,T]$. More precisely it is possible to construct a function f, $\mathcal{C}^1$ on $]0,T]$ and Log-Lipschitz-continuous on $[0,T]$, such that

$$\limsup_{t\to 0^+} t^q|f'(t)| = +\infty\,,$$

for all $q \ge 1$. Conversely it is easy to find a function g, $\mathcal{C}^1$ on $]0,T]$, continuous on $[0,T]$ but Hölder-continuous on $[0,T]$ for no $\alpha < 1$, such that

$$\limsup_{t\to 0^+} t|g'(t)| < +\infty\,.$$

In the present paper we prove that if the condition (1.3) holds with $q = 1$ then (1.1), (1.2) is $\mathcal{C}^\infty$-well-posed with a loss of derivatives (Theorem 1), if (1.3) is satisfied for a given $q > 1$ and, for instance, the coefficients a_{ij} are bounded then (1.1), (1.2) is $\gamma^{(s)}$-well-posed for all $s < q/(q-1)$ (Theorem 2) and if (1.3) is true for some $q > 1$ and the a_{ij}'s are Hölder-continuous of exponent α then the Cauchy problem is $\gamma^{(s)}$-well-posed for all $s < (q/(q-1))(1/(1-\alpha))$ (Theorem 3).

The results proved by Theorems 1, 2 and 3 are in a sense optimal. In fact we construct a coefficient a which is Hölder-continuous of exponent α for all $\alpha < 1$ and it satisfies, for all $q > 1$ and for all $t \in]0,T]$, the condition

$$|a'(t)| \le C_q t^{-q}\,,$$

and we find two $\mathcal{C}^\infty$ functions u_0, u_1 such that the Cauchy problem

$$\begin{cases} u_{tt} - a(t)u_{xx} = 0 \\ u(0,x) = u_0(x), \;\; u_t(0,x) = u_1(x) \end{cases}$$

does not have any solution (not even if we look for a solution in a neighborhood of the origin, which is only a distribution in the x variable, see Theorem 4). Similarly we show by another counter example that in the case of Gevrey-well-posedness the relations among q, α and s determined by Theorems 2 and 3 cannot be improved (Theorem 5).

Finally we state and prove two results in the case of one space variable; in this situation we show that the well-posedness can be proved when the coefficient c in the equation (1.1) depends also on x (Theorems 5 and 7).

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2. – Results and remarks

Let $T > 0$. We consider the following equation

$$u_{tt} - \sum_{i,j=1}^{n} a_{ij}(t)u_{x_i x_j} + \sum_{i=1}^{n} b_i(t)u_{x_i} + c(t)u = 0\,, \tag{2.1}$$

where the matrix $(a_{ij}(t))$ is real and symmetric for all $t \in]0,T]$, $a_{ij} \in \mathcal{C}^1(\,]0,T])$ and b_i, $c \in \mathcal{C}([0,T])$. We set, for $(t,\xi) \in]0,T] \times \mathbb{R}^n \setminus \{0\}$,

$$a(t,\xi) = \sum_{i,j=1}^{n} a_{ij}(t)\xi_i\xi_j/|\xi|^2\,,$$

and we denote by $'$ the derivative with respect to the variable t. We suppose that the equation (2.1) is strictly hyperbolic, i.e. there exists $\lambda_0 > 0$ such that

$$a(t,\xi) \ge \lambda_0\,. \tag{2.2}$$

for all $(t,\xi) \in]0,T] \times \mathbb{R}^n \setminus \{0\}$.

The first result of the paper is the following.

THEOREM 1. *Suppose that there exists* $C > 0$ *such that, for all* $(t, \xi) \in]0, T] \times \mathbb{R}^n \setminus \{0\}$,

$$t|a'(t,\xi)| \leq C\,. \tag{2.3}$$

Then the Cauchy problem for the equation (2.1) *is* $\mathcal{C}^\infty$*-well-posed.*

REMARK 1. Using the same technique of the proof of Theorem 1 it is possible to prove that the Cauchy problem for (2.1) is $\mathcal{C}^\infty$-well-posed under the following more general hypothesis (see [2]): there exist $\bar{t} \in [0, T]$ and $C > 0$ such that for all $(t, \xi) \in ([0, T] \setminus \{\bar{t}\}) \times \mathbb{R}^n \setminus \{0\}$,

$$|t - \bar{t}||a'(t,\xi)| \leq C\,.$$

For $s \geq 1$ we denote by $\gamma^{(s)}(\Omega)$ the Gevrey space of order s in the open set $\Omega \subset \mathbb{R}^n$, i.e. $f \in \gamma^{(s)}(\Omega)$ if f is a $\mathcal{C}^\infty$ function defined on Ω and for all compact sets $K \subset \Omega$ there exist C, $M > 0$ such that

$$|\partial_x^\alpha f(x)| \leq CM^{|\alpha|}(\alpha!)^s\,,$$

for all $x \in K$ and $\alpha \in \mathbb{N}^n$; in particular if $s = 1$ then $\gamma^{(1)}$ is the space of the analytic functions on Ω. We have the following result.

THEOREM 2. *Let* $q > 1$ *and* $p \in [0, 1[$*, with* $p \leq q - 1$*. Suppose that there exist* C, $C' > 0$ *such that, for all* $(t, \xi) \in]0, T] \times \mathbb{R}^n \setminus \{0\}$,

$$t^q|a'(t,\xi)| \leq C\,, \tag{2.4}$$

$$t^p|a(t,\xi)| \leq C'\,. \tag{2.5}$$

Then the Cauchy problem for the equation (2.1) *is* $\gamma^{(s)}$*-well-posed for all* $s < (q-p)/(q-1)$.

REMARK 2. We want to point out two simple consequences of Theorem 2. If $1 < q < 2$ then, taking $p = q - 1$, hypothesis (2.5) follows from (2.4). Hence, if $1 < q < 2$, condition (2.4), without any other assumption, gives the $\gamma^{(s)}$-well-posedness for all $s < 1/(q - 1)$. If a is bounded then (2.5) is verified with $p = 0$. Consequently, if $q > 1$ and a is bounded, (2.4) implies that the Cauchy problem for (2.1) is $\gamma^{(s)}$-well-posed for all $s < q/(q - 1)$.

For $\alpha \in]0, 1[$, we denote by $\mathcal{C}^{0,\alpha}$ the space of the Hölder-continuous function with exponent α. The following theorem makes a relation among the Hölder-regularity of the coefficients of the principal part, the bound on its derivatives and the order of the Gevrey space in which the well-posedness holds.

THEOREM 3. *Let* $q > 1$ *and* $\alpha \in]0, 1[$*. Let* $a_{i,j} \in \mathcal{C}^1(]0, T]) \cap \mathcal{C}^{0,\alpha}([0, T])$*. Suppose that* (2.4) *is verified.*

Then the Cauchy problem for the equation (2.1) *is* $\gamma^{(s)}$*-well-posed for all* $s < (q/(q-1))(1/(1-\alpha))$.

REMARK 3. The result of Theorem 3 can be thought as an average between Theorem 4 of [1], in which, assuming only the Hölder-continuity of the coefficients, $\gamma^{(s)}$-well-posedness is proved for $s < 1/(1-\alpha)$, and the Theorem 2 in the case of the function a bounded.

REMARK 4. Supposing $s = (q-p)/(q-1)$, under the hypotheses of Theorem 2, one can prove that for all pair of functions $u_0, u_1 \in \gamma^{(s)}(\mathbb{R})$ there exists $T' \in]0, T]$ such that the Cauchy problem for (2.1) has a unique solution $u \in \mathcal{C}^1([0, T']; \gamma^{(s)})$. The same result holds under the hypotheses of Theorem 3 if $s = (q/(q-1))(1/(1-\alpha))$.

Finally we state two theorems showing that the conditions (2.3) and (2.4) are in some sense necessary. We denote by $\mathcal{D}'(\Omega)$ and $\mathcal{D}'^{(s)}(\Omega)$ the space of the distributions and the space of the Gevrey-ultradistributions of order s on the open set Ω respectively. The results are the following.

THEOREM 4. *There exists a positive function* $a \in \mathcal{C}^\infty(\mathbb{R}\setminus\{0\}) \cap \mathcal{C}^{0,\alpha}(\mathbb{R})$ *for all* $\alpha \in]0, 1[$, *with*

$$\sup_{t>0} t^q |a'(t)| < +\infty \,, \tag{2.6}$$

for all $q > 1$, *and there exist* $u_0, u_1 \in \mathcal{C}^\infty(\mathbb{R})$ *such that the Cauchy problem*

$$\begin{cases} u_{tt} - a(t)u_{xx} = 0 \\ u(0,x) = u_0(x), \;\; u_t(0,x) = u_1(x) \end{cases} \tag{2.7}$$

has no solution in $\mathcal{C}^1([0, r[\,; \mathcal{D}'(\,]-r, r[\,))$, *for all* $r > 0$.

THEOREM 5. *Let* $q_0 > 1$ *and* $\alpha_0 \in [0, 1[$. *There exists a positive function* $a \in \mathcal{C}^\infty(\mathbb{R}\setminus\{0\}) \cap \mathcal{C}^{0,\alpha_0}(\mathbb{R})$ *(for* $\alpha_0 = 0$, $a \in \mathcal{C}^\infty(\mathbb{R}\setminus\{0\}) \cap \mathcal{C}(\mathbb{R})$*), with*

$$\sup_{t>0} t^{q_0} |a'(t)| < +\infty \,, \tag{2.8}$$

and there exist $u_0, u_1 \in \gamma^{(s)}(\mathbb{R})$ *for all* $s > s_0 = (q_0/(q_0-1))(1/(1-\alpha_0))$ *such that the Cauchy problem*

$$\begin{cases} u_{tt} - a(t)u_{xx} = 0 \\ u(0,x) = u_0(x), \;\; u_t(0,x) = u_1(x) \end{cases}$$

has no solution in $\mathcal{C}^1([0, r[\,; \mathcal{D}'^{(s)}(\,]-r, r[\,))$, *for all* $s > s_0$ *and for all* $r > 0$.

REMARK 5. Theorem 4 can be considered the converse of Theorem 1; for $\alpha_0 > 0$, Theorem 5 is related to Theorem 3, while, for $\alpha_0 = 0$, Theorem 5 gives only a counter example to the result of Theorem 2 in the case of a bounded.

3. – Results in one space dimension

In the case of one space dimension the results of Theorems 1 and 2 can be a little improved, letting the coefficient c depend also on x. We consider the equation

$$u_{tt} - a(t)u_{xx} + b(t)u_x + c(t,x)u = 0\,, \tag{3.1}$$

where $a \in \mathcal{C}^1(\,]0,T])$, $b \in \mathcal{C}^0([0,T])$, $c \in \mathcal{C}([0,T];\mathcal{C}^\infty(\mathbb{R}_x))$ and $a(t) \geq \lambda_0 > 0$ for all $t \in]0,T]$. The following results holds true.

Theorem 6. *Suppose that there exists $C > 0$ such that, for all $t \in]0,T]$,*

$$t|a'(t)| \leq C\,. \tag{3.2}$$

Then the Cauchy problem for the equation (3.1) *is $\mathcal{C}^\infty$-well-posed.*

Theorem 7. *Let $q \in]1,2[$ and $2 < s < q/(q-1)$. Suppose that a is bounded and $c \in \mathcal{C}([0,T];\gamma^{(s'/2)}(\mathbb{R}_x))$ for some s', $2 < s' < s$. Suppose moreover that there exists $C > 0$ such that, for all $t \in]0,T]$,*

$$t^q|a'(t)| \leq C\,. \tag{3.3}$$

Then the Cauchy problem for the equation (3.1) *is $\gamma^{(s)}$-well-posed.*

In the proof of the Theorem 6, in order to obtain the estimate of certain weighted $\mathcal{L}^2$-norm of the Fourier transform of the solution, we show that the weight function is actually a temperate weight function. This fact is crucial in proving the following corollary.

Corollary 1. *Let f be an entire analytic function with $f(0) = 0$. Suppose that the condition* (3.2) *holds.*

Then for all u_0, $u_1 \in \mathcal{C}_0^\infty(\mathbb{R})$ there exists $T' > 0$ such that the Cauchy problem

$$\begin{cases} u_{tt} - a(t)u_{xx} + b(t)u_x + f(u) = 0 \\ u(0,x) = u_0(x),\ u_t(0,x) = u_1(x) \end{cases} \tag{3.4}$$

has a unique solution $u \in \mathcal{C}^1([0,T'];\mathcal{C}_0^\infty(\mathbb{R}))$.

4. – Proofs of the Theorems 1, 2 and 3

In this section we prove the Theorems 1, 2 and 3. We start remarking that, possibly performing a linear change of variables, it is not restrictive to suppose that $\lambda_0 = 1$ in all of the three cases.

Let us begin with the proof of the Theorem 1. First of all we observe that (2.3) implies that the function $t \mapsto a(t,\xi)$ is in $\mathcal{L}^1(\,]0,T[\,)$ for all $\xi \in \mathbb{R}^n \setminus \{0\}$, and its $\mathcal{L}^1$-norm is uniformly bounded with respect to ξ. In fact

$$|a(T,\xi) - a(t,\xi)| \le \int_t^T |a'(s,\xi)|\,ds \le C \log\left(\frac{T}{t}\right),$$

and hence there exists $C' > 0$ such that, for all $(t,\xi) \in \,]0,T] \times \mathbb{R}^n \setminus \{0\}$,

$$|a(t,\xi)| \le C' + C \log\left(1 + \frac{1}{t}\right). \tag{4.1}$$

We define, for all $(t,\xi) \in [0,T] \times \mathbb{R}^n \setminus \{0\}$,

$$\tilde a(t,\xi) = \begin{cases} a(T,\xi) & \text{if } \; T|\xi| \le 1\,, \\ a(|\xi|^{-1},\xi) & \text{if } \; T|\xi| > 1 \;\text{ and }\; t|\xi| \le 1\,, \\ a(t,\xi) & \text{if } \; t|\xi| > 1\,, \end{cases}$$

and, for all $(t,\xi) \in \,]0,T] \times \mathbb{R}^n \setminus \{0\}$,

$$\alpha(t,\xi) = \begin{cases} |\tilde a(t,\xi) - a(t,\xi)|\,|\xi| & \text{if } \; t|\xi| \le 1\,, \\ \dfrac{|a'(t,\xi)|}{a(t,\xi)} & \text{if } \; t|\xi| > 1\,. \end{cases}$$

We remark that, for all $\xi \in \mathbb{R}^n \setminus \{0\}$, the function $t \mapsto \alpha(t,\xi)$ is in $\mathcal{L}^1(\,]0,T[\,)$ and it is also piecewise continuous. We fix $\sigma > 0$ and we set

$$k(t,\xi) = (1 + |\xi|^2)^\sigma \exp\left(-\int_0^t \alpha(s,\xi)\,ds\right).$$

We define the energy density of the solution u as

$$E(t,\xi) = (|v'(t,\xi)|^2 + (1 + \tilde a(t,\xi)|\xi|^2)|v(t,\xi)|^2)k(t,\xi)\,, \tag{4.2}$$

where $v = \mathcal{F}_x u$ is the Fourier transform of u with respect to x. We denote by $\mathcal{E}$ the energy of the solution:

$$\mathcal{E}(t) = \int_{\mathbb{R}^n} E(t,\xi)\,d\xi\,. \tag{4.3}$$

From (4.1) we deduce that, for all $(t,\xi) \in [0,T] \times \mathbb{R}^n \setminus \{0\}$,

$$|\tilde a(t,\xi)| \le C' + C \log(1 + |\xi|)\,,$$

consequently, for all $\varepsilon > 0$ there exists $\Lambda_\varepsilon > 0$ such that, for all $t \in [0,T]$,

$$\mathcal{E}(t) \le \Lambda_\varepsilon \int (1 + |\xi|^2)^{\sigma+\varepsilon}(|v'(t,\xi)|^2 + (1 + |\xi|^2)|v(t,\xi)|^2)\,d\xi\,. \tag{4.4}$$

On the other hand, if $T|\xi| \leq 1$, we have

$$\int_0^t \alpha(s,\xi)\,ds \leq \int_0^T |a(T,\xi) - a(t,\xi)|\frac{1}{T}\,dt \leq |a(T,\xi)| + \frac{1}{T}\|a(\cdot,\xi)\|_{\mathcal{L}^1(]0,T[)},$$

while, if $T|\xi| > 1$, from (2.3) and (4.1) we deduce that

$$\begin{aligned}\int_0^t \alpha(s,\xi)\,ds &\leq \int_0^T \alpha(s,\xi)\,ds \\ &\leq \int_0^{|\xi|^{-1}} |\tilde{a}(t,\xi) - a(t,\xi)||\xi|\,dt + \int_{|\xi|^{-1}}^T \frac{|a'(t,\xi)|}{a(t,\xi)}\,dt \\ &\leq C' + C\log(1+|\xi|) + |\xi|\int_0^{|\xi|^{-1}} |a(t,\xi)|\,dt + \int_{|\xi|^{-1}}^T \frac{C}{t}\,dt \\ &\leq 2C' + C(1 + \log(1+T) + 3\log(1+|\xi|)).\end{aligned}$$

Hence there exists $N > 0$ and $\tilde{\Lambda} > 0$ such that, for all $t \in [0,T]$,

$$\mathcal{E}(t) \geq \tilde{\Lambda}\int (1+|\xi|^2)^{\sigma-N}(|v'(t,\xi)|^2 + (1+|\xi|^2)|v(t,\xi)|^2)\,d\xi. \tag{4.5}$$

We compute the derivative of E with respect to t and we obtain, for $t|\xi| < 1$,

$$\begin{aligned}E'(t,\xi) = (&2\mathcal{R}e(v''\bar{v'}) + 2(1+\tilde{a}(t,\xi)|\xi|^2)\mathcal{R}e(v'\bar{v}) \\ &+ (|v'|^2 + (1+\tilde{a}(t,\xi)|\xi|^2)|v|^2)(-\alpha(t,\xi)))k(t,\xi),\end{aligned}$$

and, for $t|\xi| > 1$,

$$\begin{aligned}E'(t,\xi) = \Bigg(&2\mathcal{R}e(v''\bar{v'}) + a'(t,\xi)|\xi|^2|v|^2 + 2(1+a(t,\xi)|\xi|^2)\mathcal{R}e(v'\bar{v}) \\ &+ (|v'|^2 + (1+a(t,\xi)|\xi|^2)|v|^2)\left(-\frac{|a'(t,\xi)|}{a(t,\xi)}\right)\Bigg)k(t,\xi).\end{aligned}$$

Recalling that

$$v''(t,\xi) = -a(t,\xi)|\xi|^2 v(t,\xi) - i\sum_{j=1}^n b_j(t)\xi_j v(t,\xi) - c(t)v(t,\xi),$$

we deduce that, for $t|\xi| < 1$,

$$\begin{aligned}E'(t,\xi) \leq\ & 2(\tilde{a}(t,\xi) - a(t,\xi))|\xi|^2\mathcal{R}e(v'\bar{v})k(t,\xi) - \alpha(t,\xi)E(t,\xi) \\ &+ \left(1 + \sum_{j=1}^n |b_j(t)| + |c(t)|\right)E(t,\xi).\end{aligned}$$

and, for $t|\xi| > 1$,

$$E'(t,\xi) \le (a'(t,\xi) - |a'(t,\xi)|)|\xi|^2|v|^2k(t,\xi) - \frac{|a'(t,\xi)|}{a(t,\xi)}(|v'|^2 + |v|^2)k(t,\xi) + \left(1 + \sum_{j=1}^{n}|b_j(t)| + |c(t)|\right)E(t,\xi).$$

Consequently

$$E'(t,\xi) \le KE(t,\xi),$$

where $K = 1 + \max_{t\in[0,T]}\{\sum_{j=1}^{n}|b_j(t)| + |c(t)|\}$ and hence

$$\mathcal{E}'(t) \le K\mathcal{E}(t). \tag{4.6}$$

From (4.4), (4.5) and (4.6) the $\mathcal{C}^\infty$-well-posedness follows by standard arguments. This concludes the proof of the Theorem 1.

The proof of the Theorem 2 is close to the previous one. If $s = 1$ the result follows from [1, Theorem 4]. Let $1 < s < (q-p)/(q-1)$. We set

$$\tilde{a}(t,\xi) = \begin{cases} a(T,\xi) & \text{if } T|\xi|^{1/(qs-s)} \le 1, \\ a(|\xi|^{-1/(qs-s)},\xi) & \text{if } T|\xi|^{1/(qs-s)} > 1 \text{ and } t|\xi|^{1/(qs-s)} \le 1, \\ a(t,\xi) & \text{if } t|\xi|^{1/(qs-s)} > 1, \end{cases}$$

$$\alpha(t,\xi) = \begin{cases} |\tilde{a}(t,\xi) - a(t,\xi)|\,|\xi| & \text{if } t|\xi|^{1/(qs-s)} \le 1, \\ \dfrac{|a'(t,\xi)|}{a(t,\xi)} & \text{if } t|\xi|^{1/(qs-s)} > 1, \end{cases}$$

and

$$k(t,\xi) = \exp\left(-\int_0^t \alpha(\sigma,\xi)\,d\sigma + \beta|\xi|^{1/s}\right).$$

where β is a positive constant. We define the energy density and the energy of the solution as in (4.2) and (4.3) respectively. From (2.5) we have easily that for all $\beta' > \beta$ there exist $\Lambda_{\beta'} > 0$ such that

$$\mathcal{E}(t) \le \Lambda_{\beta'}\int e^{\beta'|\xi|^{1/s}}(|v'(t,\xi)|^2 + (1+|\xi|^2)|v(t,\xi)|^2)\,d\xi. \tag{4.7}$$

On the other hand, if $T|\xi|^{1/(qs-s)} \le 1$, again from (2.5), we deduce that

$$\int_0^t \alpha(\sigma,\xi)\,d\sigma \le \int_0^T |a(T,\xi) - a(t,\xi)||\xi|\,dt \le C'\left(\frac{2-p}{1-p}\right)T^{1-p-sq+s},$$

and, if $T|\xi|^{1/(qs-s)} > 1$, by using (2.5) and (2.4) we have

$$\int_0^t \alpha(s,\xi)\,ds \le \int_0^{|\xi|^{-1/(qs-s)}} |\tilde{a}(t,\xi) - a(t,\xi)||\xi|\,dt + \int_{|\xi|^{-1/(qs-s)}}^T \frac{|a'(t,\xi)|}{a(t,\xi)}\,dt \le C'\left(\frac{2-p}{1-p}\right)|\xi|^{1-(1-p)/(qs-s)} + \frac{C}{q-1}|\xi|^{1/s}.$$

From these inequalities, considering that $1-(1-p)/(qs-s)<1/s$, it follows that there exists $\delta>0$ such that, for all $(t,\xi)\in[0,T]\times\mathbb{R}^n\setminus\{0\}$,

$$\int_0^t \alpha(\sigma,\xi)\,d\xi \le \delta(|\xi|^{1/s}+1)\,.$$

Consequently

$$k(t,\xi)\ge \exp(-\delta+(\beta-\delta)|\xi|^{1/s})\,,$$

and

$$\mathcal{E}(t)\ge e^{-\delta}\int e^{(\beta-\delta)|\xi|^{1/s}}(|v'(t,\xi)|^2+(1+|\xi|^2)|v(t,\xi)|^2)\,d\xi\,. \tag{4.8}$$

Differentiating the energy density we obtain, for $t|\xi|^{1/(qs-s)}<1$,

$$\begin{aligned}E'(t,\xi)=&(2\mathcal{R}e(v''\bar{v'})+2(1+\tilde{a}(t,\xi)|\xi|^2)\mathcal{R}e(v'\bar{v})\\&+(|v'|^2+(1+\tilde{a}(t,\xi)|\xi|^2)|v|^2)(-\alpha(t,\xi)))k(t,\xi)\,,\end{aligned}$$

and, for $t|\xi|^{1/(qs-s)}>1$,

$$\begin{aligned}E'(t,\xi)=\Bigg(&2\mathcal{R}e(v''\bar{v'})+a'(t,\xi)|\xi|^2|v|^2+2(1+a(t,\xi)|\xi|^2)\mathcal{R}e(v'\bar{v})\\&+(|v'|^2+(1+a(t,\xi)|\xi|^2)|v|^2)\left(-\frac{|a'(t,\xi)|}{a(t,\xi)}\right)\Bigg)k(t,\xi)\,.\end{aligned}$$

Replacing v'' by its value obtained from the equation we deduce that

$$\mathcal{E}'(t)\le K\mathcal{E}(t)\,.$$

where $K=1+\max_{t\in[0,T]}\{\sum_{j=1}^n|b_j(t)|+|c(t)|\}$ and hence the conclusion follows from (4.7), (4.8) and this last inequality.

To prove the Theorem 3 we argue similarly to the two preceding proofs. Again if $s=1$ then the $\gamma^{(s)}$-well-posedness follows from [1, Theorem 4]; so we suppose that $1<s<(q/(q-1))(1/(1-\alpha))$ and we consider $r>q$ and $0<\omega<\alpha$ such that $s=(r/(r-1))(1/(1-\omega))$. Then $a\in\mathcal{C}^{0,\omega}$ and, possibly taking a different value for the constant C, the condition (2.4) implies that

$$|a'(t,\xi)|\le Ct^{-r}\,, \tag{4.9}$$

for all $(t,\xi)\in\,]0,T]\times\mathbb{R}^n\setminus\{0\}$. Let ρ be a real non-negative $\mathcal{C}^\infty$ function defined on $\mathbb{R}$ such that $\operatorname{supp}(\rho)\subset[-1,1]$ and $\int\rho(x)\,dx=1$. We extend the value of a to the whole of $\mathbb{R} \times \mathbb{R}^n \setminus \{0\}$, setting, for $t \le 0$, $a(t,\xi) = a(0,\xi)$ and, for $t \ge T$, $a(t,\xi) = a(T,\xi)$. We define

$$\tilde{a}(t,\xi) = |\xi| \int a(t-\tau,\xi)\rho(\tau|\xi|)\,d\tau\,,$$

$$\alpha(t,\xi) = |\tilde{a}(t,\xi) - a(t,\xi)||\xi| + \frac{|\tilde{a}'(t,\xi)|}{\tilde{a}(t,\xi)}\,,$$

and finally, denoting with β a positive constant,

$$k(t,\xi) = \exp\left(-\int_0^t \alpha(\sigma,\xi)\,d\sigma + \beta|\xi|^{1/s}\right)\,.$$

By the fact that a and $\tilde{a}$ are bounded we easily obtain that

$$\mathcal{E}(t) \le \Lambda \int e^{\beta|\xi|^{1/s}} (|v'(t,\xi)|^2 + (1+|\xi|^2)|v(t,\xi)|^2)\,d\xi, \tag{4.10}$$

where Λ is the maximum value for $|a|$ (here we use the same notations as in the previous cases: E and $\mathcal{E}$ are defined in (4.2) and (4.3) respectively).

To obtain an estimate from below for $\mathcal{E}(t)$ we argue in the following way. Suppose first that $|\xi| \le \max\{1, T^{r/(\omega-1)}\}$. Then

$$\int_0^t \alpha(\sigma,\xi)\,d\sigma \le \int_0^T \alpha(t,\xi)\,dt \le \int_0^T |\tilde{a}(t,\xi) - a(t,\xi)||\xi|\,dt + \int_0^T |\tilde{a}'(t,\xi)\,dt\,,$$

where

$$\int_0^T |\tilde{a}(t,\xi) - a(t,\xi)||\xi|\,dt \le 2\Lambda|\xi|T \le \frac{\delta}{2}\,,$$

and

$$\begin{aligned}
\int_0^T |\tilde{a}'(t,\xi)|\,dt &= \int_0^T |\xi|^2 \left| \int a(t-\tau,\xi)\rho'(\tau|\xi|)\,d\tau \right| dt \\
&= \int_0^T |\xi|^2 \left| \int (a(t-\tau,\xi) - a(t,\xi))\rho'(\tau|\xi|)\,d\tau \right| dt \\
&\le \int_0^T |\xi|^2 \int |a(t-\tau,\xi) - a(t,\xi)||\rho'(\tau|\xi|)|\,d\tau\,dt \\
&\le \int_0^T |\xi|^2 \int \|a\|_{\mathcal{C}^{0,\omega}} |\tau|^\omega |\rho'(\tau|\xi|)|\,d\tau\,dt \\
&\le \|a\|_{\mathcal{C}^{0,\omega}} |\xi|^{1-\omega}\, T \int |s|^\omega |\rho'(s)|\,ds \le \frac{\delta}{2}\,.
\end{aligned}$$

Hence

$$\int_0^t \alpha(\sigma,\xi)\,d\sigma \le \delta\,, \tag{4.11}$$

for some $\delta > 0$ not depending on t and ξ.

Next, let $|\xi| > \max\{1, T^{r/(\omega-1)}\}$. Then

$$\int_0^t \alpha(\sigma,\xi)\,d\sigma \le \int_0^T \alpha(t,\xi)\,dt = \int_0^{|\xi|^{(\omega-1)/r}} \alpha(t,\xi)\,dt + \int_{|\xi|^{(\omega-1)/r}}^T \alpha(t,\xi)\,dt\,.$$

We have

$$\begin{aligned}
&\int_0^{|\xi|^{(\omega-1)/r}} |\tilde{a}(t,\xi) - a(t,\xi)||\xi|\,dt \\
&\qquad \le \int_0^{|\xi|^{(\omega-1)/r}} |\xi|^2 \left| \int (a(t-\tau,\xi) - a(t,\xi))\rho(\tau|\xi|)\,d\tau \right| dt \\
&\qquad \le \int_0^{|\xi|^{(\omega-1)/r}} |\xi|^2 \int |a(t-\tau,\xi) - a(t,\xi)|\rho(\tau|\xi|)\,d\tau\,dt \\
&\qquad \le \int_0^{|\xi|^{(\omega-1)/r}} |\xi|^2 \int \|a\|_{\mathcal{C}^{0,\omega}} |\tau|^\omega |\rho(\tau|\xi|)|\,d\tau\,dt \\
&\qquad \le \|a\|_{\mathcal{C}^{0,\omega}} |\xi|^{1-\omega+(\omega-1)/r} \int |s|^\omega |\rho(s)|\,ds\,.
\end{aligned}$$

and similarly

$$\int_0^{|\xi|^{(\omega-1)/r}} |\tilde{a}'(t,\xi)|\,dt \le \|a\|_{\mathcal{C}^{0,\omega}} |\xi|^{1-\omega+(\omega-1)/r} \int |s|^\omega |\rho'(s)|\,ds\,.$$

We remark that $1-\omega+(\omega-1)/r = 1/s$, so that

$$\int_0^{|\xi|^{(\omega-1)/r}} \alpha(\sigma,\xi)\,d\sigma \le \frac{\delta}{2}|\xi|^{1/s}\,. \tag{4.12}$$

On the other hand, since $|\xi| > 1$ and $0 < (1-\omega)/r < 1$, we have that $|\xi|^{(\omega-1)/r} > |\xi|^{-1}$. Consequently we deduce from (4.9) that

$$|a(t-\tau,\xi) - a(t,\xi)| \le C(t-|\xi|^{-1})^{-r}|\xi|^{-1}\,,$$

for all $t \in [|\xi|^{(\omega-1)/r}, T]$ and $|\tau| \le |\xi|^{-1}$. Hence

$$\begin{aligned}
&\int_{|\xi|^{(\omega-1)/r}}^T |\tilde{a}(t,\xi) - a(t,\xi)||\xi|\,dt \\
&\qquad \le \int_{|\xi|^{(\omega-1)/r}}^T |\xi| \left| \int (a(t-\tau,\xi) - a(t,\xi))\rho(\tau|\xi|)\,d\tau \right| dt \\
&\qquad \le \int_{|\xi|^{(\omega-1)/r}}^T |\xi| \int C(t-|\xi|^{-1})^{-r}|\xi|^{-1}\rho(\tau|\xi|)\,d\tau\,dt \\
&\qquad \le \int_{|\xi|^{(\omega-1)/r}}^T C(t-|\xi|^{-1})^{-r}|\xi|^{-1}dt \\
&\qquad \le \frac{C}{r-1}|\xi|^{1-\omega+(\omega-1)/r}\,.
\end{aligned}$$

and by an analogous computation

$$\int_{|\xi|^{(\omega-1)/r}}^{T} |\tilde{a}'(t,\xi)|\,dt \leq \frac{C}{r-1}|\xi|^{1-\omega+(\omega-1)/r} \int |\rho'(s)|\,ds\,.$$

Then

$$\int_{|\xi|^{(\omega-1)/r}}^{T} \alpha(\sigma,\xi)\,d\sigma \leq \frac{\delta}{2}|\xi|^{1/s}\,. \tag{4.13}$$

From (4.11), (4.12) and (4.13) we finally obtain that there exists $\delta > 0$ such that, for all $(t,\xi) \in]0,T] \times \mathbb{R}^n \setminus \{0\}$,

$$\int_0^t \alpha(\sigma,\xi)\,d\sigma \leq \delta(1+|\xi|^{1/s})\,.$$

It follows that, possibly taking a bigger Λ we have

$$\mathcal{E}(t) \geq \frac{1}{\Lambda}\int e^{(\beta-\delta)|\xi|^{1/s}}(|v'(t,\xi)|^2 + (1+|\xi|^2)|v(t,\xi)|^2)\,d\xi\,. \tag{4.14}$$

The conclusion is easily reached from (4.10) and (4.14) remarking that also in this case we have

$$\mathcal{E}'(t) \leq K\mathcal{E}(t)\,.$$

whit $K = 1 + \max_{t\in[0,T]}\{\sum_{j=1}^n |b_j(t)| + |c(t)|\}$. The proof of the Theorem 3 is complete.

5. – The counter examples

In this section we will prove the Theorems 4 and 5. Although the proofs of both these two results follow closely that one of the Theorem 1 of [4] (apart from different choices of the sequences of the parameters involved in the construction of the functions a, u_0, u_1 and u), for the reader's convenience we will sketch the main parts of the argument, referring to the cited work for other details.

Let us begin with the proof of the Theorem 4. We consider a real non-negative 2π-periodic $\mathcal{C}^\infty$ function ρ defined on $\mathbb{R}$ such that $\rho(\tau) = 0$ for all τ in a neighborhood of 0 and

$$\int_0^{2\pi} \rho(\tau)\cos^2\tau\,d\tau = \pi\,.$$

We define, for all $\tau \geq 0$ and $\varepsilon \in]0, \bar{\varepsilon}]$,

$$\alpha_\varepsilon(\tau) = 1 - 4\varepsilon\rho(\tau)\sin 2\tau + 2\varepsilon\rho'(\tau)\cos^2\tau - 4\varepsilon^2\rho^2(\tau)\cos^4\tau\,, \tag{5.1}$$

$$\tilde{w}_\varepsilon(\tau) = \cos\tau \exp\left(\varepsilon\tau - 2\varepsilon\int_0^\tau \rho(s)\cos^2 s\, ds\right)\,,$$

$$w_\varepsilon(\tau) = e^{-\varepsilon\tau}\tilde{w}_\varepsilon(\tau)\,, \tag{5.2}$$

and, for all $\tau < 0$,

$$\alpha_\varepsilon(\tau) = \alpha_\varepsilon(-\tau), \quad \tilde{w}_\varepsilon(\tau) = \tilde{w}_\varepsilon(-\tau), \quad w_\varepsilon(\tau) = w_\varepsilon(-\tau)\,.$$

As $\alpha_\varepsilon(\tau) = 1$ and $w_\varepsilon(\tau) = \cos\tau$ in a neighborhood of the origin, α_ε and w_ε are $\mathcal{C}^\infty$ functions on $\mathbb{R}$. Moreover α_ε, $\tilde{w}_\varepsilon$ and w_ε verify the following properties:

$$\begin{cases} w_\varepsilon'' + \alpha_\varepsilon(\tau)w_\varepsilon = 0 \\ w_\varepsilon(0) = 1, \;\; w_\varepsilon'(0) = 0\,; \end{cases}$$

there exists $M > 0$, not depending on ε, such that for all $\tau \in \mathbb{R}$,

$$|\alpha_\varepsilon(\tau) - 1| \leq M\varepsilon, \quad |\alpha_\varepsilon'(\tau)| \leq M\varepsilon\,; \tag{5.3}$$

and finally α_ε and $\tilde{w}_\varepsilon$ are 2π-periodic in $]-\infty, 0]$ and $[0, +\infty[$.

Next we consider four monotone sequences $\{h_k\}$, $\{\eta_k\}$, $\{\rho_k\}$, $\{\varepsilon_k\}$ of positive real numbers such that

$$h_k \to +\infty, \quad \eta_k \to +\infty, \quad \varepsilon_k \to 0, \quad \rho_k \to 0\,; \tag{5.4}$$

$$\varepsilon_k \leq (2M)^{-1}, \qquad \text{for all } k \in \mathbb{N}\,; \tag{5.5}$$

$$h_k, \;\; h_k\rho_k(4\pi)^{-1} \in \mathbb{N}, \;\; \text{for all } k \in \mathbb{N}\,; \tag{5.6}$$

$$\sum_{k=0}^{+\infty} \rho_k < +\infty\,. \tag{5.7}$$

We define, for all $k \in \mathbb{N}$,

$$t_k = \frac{\rho_k}{2} + \sum_{j=k+1}^{+\infty} \rho_j\,,$$

and

$$I_k = \left[t_k - \frac{\rho_k}{2}, t_k + \frac{\rho_k}{2}\right]\,.$$

We set

$$a(t) = \begin{cases} \alpha_{\varepsilon_k}(h_k(t - t_k)) & \text{for } t \in I_k \\ 1 & \text{for } t \in \mathbb{R} \setminus \bigcup_{k=0}^{+\infty} I_k\,. \end{cases} \tag{5.8}$$

Since a is identically equal to 1 in a neighborhood of the boundary of each interval I_k, a is $\mathcal{C}^\infty$ in $\mathbb{R} \setminus \{0\}$. Moreover (5.3) and (5.5) imply that for all $t \in \mathbb{R}$, $|a(t) - 1| < 1/2$, i.e. a is positive. Suppose now that for all $\alpha < 1$,

$$\sup_{k\in\mathbb{N}} \varepsilon_k h_k^\alpha < +\infty \,, \tag{5.9}$$

then $a \in \mathcal{C}^{0,\alpha}(\mathbb{R})$ for all $\alpha < 1$. In fact in view of the periodicity of α_ε we deduce from (5.3) and (5.6),

$$\begin{aligned}
\sup_{\substack{t,s\in I_k \\ t\neq s}} \frac{|a(t)-a(s)|}{|t-s|^\alpha} &\leq \sup_{\substack{t,s\in[t_k,t_k+2\pi/h_k] \\ t\neq s}} \frac{|a(t)-a(s)|}{|t-s|^\alpha} \\
&\leq \sup_{\substack{t,s\in I_k \\ t\neq s}} \frac{|a(t)-a(s)|}{|t-s|} \left(\frac{2\pi}{h_k}\right)^{1-\alpha} \\
&\leq M\varepsilon_k (2\pi)^{1-\alpha} h_k^\alpha \,.
\end{aligned}$$

Finally, it is immediate to see that if, for all $q > 1$,

$$\sup_{k\in\mathbb{N}} \left(\sum_{j=k}^{+\infty} \rho_j\right)^q \varepsilon_k h_k < +\infty \,, \tag{5.10}$$

then the condition (2.6) is verified.

Let us come to the construction of u_0 and u_1. Let φ_k be the solution to

$$\begin{cases} \varphi_k'' + h_k^2 a(t)\varphi_k = 0 \\ \varphi_k(t_k) = \eta_k, \ \ \varphi_k'(t_k) = 0 \,. \end{cases} \tag{5.11}$$

We have that $\varphi_k \in \mathcal{C}^\infty(\mathbb{R}\setminus\{0\}) \cap \mathcal{C}^2(\mathbb{R})$ and, for all $t \in I_k$,

$$\varphi_k(t) = \eta_k w_{\varepsilon_k}(h_k(t-t_k)) = \eta_k e^{-\varepsilon_k h_k |t-t_k|} \tilde{w}_{\varepsilon_k}(h_k(t-t_k)) \,. \tag{5.12}$$

Suppose that, for all $k \in \mathbb{N}$,

$$2M \sum_{j=k+1}^{+\infty} \varepsilon_j \rho_j \leq \varepsilon_k \rho_k \,, \tag{5.13}$$

$$4M \sum_{j=0}^{k-1} \varepsilon_j h_j \rho_j \leq \varepsilon_k h_k \rho_k \,, \tag{5.14}$$

and, for all $p > 0$,

$$\lim_{k\to+\infty} p \log h_k + 4 \log \eta_k - \varepsilon_k h_k \rho_k = -\infty \,. \tag{5.15}$$

We denote by E_{φ_k}, $\tilde{E}_{\varphi_k}$ the following two different type of energy:

$$E_{\varphi_k}(t) = h_k^2|\varphi_k(t)|^2 + |\varphi_k'(t)|^2 ,$$
$$\tilde{E}_{\varphi_k}(t) = h_k^2 a(t)|\varphi_k(t)|^2 + |\varphi_k'(t)|^2 .$$

By (5.6), using the periodicity of α_ε and $\tilde{w}_\varepsilon$, we deduce that

$$E_{\varphi_k}\left(t_k \pm \frac{\rho_k}{2}\right) = \tilde{E}_{\varphi_k}\left(t_k \pm \frac{\rho_k}{2}\right) = h_k^2\eta_k^2 \exp(-\varepsilon_k h_k \rho_k) .$$

For $t \leq t_k - \rho_k/2$, we have

$$\begin{aligned} E_{\varphi_k}(t) &\leq E_{\varphi_k}\left(t_k - \frac{\rho_k}{2}\right) \exp\left(h_k \int_t^{t_k-\rho_k/2} |1-a(t)|\,dt\right) \\ &\leq h_k^2\eta_k^2 \exp(-\varepsilon_k h_k \rho_k) \exp\left(h_k \sum_{j=k+1}^{+\infty} M\varepsilon_j\rho_j\right) \\ &\leq h_k^2\eta_k^2 \exp\left(h_k\left(-\varepsilon_k\rho_k + M\sum_{j=k+1}^{+\infty}\varepsilon_j\rho_j\right)\right) . \end{aligned}$$

and by (5.13) we obtain

$$E_{\varphi_k}(t) \leq h_k^2\eta_k^2 \exp\left(-\frac{1}{2}\varepsilon_k h_k \rho_k\right) . \tag{5.16}$$

On the other hand, if $t \geq t_k + \rho_k/2$,

$$\begin{aligned} \tilde{E}_{\varphi_k}(t) &\leq \tilde{E}_{\varphi_k}\left(t_k + \frac{\rho_k}{2}\right) \exp\left(\int_{t_k+\rho_k/2}^{t} \frac{|a'(t)|}{a(t)}\,dt\right) \\ &\leq h_k^2\eta_k^2 \exp(-\varepsilon_k h_k \rho_k) \exp\left(\sum_{j=0}^{k-1} 2M\varepsilon_j h_j \rho_j\right) , \end{aligned}$$

so that, by (5.14), we infer

$$\tilde{E}_{\varphi_k}(t) \leq h_k^2\eta_k^2 \exp\left(-\frac{1}{2}\varepsilon_k h_k \rho_k\right) . \tag{5.17}$$

From (5.15), (5.16) and (5.17) we deduce that for all $p > 0$ there exists $C_p > 0$, such that

$$|\varphi_k(t)| + |\varphi_k'(t)| \leq C_p h_k^{-p} , \tag{5.18}$$

for all $t \in \mathbb{R} \setminus I_k$. We set

$$u_0(x) = \sum_{k=0}^{+\infty} \varphi_k(0) e^{ih_k x}, \quad u_1(x) = \sum_{k=0}^{+\infty} \varphi_k'(0) e^{ih_k x}, \tag{5.19}$$

and

$$u(t,x) = \sum_{k=0}^{+\infty} \varphi_k(t) e^{ih_k x}. \tag{5.20}$$

Applying a Paley-Wiener type result for the Fourier series we deduce from (5.18) that $u_0,\ u_1 \in \mathcal{C}^\infty(\mathbb{R})$ and $u \in \mathcal{C}^2(\mathbb{R} \setminus \{0\}; \mathcal{C}^\infty(\mathbb{R}))$. Moreover we have, for all $s \geq 1$,

$$\lim_{k \to +\infty} \eta_k \exp(-h_k^{1/s}) = 0. \tag{5.21}$$

In fact, suppose by contradiction that there exists $s_0 \geq 1$ such that

$$\limsup_{k \to +\infty} \log \eta_k - h_k^{1/s_0} > -\infty,$$

then, taking $s_1 > s_0$ we have

$$\limsup_{k \to +\infty} \log \eta_k - h_k^{1/s_1} = +\infty.$$

Since (5.4) and (5.9) imply that $\lim_k h_k^{1-1/s_1} \varepsilon_k \rho_k = 0$ we deduce that

$$\limsup_{k \to +\infty} \log \eta_k - h_k^{1/s_1}(h_k^{1-1/s_1} \varepsilon_k \rho_k) = \limsup_{k \to +\infty} \log \eta_k - h_k \varepsilon_k \rho_k = +\infty,$$

against (5.15). From (5.12), (5.18) and (5.21) we deduce that

$$|\varphi_k(t)| + |\varphi_k'(t)| \leq C_s \exp(h_k^{1/s}),$$

for all $t \in \mathbb{R}$ and hence u is a solution of (2.7) in $\mathcal{C}^2(\mathbb{R}; \mathcal{D}'^{(s)}(\mathbb{R}))$, where $\mathcal{D}'^{(s)}(\mathbb{R})$ is the space of the Gevrey-ultradistributions of order s. This solution is unique (see [1, Theorem 6]), so that to conclude the proof it will be sufficient to show that $u \notin \mathcal{C}^1([0, r[\,; \mathcal{D}'(\,]-r, r[\,))$ for all $r > 0$.

Suppose finally that

$$\lim_{k \to +\infty} h_k^{-p} \eta_k = +\infty \qquad \text{for all } p > 0. \tag{5.22}$$

By the recalled Paley-Wiener result we have that (5.11) (5.20) and (5.22) imply that $u \notin \mathcal{C}([0, r[\,; \mathcal{D}'(\mathbb{R}))$. Let us show this fact in a direct way. We observe that from (5.12), (5.18) it follows, for all $x \in [-\pi, \pi]$,

$$|u(t_k, x)| \le \sum_{j=0}^{+\infty} \varphi_j(t_k) \le \eta_k + C \sum_{j=0}^{+\infty} h_j^{-1},$$

and since, in view of (5.6) and (5.7), $\lim_k \eta_k = +\infty$ and $\sum_j h_j^{-1} < +\infty$, we obtain that

$$|u_k(t_k, x)| \le C\eta_k, \tag{5.23}$$

where C does not depend on k. Let χ be a $\mathcal{C}^\infty$ function with $\chi(s) = 0$ for $s \le 0$, and $\chi(s) = 1$ for $s \ge 1$; define, for all $k \in \mathbb{N}$,

$$\psi_k(x) = \chi(h_k(\pi + x))\chi(h_k(\pi - x)) \frac{e^{-ih_k x}}{\sqrt{\eta_k}}.$$

From (5.22) we have that $\psi_k \to 0$ in $\mathcal{C}_0^\infty(\mathbb{R})$, while, from (5.23),

$$\begin{aligned} \int u(t_k, x)\psi_k(x)\, dx &= 2\pi\sqrt{\eta_k} + \int_{-\pi}^{\pi} u(t_k, x) \left(\psi_k(x) - \frac{e^{-ih_k x}}{\sqrt{\eta_k}} \right) dx \\ &\ge 2\pi\sqrt{\eta_k} - \frac{2}{h_k} C\sqrt{\eta_k} \end{aligned}$$

and then $\int u(t_k, x)\psi_k(x)dx \to +\infty$.

It is possible to prove that $u \notin \mathcal{C}^1([0, r[\,; \mathcal{D}'(\,]-r, r[\,))$ arguing in a similar but more refined way and using as test function a cut off of the solution of the dual problem

$$\begin{cases} v_{tt} - a(t)v_{xx} = 0 \\ v(t_k, x) = 0, \;\; v_t(t_k, x) = \chi_k(x). \end{cases}$$

where $\{\chi_k\}$ is a suitable 2π-periodic Gevrey function (see [4, p. 118] for the details).

To end we perform a choice of sequences satisfying (5.4), (5.5), (5.6), (5.7), (5.9), (5.10), (5.13), (5.14), (5.15), (5.22). We set

$$\begin{aligned} h_k &= N^{2k} \left([\log(k+3)] \right)^k, \\ \eta_k &= (\log(k+3))^{k \log(k+3)}, \\ \rho_k &= 4\pi \left([\log(k+3)(\log(\log(k+3)))^{-1}] \right)^k \left([\log(k+3)] \right)^{-k}, \\ \varepsilon_k &= \frac{N^k L}{h_k \rho_k} = \frac{L}{4\pi N^k} \left([\log(k+3)(\log(\log(k+3)))^{-1}] \right)^{-k}, \end{aligned}$$

where $[x]$ denotes the maximum integer $\leq x$ and N, $1/L$ are integers sufficiently large. The proof of the Theorem 4 is concluded.

The proof of the Theorem 5 is essentially the same as the previous one. We stress only the different points. We consider α_ε and w_ε as defined in (5.1) and (5.2) respectively and we introduce four sequences $\{h_k\}$, $\{\eta_k\}$, $\{\rho_k\}$ and $\{\varepsilon_k\}$ such that (5.4), (5.5), (5.6) and (5.7) are satisfied. We define a as in (5.8) and we require that

$$\sup_{k\in\mathbb{N}} \varepsilon_k h_k^{\alpha_0} < +\infty\,. \tag{5.24}$$

This is enough to have that $a \in \mathcal{C}^{0,\alpha_0}(\mathbb{R})$. We remark that if $\alpha_0 = 0$, then (5.4) implies that $a \in \mathcal{C}(\mathbb{R})$ and moreover in this case (5.24) is a consequence of (5.4). To obtain the condition (2.8) it will be sufficient to suppose that

$$\sup_{k\in\mathbb{N}} \left(\sum_{j=k}^{+\infty} \rho_j\right)^{q_0} \varepsilon_k h_k < +\infty\,. \tag{5.25}$$

Let φ_k be the solution to (5.11). We suppose that (5.13) and (5.14) are satisfied. Moreover we impose that

$$\lim_{k\to+\infty} h_k^{1/s} + 4\log\eta_k - \varepsilon_k h_k \rho_k = -\infty\,, \tag{5.26}$$

for all $s > s_0 = (q_0/(q_0-1))(1/(1-\alpha_0))$. Since also in this case (5.16) and (5.17) are verified, condition (5.26) implies that for all $s > s_0$ there exists a positive constant C_s such that

$$|\varphi_k(t)| + |\varphi_k'(t)| \leq C_s \exp(-h_k^{1/s})\,, \tag{5.27}$$

for all $t \in \mathbb{R} \setminus I_k$. The functions u_0, u_1 and u are now defined like in (5.19) and (5.20). As a consequence of the Paley-Wiener theorem it follows from (5.27) that u_0, $u_1 \in \gamma^{(s)}(\mathbb{R})$ and $u \in \mathcal{C}^2(\mathbb{R}\setminus\{0\}; \gamma^{(s)}(\mathbb{R}))$ for all $s > s_0$. Moreover we claim that

$$\lim_{k\to+\infty} \eta_k \exp(-h_k^{1/s}) = 0\,, \tag{5.28}$$

for all $s < s_0$. Let us prove (5.28) by contradiction. Suppose that there exists $s < s_0$ such that

$$\limsup_{k\to+\infty} \log\eta_k - h_k^{1/s} > -\infty\,,$$

then taking $s < s' < s_0$ we have that

$$\limsup_{k\to+\infty} \log\eta_k - h_k^{1/s'} = +\infty\,.$$

Since $s' < s_0$ there exist $q > q_0$ and $\alpha < \alpha_0$ such that $s' = (q/(q-1))(1/(1-\alpha))$. From (5.24) we deduce that

$$\lim_{k\to+\infty} (\varepsilon_k h_k^{\alpha})^{(q-1)/q} = 0\,,$$

and from (5.25) we infer that

$$\lim_{k\to+\infty} \rho_k(\varepsilon_k h_k)^{1/q} = 0\,.$$

Consequently

$$\limsup_{k\to+\infty} \log\eta_k - \varepsilon_k h_k \rho_k = \limsup_{k\to+\infty} \log\eta_k - h_k^{1/s'}((\varepsilon_k h_k^{\alpha})^{(q-1)/q}(\varepsilon_k h_k)^{1/q} = +\infty\,,$$

and this last line is absurd in view of (5.26). A consequence of (5.28) is that u is in $\mathcal{C}^2(\mathbb{R}; \mathcal{D}'^{(s)}(\mathbb{R}))$ for all $s < s_0$.

Supposing that, for all $s > s_0$,

$$\lim_{k\to+\infty} \eta_k \exp(-h_k^{1/s}) = +\infty\,, \tag{5.29}$$

it is possible to see that $u \notin \mathcal{C}^1([0, r[\,; \mathcal{D}'^{(s)}(\,]-r, r[\,))$ for all $s > s_0$ and for all $r > 0$. The proof of this last claim is based, as in the case of the Theorem 4, on a duality argument. We choose the sequences $\{h_k\}$, $\{\eta_k\}$, $\{\rho_k\}$ and $\{\varepsilon_k\}$ in the following way: if $\alpha_0 = 0$,

$$\begin{aligned}
h_k &= N^k\,,\\
\eta_k &= \exp(N^{k/s_0}k^{-q_0})\,,\\
\rho_k &= 4\pi\left[N^{(1-1/q_0)k}\right]N^{-k}\,,\\
\varepsilon_k &= Lk^{-q_0}\,,
\end{aligned}$$

and if $\alpha_0 > 0$,

$$\begin{aligned}
h_k &= N^k\,,\\
\eta_k &= \exp(N^{k/s_0}k^{-1})\,,\\
\rho_k &= 4\pi\left[N^{(1-(1-\alpha_0)/q_0)k}\right]N^{-k}\,,\\
\varepsilon_k &= LN^{-\alpha_0 k}\,,
\end{aligned}$$

where N and $1/L$ are integers sufficiently large. We let to the reader to verify that with these choices the conditions (5.4), (5.5), (5.6), (5.7), (5.24), (5.25), (5.13), (5.14), (5.26) and (5.29) are satisfied. The proof of Theorem 5 is complete.

6. – Proofs of the Theorems 6 and 7

In this section we outline the main points of the proofs of the Theorems 6 and 7 and of the Corollary 1. We start with the proofs of the two theorems. First of all we remark that, by the finite speed of propagation (which is a consequence of the fact that $a \in \mathcal{L}^1$), we can suppose in both cases, without loss of generality, that c has compact support in the x variable. To prove Theorem 6 we define

$$\tilde{a}(t,\xi) = \begin{cases} a(T) & \text{if } T|\xi| \le 1\,, \\ a(|\xi|^{-1}) & \text{if } T|\xi| > 1 \text{ and } t|\xi| \le 1\,, \\ a(t) & \text{if } t|\xi| > 1\,, \end{cases}$$

and

$$\alpha(t,\xi) = \begin{cases} |\tilde{a}(t,\xi) - a(t)|\,|\xi| & \text{if } t|\xi| \le 1\,, \\ \dfrac{|a'(t)|}{a(t)} & \text{if } t|\xi| > 1\,. \end{cases}$$

We set

$$k(t,\xi) = (1+|\xi|^2)^\sigma \exp\left(-\int_0^t \alpha(s,\xi)\,ds\right)\,,$$

where σ is a positive constant. Defining $\mathcal{E}$ as in (4.3) and arguing as in the proof of Theorem 1 we obtain that the estimates (4.4) and (4.5) hold. Moreover we have

$$\mathcal{E}'(t) \le \tilde{K}\mathcal{E}(t) + \int |\mathcal{F}_x(cu)|^2\, k(t,\xi)\,d\xi\,, \tag{6.1}$$

where $\tilde{K} = 1 + \max_{t\in[0,T]}\{\sum_{j=1}^n |b_j(t)|\}$. Suppose now that there exist C, $N > 0$, with N independent form σ, such that, for all $t \in [0,T]$ and for all ξ, $\eta \in \mathbb{R}$,

$$k(t,\xi+\eta) \le Ck(t,\xi)(1+|\eta|)^N\,, \tag{6.2}$$

(this will mean in particular that k is a temperate weight function). Then, if σ is sufficiently large, we deduce that

$$\begin{aligned}\int \frac{k(t,\xi)}{k(t,\eta)k(t,\xi-\eta)}\,d\eta &\le C\int \frac{(1+|\eta|)^N}{k(t,\eta)}\,d\eta \\ &\le C\int (1+|\eta|)^N \exp\left(\int_0^s \alpha(s,\eta)\right)\,d\eta \le C'\,.\end{aligned}$$

Therefore denoting by $|||c(t)|||$ the value

$$|||c(t)||| = \left(\int |\mathcal{F}_x c(t,\xi)|^2 k(t,\xi)\,d\xi\right)^{1/2}\,,$$

we have

$$
\begin{aligned}
|||c(t)u(t)|||^2 &= \int |\mathcal{F}_x(cu)(t,\xi)|^2 k(t,\xi)\,d\xi \\
&\le \frac{1}{2\pi} \iint \mathcal{F}_x c(\xi-\eta) k(\xi-\eta)^{1/2} v(\eta) k(\eta)^{1/2} \\
&\quad \times \left(\frac{k(\xi)}{k(\eta)k(\xi-\eta)} \right)^{1/2} \overline{\mathcal{F}_x(cu)(\xi)} k(\xi)^{1/2}\,d\eta\,d\xi \\
&\le \frac{1}{2\pi} \left(\iint |\mathcal{F}_x c(\xi-\eta)|^2 k(\xi-\eta) |v(\eta)|^2 k(\eta) d\eta d\xi \right)^{1/2} \\
&\quad \times \left(\iint \frac{k(\xi)}{k(\eta)k(\xi-\eta)} |\mathcal{F}_x(cu)|^2 k(\xi)\,d\eta\,d\xi \right)^{1/2} \\
&\le \frac{1}{2\pi} \left(\int |\mathcal{F}_x c(\xi)|^2 k(\xi)\,d\xi \int |v(\eta)|^2 k(\eta)\,d\eta \right)^{1/2} \\
&\quad \times \left(\int \left(\int \frac{k(\xi)}{k(\eta)k(\xi-\eta)}\,d\eta \right) |\mathcal{F}_x(cu)|^2 k(\xi)\,d\xi \right)^{1/2} \\
&\le \frac{1}{2\pi} |||c(t)|||\; |||u(t)|||\, C' |||c(t)u(t)|||\,,
\end{aligned}
$$

and finally

$$
|||c(t)u(t)||| \le \tilde{C} |||c(t)||| \, |||u(t)|||\,,
$$

where $\tilde{C}$ does not depend on c or u. Using this estimate and the fact that $|||u(t)||| \le \mathcal{E}(t)$, we obtain from (6.1) that

$$
\mathcal{E}'(t) \le \left(\tilde{K} + \tilde{C}^2 \max_{t\in[0,T]} \{|||c(t)|||^2\} \right) \mathcal{E}(t)\,. \tag{6.3}
$$

From (4.4), (4.5) and (6.3) the $\mathcal{C}^\infty$-well-posedness follows.

To end the proof it is sufficient to show (6.2). Let us denote by h the following function

$$
h(t,\xi) = \exp\left(\int_0^t \alpha(s,\xi)\,ds \right)\,.
$$

We show now that there exist C, $N > 0$ such that, for all $t \in [0,T]$ and for all ξ, $\eta \in \mathbb{R}$, with $|\xi| \ge \max\{1/T, 1\}$, we have

$$
h(t,\xi+\eta) \le C h(t,\xi)(1+|\eta|)^N\,. \tag{6.4}
$$

We prove (6.4) considering many different cases depending on the values of t, ξ and η. In the following C and C' will denote constants not depending on t, ξ and η, possibly having different values in different lines.

If $T|\xi+\eta| \le 1$ then

$$
\int_0^t \alpha(s,\xi+\eta)\,ds \le a(T) + \frac{1}{T}\|a\|_{\mathcal{L}^1}\,.
$$

If $T|\xi+\eta|>1$ and $t|\xi+\eta|\le 1$, $10|\eta|\le|\xi|^{1/2}$ and $t|\xi|\le 1$ then

$$\begin{aligned}\int_0^t \alpha(s,\xi+\eta)\,ds &\le \int_0^t |\tilde a(s,\xi+\eta)-a(s)|\,|\xi+\eta|\,ds\\ &\le \int_0^t |a(|\xi+\eta|^{-1})-a(s)||\xi+\eta|ds \le \int_0^t |a(|\xi|^{-1})-a(s)||\xi|ds\\ &\quad+\int_0^{|\xi+\eta|^{-1}} |a(|\xi|^{-1})-a(s)|\,|\eta|\,ds\\ &\quad+\int_0^{|\xi+\eta|^{-1}} |a(|\xi+\eta|^{-1})-a(|\xi|^{-1})|\,|\xi+\eta|\,ds\\ &\le \int_0^t \alpha(s,\xi)\,ds+\int_0^{|\xi+\eta|^{-1}} |a(|\xi|^{-1})-a(s)|\,|\eta|\,ds\\ &\quad+|a(|\xi+\eta|^{-1})-a(|\xi|^{-1})|\,.\end{aligned}$$

We have

$$\begin{aligned}\int_0^{|\xi+\eta|^{-1}} |a(|\xi|^{-1})-a(s)||\eta|\,ds &\le (a(|\xi|^{-1})+C(1+\log(1+|\xi+\eta|)))\frac{|\eta|}{|\xi+\eta|}\\ &\le C(1+\log(1+|\xi|+|\eta|))\frac{|\eta|}{|\xi+\eta|}\\ &\le C(1+\log(1+|\xi|))|\xi|^{-1/2}\le C'\end{aligned}$$

and

$$|a(|\xi+\eta|^{-1})-a(|\xi|^{-1})|\le C\left|\log\left(\frac{|\xi+\eta|}{|\xi|}\right)\right|\le C'$$

so that

$$\int_0^t \alpha(s,\xi+\eta)\,ds\le\int_0^t \alpha(s,\xi)\,ds+C\,. \tag{6.5}$$

If $T|\xi+\eta|>1$ and $t|\xi+\eta|\le 1$, $10|\eta|\le|\xi|^{1/2}$ and $t|\xi|>1$ then

$$\begin{aligned}\int_0^t \alpha(s,\xi+\eta)\,ds &\le \int_0^{|\xi|^{-1}} |a(|\xi+\eta|^{-1})-a(s)|\,|\xi+\eta|\,ds\\ &\quad+\int_{|\xi|^{-1}}^{|\xi+\eta|^{-1}} |a(|\xi+\eta|^{-1})-a(s)|\,|\xi+\eta|\,ds\,.\end{aligned}$$

Arguing as in the previous case we obtain

$$\int_0^{|\xi|^{-1}} |a(|\xi+\eta|^{-1})-a(s)|\,|\xi+\eta|\,ds\le\int_0^t \alpha(s,\xi)\,ds+C\,,$$

and

$$\begin{aligned}\int_{|\xi|^{-1}}^{|\xi+\eta|^{-1}} &|a(|\xi+\eta|^{-1}) - a(s)||\xi+\eta|\, ds \\ &\le C\big||\xi+\eta|^{-1} - |\xi|^{-1}\big||\xi+\eta|(1+\log(1+|\xi+\eta|) \\ &\quad + \log(1+|\xi|)) \le C\frac{|\eta|}{|\xi|}(1+\log(1+|\xi|+|\eta|)) \le C',\end{aligned}$$

and (6.5) follows from these inequalities.

If $T|\xi+\eta| > 1$, $t|\xi+\eta| \le 1$, $10|\eta| > |\xi|^{1/2}$ then

$$\begin{aligned}\int_0^t \alpha(s,\xi+\eta)\, ds &\le \int_0^{|\xi+\eta|^{-1}} |a(|\xi+\eta|^{-1}) - a(s)|\,|\xi+\eta|\, ds \\ &\le |a(|\xi+\eta|^{-1})| + |\xi+\eta| \int_0^{|\xi+\eta|^{-1}} |a(s)|\, ds \\ &\le C(1+\log(1+|\xi+\eta|)) \\ &\le C(1+\log(1+|\eta|))\,.\end{aligned}$$

If $t|\xi+\eta| > 1$ (and consequently $T|\xi+\eta| > 1$), $10|\eta| \le |\xi|^{1/2}$ and $t|\xi| \le 1$ then

$$\int_0^t \alpha(s,\xi+\eta)\, ds \le \int_0^{|\xi+\eta|^{-1}} |a(|\xi+\eta|^{-1}) - a(s)|\,|\xi+\eta|\, ds + \int_{|\xi+\eta|^{-1}}^t \frac{|a'(s)|}{a(s)}\, ds\,.$$

As before we obtain that

$$\int_0^{|\xi+\eta|^{-1}} |a(|\xi+\eta|^{-1}) - a(s)|\,|\xi+\eta|\, ds \le \int_0^t \alpha(s,\xi)\, ds + C\,,$$

and from (3.2)

$$\int_{|\xi+\eta|^{-1}}^t \frac{|a'(s)|}{a(s)}\, ds \le \int_{|\xi+\eta|^{-1}}^{|\xi|^{-1}} \frac{C}{s}\, ds \le C'$$

so that

$$\int_0^t \alpha(s,\xi+\eta)\, ds \le \int_0^t \alpha(s,\xi)\, ds + C\,.$$

If $t|\xi+\eta| > 1$, $10|\eta| \le |\xi|^{1/2}$ and $t|\xi| > 1$ then

$$\begin{aligned}\int_0^t \alpha(s,\xi+\eta) ds &\le \int_0^{|\xi+\eta|^{-1}} |a(|\xi+\eta|^{-1}) - a(s)||\xi+\eta| ds + \int_{|\xi+\eta|^{-1}}^t \frac{|a'(s)|}{a(s)} ds \\ &\le \int_0^{|\xi|^{-1}} |a(|\xi+\eta|^{-1}) - a(s)||\xi+\eta| ds + \int_{|\xi|^{-1}}^t \frac{|a'(s)|}{a(s)} ds \\ &\quad + \left| \int_{|\xi|^{-1}}^{|\xi+\eta|^{-1}} |a(|\xi+\eta|^{-1}) - a(s)||\xi+\eta| + \frac{|a'(s)|}{a(s)} ds \right| \\ &\le \int_0^t \alpha(s,\xi) ds + C\,.\end{aligned}$$

Finally if $t|\xi+\eta|>1$, $10|\eta|>|\xi|^{1/2}$ then

$$\int_0^t \alpha(s,\xi+\eta)ds \le \int_0^{|\xi+\eta|^{-1}} |a(|\xi+\eta|^{-1})-a(s)||\xi+\eta|ds + \int_{|\xi+\eta|^{-1}}^t \frac{|a'(s)|}{a(s)}ds \le C(1+\log(1+|\eta|))\,.$$

Hence there exist K, $M>0$ such that for all $t\in[0,T]$ and for all ξ, $\eta\in\mathbb{R}$ with $|\xi|\ge\max\{1/T,1\}$,

$$\int_0^t \alpha(s,\xi+\eta)\,ds \le K+M\log(1+|\eta|)+\int_0^t \alpha(s,\xi)ds\,,$$

and (6.4) follows. By the properties of the temperate weight functions (see e.g. [5, Ch. 2]), (6.2) is an easy consequence of (6.4). This concludes proof of the Theorem 6.

Let us come briefly to the proof of Theorem 7. First of all we remark that it is sufficient to show the result supposing that $s>1/(q-1)$. In fact if $s\le 1/(q-1)$ there exists $q'>q$ such that $1/(q'-1)<s<q'/(q'-1)$ and obviously the condition (3.3) implies that $t^{q'}|a'(t)|\le C'$.

We set

$$\tilde a(t,\xi)=\begin{cases} a(T) & \text{if } T|\xi|^{1/(qs-s)}\le 1\,,\\ a(|\xi|^{-1/(qs-s)}) & \text{if } T|\xi|^{1/(qs-s)}>1 \text{ and } t|\xi|^{1/(qs-s)}\le 1\,,\\ a(t) & \text{if } t|\xi|^{1/(qs-s)}>1\,,\end{cases}$$

and

$$\alpha(t,\xi)=\begin{cases} |\tilde a(t,\xi)-a(t)|\,|\xi| & \text{if } t|\xi|^{1/(qs-s)}\le 1\,,\\ \dfrac{|a'(t)|}{a(t)} & \text{if } t|\xi|^{1/(qs-s)}>1\,.\end{cases}$$

We define

$$k(t,\xi)=\exp\left(-\int_0^t \alpha(\sigma,\xi)\,d\sigma+\beta|\xi|^{1/s}\right)\,,$$

with β positive constant, and $\mathcal{E}$ like in (4.3). As in the proof of Theorem 2 the estimates (4.7) and (4.8) follow. Similarly we deduce that

$$\mathcal{E}'(t)\le \tilde K\,\mathcal{E}(t)+\int |\mathcal{F}_x(cu)|^2\,k(t,\xi)\,d\xi\,,$$

where $\tilde K=1+\max_{t\in[0,T]}\{\sum_{j=1}^n |b_j(t)|\}$. Remarking now that (see the Appendix) there exist C, $\varepsilon>0$ such that for all $t\in[0,T]$ and for all ξ, $\eta\in\mathbb{R}$,

$$k(t,\xi+\eta)\le Ck(t,\xi)e^{\varepsilon|\eta|^{2/s}}\,, \tag{6.6}$$

we deduce as in the previous proof that

$$\int |\mathcal{F}_x(cu)(t,\xi)|^2 k(t,\xi)\, d\xi \le \tilde{C}^2 \|\!|c(t)|\!\|^2 \mathcal{E}(t)\,,$$

where

$$\|\!|c(t)|\!\| = (\int |\mathcal{F}_x c(t,\xi)|^2 (1+|\xi|^2) e^{\varepsilon|\xi|^{2/s}}\, d\xi\,.$$

Hence

$$\mathcal{E}'(t) \le (\tilde{K} + \tilde{C}^2 \max_{t\in[0,T]} \|\!|c(t)|\!\|^2)\mathcal{E}(t)\,,$$

and the $\gamma^{(s)}$-well-posedness follows.

Let us finally sketch the proof of the Corollary 1. We set $u^{(0)}(t,x) = u_0(x)$ and, for $j = 1, 2, \ldots,$ we define $u^{(j)}(t,x)$ as the solution to the following Cauchy problem

$$\begin{cases} u^{(j)}_{tt} - a(t)u^{(j)}_{xx} + b(t)u^{(j)}_x + f(u^{(j-1)}) = 0 & \text{in } [0,T]\times\mathbb{R}\,, \\ u^{(j)}(0,x) = u_0(x), \;\; u^{(j)}_t(0,x) = u_1(x) & \text{in } \mathbb{R}\,. \end{cases}$$

Using the notations introduced in the proof of the Theorem 6 we denote by D the following constant

$$D = \max\left\{2,\;\; 2e^{\tilde{K}/2}\int_{\mathbb{R}} (|\mathcal{F}_x(u_1)|^2 + (1+\sup_{t\in[0,T]} \tilde{a}(t,\xi)|\xi|^2)|\mathcal{F}_x(u_0)|^2)k(0,\xi)d\xi\right\}\,.$$

Setting, for $j\in\mathbb{N}$,

$$\mathcal{E}_{u^{(j)}}(t) = \int_{\mathbb{R}} (|\mathcal{F}_x(u^{(j)})'|^2 + (1+\tilde{a}(t,\xi)|\xi|^2)|\mathcal{F}_x(u^{(j)})|^2)k(t,\xi)\, d\xi\,,$$

we easily deduce that

$$\sup_{t\in[0,T]} \mathcal{E}_{u^{(0)}}(t) \le D/2\,. \tag{6.7}$$

Moreover, as we have

$$\|\!|u^{(j)}|\!\| = \left(\int_{\mathbb{R}} |\mathcal{F}_x u^{(j)}|^2 k(t,\xi)\, d\xi\right)^{\frac{1}{2}}\,,$$

we deduce that $\|\!|u^{(j)}|\!\| \le \mathcal{E}_{u^{(j)}}$.

We recall that the fact that k is a temperate weight has the consequence that there exists $\tilde{C} > 0$ such that

$$\|\!|w^j|\!\| \le \tilde{C}^{j-1}\|\!|w|\!\|^j\,,$$

for all $w \in \mathcal{C}^1([0, T']; \mathcal{C}_0^\infty(\mathbb{R}))$ and for all $j \geq 1$. Let us remark that $\tilde{C}$ does not depend on w but only on the weight k. On the other hand, since f is an entire analytic function, for all $r > 0$ there exists $C_r > 0$ such that

$$\left| \frac{d^j f}{du^j}(0) \right| \leq C_r r^j j! \,.$$

Let now $r = 1/(2\tilde{C}D)$ and $M = \max\{1, \ (2C_r r)^2\}$; we claim that for all $j \in \mathbb{N}$,

$$\sup_{t \in [0, T']} \mathcal{E}_{u^{(j)}}(t) \leq D \,, \tag{6.8}$$

where $T' = \min\{T, 1/(2M)\}$.

For $j = 0$ the inequality (6.8) is a trivial consequence of (6.7). Suppose now that

$$\sup_{t \in [0, T']} \mathcal{E}_{u^{(j-1)}}(t) \leq D \,.$$

Arguing as in the proof of the Theorem 6 we have that

$$\begin{aligned} \mathcal{E}'_{u^{(j)}}(t) &\leq \tilde{K} \mathcal{E}_{u^{(j)}}(t) + |||f(u^{(j-1)})|||^2 \\ &\leq \tilde{K} \mathcal{E}_{u^{(j)}}(t) + \left(\sum_{k=1}^{+\infty} \frac{\left| \frac{d^k f}{du^k}(0) \right|}{k!} |||(u^{(j-1)})^k||| \right)^2 \\ &\leq \tilde{K} \mathcal{E}_{u^{(j)}}(t) + \left(C_r r \sum_{k=1}^{+\infty} (r\tilde{C} |||u^{(j-1)}|||)^{k-1} \right)^2 |||u^{(j-1)}|||^2 \\ &\leq \tilde{K} \mathcal{E}_{u^{(j)}}(t) + M \mathcal{E}_{u^{(j-1)}}(t) \,. \end{aligned}$$

Using Gronwall's lemma we deduce

$$\begin{aligned} \sup_{t \in [0, T']} \mathcal{E}_{u^{(j)}}(t) &\leq e^{\tilde{K}T'} \mathcal{E}_{u^{(j)}}(0) + M \int_0^{T'} \mathcal{E}_{u^{(j-1)}}(\tau) \, d\tau \\ &\leq e^{\tilde{K}/(2M)} \mathcal{E}_{u^{(j)}}(0) + MT' \sup_{t \in [0, T']} \mathcal{E}_{u^{(j-1)}}(t) \\ &\leq D/2 + D/2 = D \,, \end{aligned}$$

and the claim (6.8) follows by a recurrence argument.

We define, for $j = 1, 2, \ldots$, $w^{(j)} = u^{(j+1)} - u^{(j)}$, and we remark that $w^{(j)}$ is the solution to the Cauchy problem

$$\begin{cases} w_{tt}^{(j)} - a(t) w_{xx}^{(j)} + b(t) w_x^{(j)} + f(u^{(j)}) - f(u^{(j-1)}) = 0 & \text{in } [0, T] \times \mathbb{R} \,, \\ w^{(j)}(0, x) = 0, \ \ w_t^{(j)}(0, x) = 0 & \text{in } \mathbb{R} \,. \end{cases}$$

Since

$$\begin{aligned} f(u^{(j)}) - f(u^{(j-1)}) &= \sum_{k=1}^{+\infty} \frac{\frac{d^k f}{du^k}(0)}{k!}((u^{(j)})^k - (u^{(j-1)})^k) \\ &= \sum_{k=1}^{+\infty} \frac{\frac{d^k f}{du^k}(0)}{k!} \left(\sum_{l=0}^{k-1} ((u^{(j)})^{k-1-l}(u^{(j-1)})^l) \right) w^{(j-1)} , \end{aligned}$$

we obtain that, for all $t \in [0, T']$,

$$\begin{aligned} \mathcal{E}'_{w^{(j)}}(t) &\le \tilde{K}\mathcal{E}_{w^{(j)}}(t) + \| f(u^{(j)}) - f(u^{(j-1)}) \|^2 \\ &\le \tilde{K}\mathcal{E}_{w^{(j)}}(t) + \left(\sum_{k=1}^{+\infty} \frac{\left| \frac{d^k f}{du^k}(0) \right|}{k!} \sum_{l=0}^{k-1} \| (u^{(j)})^{k-1-l}(u^{(j-1)})^l w^{(j-1)} \| \right)^2 \\ &\le \tilde{K}\mathcal{E}_{w^{(j)}}(t) + \left(C_r r \sum_{k=1}^{+\infty} k(r\tilde{C}D)^{k-1} \right)^2 \| w^{(j-1)} \|^2 \\ &\le \tilde{K}\mathcal{E}_{w^{(j)}}(t) + 4M\mathcal{E}_{w^{(j-1)}}(t) . \end{aligned}$$

Observing that $\sup_{t\in[0,T']} \mathcal{E}_{w^{(1)}}(t) \le 2\sup_{t\in[0,T']}(\mathcal{E}_{u^{(2)}}(t) + \mathcal{E}_{u^{(1)}}(t)) \le 4D$ and $\mathcal{E}_{w^{(j)}}(0) = 0$ for all $j = 1, 2, \dots,$ again using Gronwall's lemma, we deduce

$$\begin{aligned} \sup_{t\in[0,T']} \mathcal{E}_{w^{(j)}}(t) &\le 4M \int_0^{T'} \mathcal{E}_{w^{(j-1)}}(\tau_1)\, d\tau_1 \\ &\le (4M)^2 \int_0^{T'} \int_0^{\tau_1} \mathcal{E}_{w^{(j-2)}}(\tau_2)\, d\tau_2\, d\tau_1 \\ &\le (4M)^{j-1} \int_0^{T'} \int_0^{\tau_1} \cdots \int_0^{\tau_{j-2}} \mathcal{E}_{w^{(1)}}(\tau_{j-1})\, d\tau_{j-1} \dots d\tau_2\, d\tau_1 \\ &\le (4M)^{j-1} 4D \frac{(T')^{j-1}}{(j-1)!} . \end{aligned}$$

Thus we find that

$$\sum_{j=1}^{+\infty} \| u^{(j+1)} - u^{(j)} \| \le \sum_{j=1}^{+\infty} \left(\sup_{t\in[0,T']} \mathcal{E}_{w^{(j)}}(t) \right)^{\frac{1}{2}} \le \sum_{j=1}^{+\infty} \left(\frac{2^{j+1}D}{(j-1)!} \right)^{\frac{1}{2}} < +\infty .$$

This implies that the sequence $\{u^{(j)}\}$ converges, in the weighted $\mathcal{L}^2$-space, to the solution of the Cauchy problem (3.4).

A. – Appendix

Let a be a bounded function such that $a \in \mathcal{C}^1(]0,T])$ and suppose that $q \in]1,2[$, $s \in]2, q/(q-1)[$, $s > 1/(q-1)$ and a satisfies the condition (3.3); let $\tilde{a}$ and α be defined as in the proof of the Theorem 7. We claim that there exists $\varepsilon > 0$ such that, for all $t \in [0,T]$ and for all ξ, $\eta \in \mathbb{R}$, with $T|\xi|^{1/(qs-s)} > 1$,

$$\int_0^t \alpha(\sigma, \xi+\eta)\, d\sigma \le \int_0^t \alpha(\sigma,\xi)\, d\sigma + \varepsilon(1+|\eta|^{2/s})\,. \tag{A.1}$$

The inequality (A.1) will be sufficient to obtain (6.6). In the following Λ will denote the supremum of $|a|$ and C will denote a constant not depending on t, ξ and η, possibly having different values in different lines.

If $T|\xi+\eta|^{1/(qs-s)} \le 1$ then

$$\int_0^t \alpha(\sigma,\xi+\eta)\, d\sigma \le 2\Lambda T|\xi+\eta| \le 2\Lambda T^{1+s-qs}\,.$$

If $T|\xi+\eta|^{1/(qs-s)} > 1$, $t|\xi+\eta|^{1/(qs-s)} \le 1$, $10|\eta| \le |\xi|$ and $t|\xi|^{1/(qs-s)} \le 1$ then

$$\begin{aligned}
\int_0^t \alpha(\sigma,\xi+\eta)\, d\sigma &\le \int_0^t |\tilde{a}(\sigma,\xi+\eta) - a(\sigma)|\, |\xi+\eta|\, d\sigma \\
&\le \int_0^t |a(|\xi+\eta|^{1/(s-qs)}) - a(\sigma)|\, |\xi+\eta|\, d\sigma \\
&\le \int_0^t |a(|\xi|^{1/(s-qs)}) - a(\sigma)||\xi| d\sigma + \\
&\quad + \int_0^{|\xi+\eta|^{1/(s-qs)}} |a(|\xi|^{1/(s-qs)}) - a(\sigma)||\eta| d\sigma \\
&\quad + \int_0^{|\xi+\eta|^{1/(s-qs)}} |a(|\xi+\eta|^{1/(s-qs)}) - a(|\xi|^{1/(s-qs)})||\xi+\eta| d\sigma \\
&\le \int_0^t \alpha(\sigma,\xi) d\sigma + \int_0^{|\xi+\eta|^{1/(s-qs)}} |a(|\xi|^{1/(s-qs)}) - a(\sigma)|\, |\eta|\, d\sigma \\
&\quad + |a(|\xi+\eta|^{1/(s-qs)}) - a(|\xi|^{1/(s-qs)})|\, |\xi+\eta|^{1+1/(s-qs)}\,.
\end{aligned}$$

We have

$$\int_0^{|\xi+\eta|^{1/(s-qs)}} |a(|\xi|^{1/(s-qs)}) - a(\sigma)||\eta| d\sigma \le 2\Lambda|\xi+\eta|^{1/(s-qs)}|\eta| \le 2\Lambda|\eta|^{1+1/(s-qs)}\,,$$

and from (3.3) we deduce

$$|a(|\xi+\eta|^{1/(s-qs)}) - a(|\xi|^{1/(s-qs)})| \le C|\xi+\theta\eta|^{-1+1/s}|\eta|\,,$$

with $\theta \in]0,1[$, so that

$$|a(|\xi+\eta|^{1/(s-qs)}) - a(|\xi|^{1/(s-qs)})||\xi+\eta|^{1+1/(s-qs)} \le C|\eta|^{1+1/(s-qs)+1/s},$$

and finally

$$\int_0^t \alpha(\sigma,\xi+\eta)\,d\sigma \le \int_0^t \alpha(\sigma,\xi)\,d\sigma + C(|\eta|^{1+1/(s-qs)} + |\eta|^{1+1/(s-qs)+1/s}).$$

If $T|\xi+\eta|^{1/(qs-s)} > 1$, $t|\xi+\eta|^{1/(qs-s)} \le 1$, $10|\eta| \le |\xi|$ and $t|\xi|^{1/(qs-s)} > 1$ then

$$\int_0^t \alpha(\sigma,\xi+\eta)\,d\sigma \le \int_0^{|\xi|^{1/(s-qs)}} |a(|\xi+\eta|^{1/(s-qs)}) - a(\sigma)|\,|\xi+\eta|\,d\sigma + \int_{|\xi|^{1/(s-qs)}}^{|\xi+\eta|^{1/(s-qs)}} |a(|\xi+\eta|^{1/(s-qs)}) - a(\sigma)|\,|\xi+\eta|\,d\sigma.$$

Arguing as in the previous case we obtain

$$\int_0^{|\xi|^{1/(s-qs)}} |a(|\xi+\eta|^{1/(s-qs)}) - a(\sigma)|\,|\xi+\eta|\,d\sigma \le \int_0^t \alpha(\sigma,\xi)\,d\sigma + C(|\eta|^{1+1/(s-qs)} + |\eta|^{1+1/(s-qs)+1/s}).$$

On the other hand we have

$$||\xi+\eta|^{1/(s-qs)} - |\xi|^{1/(s-qs)}| \le C|\xi|^{-1+1/(s-qs)}|\eta|, \tag{A.2}$$

so that

$$\int_{|\xi|^{1/(s-qs)}}^{|\xi+\eta|^{1/(s-qs)}} |a(|\xi+\eta|^{1/(s-qs)}) - a(\sigma)|\,|\xi+\eta|\,d\sigma \le C|\eta|^{1+1/(s-qs)},$$

and consequently

$$\int_0^t \alpha(\sigma,\xi+\eta)\,d\sigma \le \int_0^t \alpha(\sigma,\xi)\,d\sigma + C(|\eta|^{1+1/(s-qs)} + |\eta|^{1+1/(s-qs)+1/s}).$$

If $T|\xi+\eta|^{1/(qs-s)} > 1$, $t|\xi+\eta|^{1/(qs-s)} \le 1$, $10|\eta| > |\xi|$ then

$$\int_0^t \alpha(\sigma,\xi+\eta)d\sigma \le \int_0^{|\xi+\eta|^{1/(s-qs)}} |a(|\xi+\eta|^{1/(s-qs)}) - a(\sigma)||\xi+\eta|d\sigma \le 2\Lambda|\xi+\eta|^{1+1/(s-qs)} \le C|\eta|^{1+1/(s-qs)}.$$

If $t|\xi+\eta|^{1/(qs-s)} > 1$, $10|\eta| \le |\xi|$ and $t|\xi|^{1/(qs-s)} \le 1$ then

$$\int_0^t \alpha(\sigma, \xi+\eta)\, d\sigma \le \int_0^{|\xi+\eta|^{1/(s-qs)}} |a(|\xi+\eta|^{1/(s-qs)}) - a(\sigma)||\xi+\eta| d\sigma + \int_{|\xi+\eta|^{1/(s-qs)}}^t \frac{|a'(\sigma)|}{a(\sigma)} d\sigma\,.$$

As before we obtain that

$$\int_0^{|\xi+\eta|^{1/(s-qs)}} |a(|\xi+\eta|^{1/(s-qs)}) - a(\sigma)|\,|\xi+\eta|\, d\sigma \le \int_0^t \alpha(\sigma,\xi)\, d\sigma + C(|\eta|^{1+1/(s-qs)} + |\eta|^{1+1/(s-qs)+1/s})\,,$$

and noticing, in view of (3.3), that for all $\sigma \in [|\xi+\eta|^{1/(s-qs)}, |\xi|^{1/(s-qs)}]$ we have

$$\frac{|a'(\sigma)|}{a(\sigma)} \le C|\xi|^{q/(qs-s)}\,,$$

we deduce from (A.2) that

$$\int_{|\xi+\eta|^{1/(s-qs)}}^t \frac{|a'(\sigma)|}{a(\sigma)}\, d\sigma \le C|\xi|^{q/(qs-s)}|\xi|^{-1+1/(s-qs)}|\eta| \le C|\eta|^{1/s}\,.$$

If $t|\xi+\eta|^{1/(qs-s)} > 1$, $10|\eta| \le |\xi|$ and $t|\xi|^{1/(qs-s)} > 1$ then

$$\begin{aligned}\int_0^t \alpha(\sigma,\xi+\eta) d\sigma &\le \int_0^{|\xi+\eta|^{1/(s-qs)}} |a(|\xi+\eta|^{1/(s-qs)}) - a(\sigma)||\xi+\eta| d\sigma \\ &\quad + \int_{|\xi+\eta|^{1/(s-qs)}}^t \frac{|a'(\sigma)|}{a(\sigma)}\, d\sigma \\ &\le \int_0^{|\xi|^{1/(s-qs)}} |a(|\xi+\eta|^{1/(s-qs)}) - a(\sigma)||\xi+\eta| d\sigma \\ &\quad + \int_{|\xi|^{1/(s-qs)}}^t \frac{|a'(\sigma)|}{a(\sigma)}\, d\sigma \\ &\quad + \left| \int_{|\xi|^{1/(s-qs)}}^{|\xi+\eta|^{1/(s-qs)}} |a(|\xi+\eta|^{1/(s-qs)}) - a(\sigma)||\xi+\eta| + \frac{|a'(\sigma)|}{a(\sigma)} d\sigma \right| \\ &\le \int_0^t \alpha(\sigma,\xi) d\sigma + C(|\eta|^{1+1/(s-qs)} + |\eta|^{1+1/(s-qs)+1/s} + |\eta|^{1/s})\,.\end{aligned}$$

Finally if $t|\xi+\eta|^{1/(qs-s)} > 1$, $10|\eta| > |\xi|$ then, using also the fact that $s > 1/(q-1)$, we have

$$\int_0^t \alpha(\sigma,\xi+\eta)d\sigma \leq \int_0^{|\xi+\eta|^{1/(s-qs)}} |a(|\xi+\eta|^{1/(s-qs)}) - a(\sigma)||\xi+\eta| d\sigma$$

$$+\int_{|\xi+\eta|^{1/(s-qs)}}^t \frac{|a'(\sigma)|}{a(\sigma)} d\sigma \leq 2\Lambda|\xi+\eta|^{1+1/(s-qs)} + C|\xi+\eta|^{1/s}$$

$$\leq C(|\eta|^{1+1/(s-qs)} + |\eta|^{1/s}.$$

The conclusion is now easily reached remarking that $1+1/(s-qs) < 1/s$.

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