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On Pseudosymmetric Systems with One Space Variable

TATSUO NISHITANI – SERGIO SPAGNOLO

Abstract. We investigate the Cauchy problem for a system of the form \( \partial_t u = A(x)\partial_x u + f(t, x) \), where \( A(x) \) is a pseudosymmetric matrix with analytic entries \( a_{ij}(x), i, j = 1, \ldots, N \). We prove the well-posedness at each point \( x_0 \) where \( a_{ij}(x_0) \cdot a_{ji}(x_0) = 0 \) for all \( i, j \). In the case \( N = 3 \), it is sufficient to assume such a condition for \( i = j \).

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Introduction

The class of pseudo symmetric systems was introduced by D’Ancona and Spagnolo [3] as the natural extension to the vector case of weakly hyperbolic equations

\[
\begin{align*}
\partial^2_t u &= \sum_{i,j}^{1,n} a_{ij}(t, x) \partial_{x_i} \partial_{x_j} u, \\
\sum_{i,j}^{1,n} a_{ij} \xi_i \xi_j &\geq 0.
\end{align*}
\]

(1)

The \( N \times N \) system in \( \mathbb{R}_t \times \mathbb{R}_x^n \)

\[
\partial_t u = i A(t, x, D_x) u \quad (D_x = i^{-1}\partial_x),
\]

(2)

where \( A(t, x, \xi) = (a_{hk})_{h,k=1,\ldots,N} \) is a matrix symbol, homogeneous of degree 1, is called pseudosymmetric when the following conditions are fulfilled for all choices of the indices \( h, k, h_1, \ldots, h_v \in \{1, \ldots, N\} \):

\[
\begin{align*}
a_{hk} \cdot a_{kh} &\geq 0 \\
a_{h_1h_2}a_{h_3h_4} \cdots a_{h_vh_1} &= a_{h_1h_v} \cdots a_{h_vh_2}a_{h_2h_1}.
\end{align*}
\]

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These conditions are trivially satisfied by the Hermitian matrices, as well as by the triangular matrices with real entries in the diagonal. The $2 \times 2$ matrix

$$A = \begin{pmatrix} d_1 & a \\ b & d_2 \end{pmatrix}$$

is pseudosymmetric, i.e., satisfies (3)-(4), if and only if

$$d_1, d_2 \in \mathbb{R}, \quad ab \geq 0.$$  

In particular, each equation of type (1) is equivalent to a pseudosymmetric system of type (2), where $A(t, x, \xi)$ is as in (5) with

$$d_1 = d_2 = 0, \quad a = |\xi|, \quad b = \sum a_{ij}(t, x) \xi_i \xi_j |\xi|^{-1}.$$  

For $N = 3$, the matrix

$$A = \begin{pmatrix} d_1 & a & c' \\ a' & d_2 & b \\ c & b' & d_3 \end{pmatrix}$$

is pseudosymmetric if and only if

$$d_1, d_2, d_3 \in \mathbb{R}, \quad aa' \geq 0, \quad bb' \geq 0, \quad cc' \geq 0, \quad abc = \overline{ab} \overline{bc}.$$  

The nature of the pseudosymmetry is made clear by the following result (see [3]):

- A (constant) matrix $A$ is pseudosymmetric if and only if, for all $\epsilon > 0$, it is possible to find a diagonal matrix $\Lambda_\epsilon$ with entries $\geq 0$ for which

$$\|\Lambda_\epsilon A \Lambda_\epsilon^{-1} - A^{-1} A^* \Lambda_\epsilon\| \rightarrow 0 \quad \text{as} \quad \epsilon \rightarrow 0.$$  

In the special case when all the non-diagonal entries of $A$ are different from zero, we can find a diagonal matrix $\Lambda$ which symmetrizes $A$, i.e., such that $\Lambda A \Lambda^{-1}$ is Hermitian.

Therefore, in the terminology used in [2], the pseudosymmetric matrices are simply the matrices which admit a quasi-symmetrizer of diagonal type.

As a consequence of the above characterization, one can easily prove that each pseudosymmetric matrix is hyperbolic, i.e., has real eigenvalues. Moreover, one expects that some of the wellposedness properties of the second order equations, extend to the pseudosymmetric systems. This is a case of systems with coefficients depending only on time; indeed we have (see [3]):
The Cauchy Problem for any $N \times N$ pseudosymmetric system of the form

$$\partial_t u = \sum_{j=1}^{n} A_j(t) \partial_{x_j} u$$

is well posed in $\mathcal{C}^\infty$, provided the matrices $A_1(t), \ldots, A_n(t)$ are analytic in $t$.\(^{(1)}\)

The aim of this paper is to investigate the Cauchy Problem for pseudosymmetric systems with coefficients depending on one space variable, that is, of the form

$$\partial_t u = A(x) \partial_x u + f(t, x), \quad \text{in} \quad \mathbb{R}_t \times \mathbb{R}_x,$$

where $A(x)$ is a $N \times N$ matrix with analytic entries. The situation is rather different from the case of time dependent coefficients: in the scalar case, for any equation of type (1) with (smooth) coefficients $a_{ij} \equiv a_{ij}(x)$, one has the $\mathcal{C}^\infty$ wellposedness, but such a conclusion is no longer valid in the vector case, even for analytic coefficients. For instance, the Cauchy Problem for the system

$$\partial_t u = \begin{pmatrix} 1 + x & x \\ -x & 1 - x \end{pmatrix} \partial_x u$$

is not well posed in $\mathcal{C}^\infty$.

The class of $2 \times 2$ systems with analytic coefficients in $\mathbb{R}_t \times \mathbb{R}_x$, of the form

$$\partial_t u = \begin{pmatrix} d_1(t, x) & a(t, x) \\ b(t, x) & d_2(t, x) \end{pmatrix} \partial_x u,$$

was extensively studied by Nishitani, who proved in particular (see [4]):

- A sufficient condition for the $\mathcal{C}^\infty$ wellposedness of (7) is

$$\left(C(d_1 - d_2)^2 + 4ab\right)(t, x) \geq 0$$

for some constant $C < 1$.

Let $C \neq 0$ be a constant. Therefore, the Cauchy Problem for the system

$$\partial_t u = \begin{pmatrix} C + d(x) & a(x) \\ b(x) & C - d(x) \end{pmatrix} \partial_x u$$

where

$$\left(ab + d^2\right)(x) \equiv 0,$$

is not $\mathcal{C}^\infty$ well posed, unless $a \equiv b \equiv d \equiv 0$.

\(^{(1)}\)For the scalar equations of type (1), with coefficients $a_{ij} \equiv a_{ij}(t)$ depending on time, the result was proved in [1].
We note that the conditions (8) and (9) are stronger than the hyperbolicity condition
\[(d_1 - d_2)^2 + 4ab \geq 0,\]
which expresses that the matrix (5) has real eigenvalues. On the other hand the pseudosymmetry condition \(ab \geq 0\) is stronger than (8), but is inconsistent with (9) unless \(a(x), b(x), c(x)\) are all identically zero.

Passing to the case of systems of type (6) with size \(N \geq 3\), it is natural to ask whether the pseudosymmetry's assumptions (3)-(4), together with the analytic regularity of the coefficients, are sufficient to ensure the \(C^\infty\) wellposedness. We are not able to give a general answer to such a question, however we can prove the wellposedness under some additional assumptions.

Before stating our result, let us remark that there is \(C^\infty\) wellposedness near each point \(x_0 \in \mathbb{R}\) where the non-diagonal entries of the matrix \(A(x)\) are all different from zero; indeed in such a case, thanks to the pseudosymmetry, \(A(x)\) results in being smoothly symmetrizable (see Proposition 1.1 below). Therefore, we can put ourselves near a point \(x_0\) where some of the \(a_{ij}\)'s with \(i \neq j\) is vanishing. We prove the following result (see Theorems 2.1 and 3.1 below):

**Theorem.** Let \(A(x)\) be a pseudosymmetric analytic matrix. Therefore, the Cauchy Problem for (6) is \(C^\infty\) well posed near \(x_0\) provided that

\[a_{ij}(x_0) \cdot a_{ji}(x_0) = 0, \quad \forall i, j = 1, \ldots, N.\]

In the case \(N = 3\), it is sufficient to assume that

\[a_{jj}(x_0) = 0, \quad \forall j = 1, \ldots, N.\]

**Example.** Consider the system

\[
\partial_t u = \begin{pmatrix}
  x^{\delta_1} & x^\alpha & x^{\gamma'} \\
  x^{\alpha'} & x^{\delta_2} & x^\beta \\
  x^{\gamma} & x^{\beta'} & x^{\delta_3}
\end{pmatrix} \partial_x u,
\]

where the exponents are non-negative integers such that

\[\alpha + \alpha', \beta + \beta', \gamma + \gamma' \text{ are even}, \quad \alpha + \beta + \gamma = \alpha' + \beta' + \gamma', \quad \delta_j \geq 1.\]

Therefore, the Cauchy Problem is \(C^\infty\) well posed near \(x = 0\).

**Notations.** All the functions considered in the rest of this paper will have real values. Given an open interval \(I \subseteq \mathbb{R}\), we denote by \(A(I) \equiv A(I; \mathbb{R})\) the class of analytic functions on \(I\). For a function \(\phi(x)\) on \(I\), \(\phi \not\equiv 0\) means that \(\phi(x)\) is not identically zero.
1. Preparatory lemmas

Let $A(x)$ be a pseudosymmetric matrix with entries $a_{ij}(x) \in A(I)$, $i, j = 1, \ldots, N$, where $I \subseteq \mathbb{R}$ is an open interval.

**Proposition 1.1.** It is possible to find $k_{ij}(x), \lambda_j(x) \in A(I)^{(2)}$ in such a way that

\begin{align*}
(1.1) \quad k_{ij}(x) &= k_{ji}(x), \quad k_{ij}^2(x) = a_{ij}(x)a_{ji}(x), \\
(1.2) \quad \lambda_i(x)a_{ij}(x) &= k_{ij}(x)\lambda_j(x),
\end{align*}

for all $i, j = 1, \ldots, N$, and

\[ \lambda_{j_0} \neq 0 \quad \text{for some} \quad j_0 \in \{1, \ldots, N\}. \]

*If the $a_{ij}$'s have at most one isolated zero $x_0 \in I$, more precisely if

\begin{align*}
(1.3) \quad \forall i \neq j & : \quad \text{either} \quad a_{ij} \equiv 0, \quad \text{or} \quad a_{ij}(x) \neq 0 \quad \forall x \in I \setminus \{x_0\},
\end{align*}

then we can find the $\lambda_j$'s, as above, such that each $\lambda_j(x)$ may vanish only at $x = x_0$ (unless $\lambda_j \equiv 0$), and $\lambda_{j_0} \equiv 1$. If, for all $(i, j)$ with $i \neq j$, we have $a_{ij}(x) \neq 0 \forall x \in I$, then we can find the $\lambda_j$'s such that $\lambda_j(x) \neq 0 \forall x \in I$. Hence $A(x)$ is a smoothly symmetrizable matrix in a neighborhood of $x_0$.*

**Remark 1.1** As a consequence of (1.1)-(1.2), we have

\begin{align*}
(1.4) \quad \lambda_i^2(x)a_{ij}(x) &= \lambda_j^2(x)a_{ji}(x).
\end{align*}

Setting

\begin{align*}
(1.5) \quad \Lambda(x) &= \begin{pmatrix} \lambda_1(x) \\ \vdots \\ \lambda_N(x) \end{pmatrix}, \quad K(x) = (k_{ij}(x))_{i,j=1,...,N},
\end{align*}

we can rewrite (1.1), (1.2), and (1.4), in the forms

\begin{align*}
(1.6) \quad K(x) &= K^*(x), \quad \Lambda(x)A(x) = K(x)\Lambda(x), \quad \Lambda^2(x)A(x) = (\Lambda^2(x)A(x))^*.
\end{align*}

In order to prove Proposition 1.1, we shall use the following elementary result (a proof of which will be given in the Appendix).

(2) These $\lambda_j$'s correspond to the square roots of those defined in [3].
LEMMA 1.1 (square root). Let \( I \subseteq \mathbb{R} \) be an open interval, and \( f \in \mathcal{A}(I) \) be such that \( f(x) \geq 0 \) for all \( x \in I \). Then there exists some \( \phi \in \mathcal{A}(I) \) for which

\[
\phi^2(x) = f(x).
\]

Such a \( \phi \) is unique up to the factor \(-1\).

PROOF OF PROPOSITION 1.1 If \( A(x) \) is a diagonal matrix, we simply take \( K = A \) and \( \Lambda = I_N \), the identity matrix. Thus, we'll assume that \( a_{ij} \neq 0 \) for some \( i \neq j \), and we define the analytic function

\[
\theta(x) = \prod_{i \neq j, a_{ij} \neq 0} a_{ij}(x).
\]

We first deal with a special case:

**Case I:** \( a_{ij} \neq 0 \) for all \( i \neq j \).

Let us fix an arbitrary point \( \bar{x} \in I \) where \( \theta(x) \) is not vanishing, i.e., such that

\[
a_{ij}(\bar{x}) \neq 0 \quad \forall i \neq j,
\]

and let us define the functions \( k_{ij} \in \mathcal{A}(I) \) as:

\[
k_{ii}(x) = a_{ii}(x)
\]

\[
k_{ij}^2(x) = a_{ij}(x)a_{ji}(x), \quad k_{ij}(\bar{x})a_{ij}(\bar{x}) > 0.
\]

Note that we have also \( k_{ji}(\bar{x})a_{ij}(\bar{x}) > 0 \), since \( a_{ij}(\bar{x})a_{ji}(\bar{x}) > 0 \) by the pseudosymmetry of \( A(x) \), and that (1.11) defines the analytic function \( k_{ij} \) in a unique way by Lemma 1.1. Hence \( k_{ij} = k_{ji} \).

We now define the functions \( \lambda_j \)'s as

\[
\lambda_j(x) = \theta(x) \frac{k_{jN}(x)}{a_{jN}(x)} \quad \text{for} \quad 1 \leq j < N, \quad \lambda_N(x) = \theta(x),
\]

Clearly, we have \( \lambda_j \in \mathcal{A}(I) \). It remains to prove the equality (1.2). Such equality becomes, squaring each term and using (1.12),

\[
\theta^2 \frac{k_{iN}^2}{a_{iN}^2} a_{ij}^2 = k_{ij}^2 \theta^2 \frac{k_{jN}^2}{a_{jN}^2},
\]

and this turns to be \( \theta^2 a_{iN} a_{Nj} a_{iN}^2 / a_{jN}^2 = a_{ij} a_{ji} \theta^2 a_{jN} a_{Nj} / a_{jN}^2 \) in view of (1.11). Now, by the pseudosymmetry of \( A(x) \) we know that

\[
a_{ij} a_{jN} a_{Ni} = a_{ji} a_{iN} a_{Nj},
\]
hence we have proved
\[(\lambda_i a_{ij})^2 = (k_{ij} \lambda_j)^2.\]

This implies, by analyticity, that
\[\lambda_i a_{ij} = \epsilon k_{ij} \lambda_j \quad \text{for} \quad \epsilon = \pm 1.\]

But
\[
\epsilon = \frac{\lambda_i a_{ij}(\bar{x})}{k_{ij} \lambda_j} = \frac{\theta(\bar{x}) k_{iN}(\bar{x}) a_{iN}(\bar{x})^{-1} a_{ij}(\bar{x})}{k_{ij}(\bar{x}) \theta(\bar{x}) k_{jN}(\bar{x}) a_{jN}(\bar{x})^{-1}} = \frac{k_{iN}(\bar{x})}{a_{iN}(\bar{x})} \cdot \frac{a_{ij}(\bar{x})}{k_{ij}(\bar{x})} \cdot \frac{a_{jN}(\bar{x})}{k_{jN}(\bar{x})} > 0
\]
by (1.11), hence \(\epsilon = 1\), and we find (1.9).

We remark that the functions \(\lambda_1^2, \ldots, \lambda_N^2\) are uniquely defined up to the factor \(\theta(x)\), indeed we have
\[
\frac{\lambda_i^2}{\lambda_j^2} = \frac{a_{ji}}{a_{ij}}.
\]

On the other hand, due to the arbitrariness in the choice of point \(\bar{x}\) in (1.9), the functions \(k_{ij}(x)\) for \(i \neq j\) are determined up to a factor \((-1)^{ij}\). Similarly, each of the \(\lambda_j\)'s is free from the factor \(\pm 1\); for instance, if \(\Lambda(x)\) and \(K(x)\) satisfy (1.6), another choice is given by \(\tilde{\Lambda}(x)\) and \(\tilde{K}(x)\), where
\[
\tilde{\lambda}_1 = -\lambda_1, \quad \tilde{\lambda}_j = \lambda_j \quad \text{for} \quad j \geq 2,
\]
\[
\tilde{k}_{1j} = \tilde{k}_{j1} = -k_{1j} \quad \text{for} \quad j \geq 2, \quad \tilde{k}_{ij} = k_{ij} \quad \text{otherwise}.
\]

\textbf{Case II: for all } i \neq j \text{ we have } i \sim j \text{ in the sense of [3], that is,}

\begin{equation}
\begin{aligned}
\text{either} & \quad a_{ij} a_{ji} \neq 0, \\
\text{or} & \quad a_{ih_1 h_2 \ldots h_v j} a_{j h_v \ldots h_{1i}} \neq 0 \\
& \quad \text{for some chain } \{h_1, \ldots, h_v\} \text{ connecting } i \text{ with } j.
\end{aligned}
\end{equation}

Note that in the last case, i.e., when \(a_{ij} a_{ji} \equiv 0\), we have necessarily \(a_{ij} \equiv a_{ji} \equiv 0\); indeed the pseudosymmetry gives
\[a_{ij} \cdot (a_{jh_v \ldots h_{1i}}) = (a_{ih_1 \ldots h_{1i}}) \cdot a_{ji},\]
and hence we obtain the result, because \((a_{jh_v \ldots h_{1i}}) \neq 0, (a_{ih_1 \ldots h_{1i}}) \neq 0\).

To define the functions \(k_{ij}(x)\), we choose some \(\bar{x} \in I\) where \(\theta(\bar{x}) \neq 0\), that is for which

\begin{equation}
\forall i \neq j : \quad \text{either} \quad a_{ij} \equiv 0, \quad \text{or} \quad a_{ij}(\bar{x}) \neq 0,
\end{equation}
and we define $k_{ij}$ as in (1.10)-(1.11). Then, we choose some index $p_0 \in \{1, \ldots, N\}$, and we define

\[ (1.15) \quad \lambda_j(x) = \theta(x) \frac{k_{j h_1}(x) k_{h_1 h_2}(x) \cdots k_{h_v p_0}(x)}{a_{j h_1}(x) a_{h_1 h_2}(x) \cdots a_{h_v p_0}(x)} \quad \text{for} \quad j \neq p_0, \quad \lambda_{p_0}(x) = \theta(x), \]

$\{h_1, \ldots, h_v\}$ being any chain connecting $j$ and $p_0$ for which

\[ a_{j h_1} a_{h_1 h_2} \cdots a_{h_v p_0} a_{p_0 h_v} \cdots a_{h_1 j} \neq 0. \]

By virtue of the pseudosymmetry, such a definition is independent of the choice of $\{h_1, \ldots, h_v\}$. Indeed, introducing the meromorphic functions

\[ \beta_{ij}(x) = \frac{a_{ij}(x)}{a_{ji}(x)}, \quad \text{for all } (i, j) \text{ for which } a_{ij} \cdot a_{ji} \neq 0, \]

we derive, from (1.15) and (1.16), that

\[ (1.16) \quad \lambda_j^2 = \frac{\theta^2}{\beta_{j h_1} \beta_{h_1 h_2} \cdots \beta_{h_v p_0}}. \]

Now $\beta_{ji} = \beta_{ij}^{-1}$, and more generally $\beta_{h_1 h_2} \beta_{h_2 h_3} \cdots \beta_{h_v h_1} = 1$ for all cycles; thus, if $\{h'_1, \ldots, h'_v\}$ is another chain connecting $j$ with $p_0$ in the sense of (1.13), setting

\[ \tilde{\lambda}_j^2 = \frac{\theta^2}{\beta_{j h'_1} \beta_{h'_1 h'_2} \cdots \beta_{h'_v p_0}}, \]

we have

\[ \frac{\lambda_j^2}{\tilde{\lambda}_j^2} = (\beta_{j h'_1} \beta_{h'_1 h'_2} \cdots \beta_{h'_v p_0}) \cdot (\beta_{p_0 h_v} \cdots \beta_{h_1 j}) = 1. \]

But, for $x = \bar{x}$, we have

\[ \lambda_j(\bar{x}) = \theta(\bar{x}) \cdot \left( \begin{array}{c} k_{h_1 j}(\bar{x}) \\ a_{j h_1}(\bar{x}) \end{array} \right) \cdots \left( \begin{array}{c} k_{p_0 h_v}(\bar{x}) \\ a_{h_v p_0}(\bar{x}) \end{array} \right) = C \theta(\bar{x}) \]

with $C > 0$ by (1.11), and similarly we have $\tilde{\lambda}_j(\bar{x}) = \tilde{C} \theta(\bar{x})$ with $\tilde{C} > 0$. Thus we conclude that $\lambda_j(x)/\tilde{\lambda}_j(x) = 1$. In a similar way we see that the definition (1.15) of $\lambda_j$ is independent of the choice of the index $p_0 \in \{1, \ldots, N\}$.

It remains to prove (1.2). We first prove (1.4). Let $\{h_1, \ldots, h_v\}$ be a chain connecting $j$ with $p_0$ in the sense of (1.13), and $\{h'_1, \ldots, h'_v\}$ be a chain
connecting $i$ with $p_0$: by (1.16) we have, in the sense of the meromorphic functions (note that all the functions here involved are not identically zero),

$$\frac{\lambda_i^2}{\lambda_j^2} = \frac{\beta_{j_1 h_1} \beta_{h_1 h_2} \cdots \beta_{h_v p_0}}{\beta_{i h_1'} \beta_{h_1' h_2'} \cdots \beta_{h_v' p_0}} = (\beta_{j_1 h_1 h_2} \cdots \beta_{h_v p_0}) \cdot (\beta_{p_0 h_1'} \cdots \beta_{h_v' i})$$

$$= \frac{1}{\beta_{ij}} = \frac{a_{ij}}{a_{ij}}.$$

To derive (1.2) we have only to observe that

$$\left(\frac{k_{ij}}{a_{ij}}\right)^2 = \frac{a_{ij}a_{ji}}{a_{ij}^2} = \frac{a_{ji}}{a_{ij}},$$

and $\lambda_i(\bar{x})/\lambda_j(\bar{x}) > 0$ by the pseudosymmetry, while $k_{ji}(\bar{x})/a_{ij}(\bar{x}) > 0$ by the definition (1.11). This completes the proof of Proposition 1.1 in the Case II. Note that in this case, no one of the $\lambda_j$'s results in being identically zero.

**Case III: the general case.**

As in Case II, having fixed a point $\bar{x} \in I$ where (1.14) holds, we define the functions $k_{ij}(x)$ by (1.10)-(1.11). Next we introduce on the set $\{1, \ldots, N\}$ an equivalence relation:

$$i \sim j \iff \text{either } i = j \text{ or (1.13) is fulfilled}.$$

Case II is the case in which all indices are equivalent. If $\alpha$ and $\beta$ are two classes of equivalence, we say (cf. [3]) that

$$\alpha > \beta$$

if, for some $p \in \alpha, \ q \in \beta$, we have

$$(1.17) \quad a_{pq} \neq 0, \quad a_{qp} \equiv 0.$$

We note that, in such a case, we have also

$$a_{q'}p \equiv 0 \text{ for all } q' \in \beta;$$

indeed if $\{h_1, \ldots, h_v\}$ connects $q$ with $q'$, then we can write

$$a_{pq} \cdot (a_{qh_1} \cdots a_{h_vq'}) \cdot a_{q'p} \equiv a_{pq} \cdot (a_{q'h_v} \cdots a_{h_1q'}) \cdot a_{qp} \equiv 0$$

and this proves $a_{q'p} \equiv 0$ because $(a_{qh_1} \cdots a_{h_vq'}) \neq 0$. We also observe that (1.17) does not define an (even partial) order relation on the quotient set $\{1, \ldots, N\}/ \sim$, since the transitive property fails. However ">" is endowed
with an important property which follows easily from the pseudosymmetry: for any cycle \( \{a_1, \ldots, a_v, a_1\} \) of classes one cannot have

\[ a_1 > a_2 > \ldots > a_v > a_1. \]

As a consequence, there exists always a "minimal" class, that is, a class \( \alpha \) for which there is no \( \beta \) with \( \alpha > \beta \).

Now, let us define the functions \( \lambda_j \)'s. Inside each equivalence class \( \alpha \) we can proceed as in the case (II), but we have to distinguish the two types of \( \alpha \):

(i) there exists \( \beta \) such that \( \alpha > \beta \)
(ii) there is no \( \beta \) such that \( \alpha > \beta \).

In the first case, for every \( p \in \alpha \) there is some \( q \in \beta \) for which (1.17) holds; thus, in order to get (1.1), and hence also \( \lambda_p^2 a_{pq} = \lambda_q^2 a_{qp} \), we must define \( \lambda_j = 0 \) for every \( j \in \alpha \). In the second case, we choose an index \( p_a \in \alpha \), and we define the \( \lambda_j \)'s for \( j \in \alpha \) just as in Case II, that is by (1.15) with \( p_a \) in place of \( p_0 \). The relations

\[ \lambda_j^2 a_{ij} = \lambda_j^2 a_{ji} \]

are always fulfilled. This is clear if \( i \sim j \), by the same arguments used in Case II. If \( i \not\sim j \) we have two possibilities: either \( a_{ij} = a_{ji} = 0 \), in which case (1.18) is trivial, or \( a_{ij} \neq 0 \) and \( a_{ji} \equiv 0 \), which means \([i] > [j]\). Since we have defined \( \lambda_i \equiv 0 \), then (1.18) is again true. The fact that there is always some class \( \alpha \) of type (ii), ensures that we can find a matrix \( \Lambda(x) = \text{diag}[\lambda_1(x), \ldots, \lambda_N(x)] \) which is not identically zero.

Let us now prove the last part of Proposition 1.1. Assume that (1.3) holds at some point \( x_0 \in I \), that is, \( \theta(x) = 0 \) only for \( x = x_0 \). Therefore, going back to the definitions of the functions \( k_{ij} \)'s and \( \lambda_j \)'s, we see that each of these functions may vanish only at \( x_0 \) (unless it is identically zero). Hence, we can write

\[ \lambda_j(x) = (x - x_0)^{v_j} \mu_j(x), \quad j = 1, \ldots, N, \]

with \( v_j \) integers \( \geq 0 \), where either \( \mu_j(x) \equiv 0 \), or \( \mu_j(x) \neq 0 \) for all \( x \in I \). Taking \( v_{j_0} = \min \{v_j \mid \mu_j \neq 0\} \), we can define

\[ \tilde{\lambda}_j(x) = \frac{\lambda_j(x)}{\lambda_{j_0}(x)} = (x - x_0)^{v_j - v_{j_0}} \frac{\mu_j(x)}{\mu_{j_0}(x)} \in \mathcal{A}(I) \]

and \( \tilde{\lambda}_{j_0}(x) \equiv 1 \) on \( I \). Finally, if the non-diagonal entries of \( A(x) \) do not vanish at any point of \( I \), we can resort to the arithmetic square root and take

\[ k_{ij} = \sqrt{a_{ij} a_{ji}}, \quad \lambda_j = \sqrt{a_{nj} a_{jn}} \quad \text{for} \quad 1 \leq j \leq N - 1, \quad \lambda_N \equiv 1. \]

This concludes the proof of Proposition 1.1.
2. - Cauchy Problem

Given an open interval $I \subseteq \mathbb{R}$, and a pseudosymmetric matrix $A(x) = (a_{ij}(x))_{i,j=1,\ldots,N}$ with analytic entries in $I$, let us consider the Cauchy Problem

\begin{equation}
\begin{cases}
\partial_t u = A(x) \partial_x u + f(t,x) \\
 u(0,x) = u_0(x).
\end{cases}
\end{equation}

**Theorem 2.1.** Assume that

\begin{equation}
\begin{array}{c}
a_{ij}(x_0) a_{ji}(x_0) = 0, \quad \forall i, j = 1, \ldots, N;
\end{array}
\end{equation}

therefore (2.1) is $C^\infty$ well posed in a neighborhood of $x_0$.

If $a_{ij}(x_0) \neq 0$ for all $i \neq j$, the same conclusion holds without the assumption (2.2).

**Remark 2.2.** Since $a_{ij}a_{ji} \geq 0$, the condition (2.2) is equivalent to require that the non-negative function

\[ \tau(x) = \sum_{i,j}^{1,N} a_{ij}(x) a_{ji}(x) \]

vanishes at the point $x_0$. Note that $\tau(x)$ is the trace of the matrix $A^2(x)$.

**Proof of Theorem 1.1.** To say that (2.1) is $C^\infty$ well posed near $x_0$, means that there are two neighborhoods $W, W'$ of $(x_0, 0)$ such that, for each $u_0 \in C^\infty(W \cap \{t = 0\})$ and $f \in C^\infty(W)$, there is a unique solution $u \in C^\infty(W')$. We shall prove a more precise result:

Let us restrict ourselves to an interval $I_0 = [x_0 - r_0, x_0 + r_0] \subset I$ where the function $\theta(x)$ defined in (1.8) has $x_0$ as its unique zero, that is, where (1.3) holds, and let us define the cone

\begin{equation}
\Gamma(I_0, \kappa) = \left\{ (x, t) : |x - x_0| \leq r_0 - \kappa t \right\},
\end{equation}

with

\begin{equation}
\kappa = N \max_{i,j} \sup_{I_0} |a_{ij}(x)|.
\end{equation}

Then, for each $u_0 \in C^\infty(I_0)$ and $f \in C^\infty(\Gamma(I_0, \kappa))$, (2.1) has a solution $u \in C^\infty(\Gamma(I_0, \kappa))$.

In order to prove such a result, we shall derive an *apriori* estimate for any smooth solution of

\begin{equation}
\partial_t u = A(x) \partial_x u + f(t,x) \quad \text{on} \quad \Gamma(I_0, \kappa).
\end{equation}
By Proposition 1.1 we choose two analytic matrices, \( K(x) = (k_{ij}(x)) \), \( \Lambda(x) = (\lambda_i(x) \delta_{ij}) \), in such a way that
\[
K = K^*, \quad \Lambda A = \Lambda \Lambda, \quad \Lambda^2 A = (\Lambda^2 A)^* ,
\]
and
\[
\lambda_1(x) \equiv 1 \text{ on } I_0.
\]

Effecting the transformation
\[
v = \Lambda(x)u ,
\]
we obtain
\[
\partial_t v = \Lambda \partial_t u = \Lambda(\Lambda \partial_x u + f) = K \Lambda \partial_x u + \Lambda f, \quad \partial_x v = \Lambda \partial_x u + \Lambda' u,
\]
hence (2.5) becomes
\[
\partial_t v = K(x) \partial_x v - K(x) \Lambda'(x) u + \Lambda(x) f(t, x).
\]
Now we have defined \( k^2_{ij} = a_{ij}a_{ji} \), thus the assumption (2.2) means that
\[
(2.6) \quad K(x_0) = 0. \quad
\]
Hence, recalling (1.19) and the analyticity of \( K(x) \), one can write
\[
K(x) \Lambda'(x) = T(x) \Lambda(x)
\]
for some matrix \( T(x) \in \mathcal{A}(I_0, \mathbb{R}^{N \times N}) \); indeed (2.6) ensures that \( k_{ij}(x) \lambda_j'(x)/\lambda_j(x) \) is analytic unless \( \lambda_j \equiv 0 \). Thus we find
\[
(2.7) \quad \partial_t v = K(x) \partial_x v - T(x) v + \Lambda(x) f(t, x) \quad (v = \Lambda u).
\]

Let us now define, for \( 0 \leq t < r_0/\kappa \), the energy function
\[
E_0(t) = \int_{-r(t)}^{r(t)} |v(t)|^2 dx \equiv \int_{-r(t)}^{r(t)} |\Lambda u(t)|^2 dx, \quad \text{where} \quad r(t) = r_0 - \kappa t .
\]
We get an apriori estimate for such a function: to this end we study \( E'_0(t) \), which becomes, by (2.7),
\[
E'_0(t) = -\int_{-r(t)}^{r(t)} (K' v, v) dx + 2 \int_{-r(t)}^{r(t)} \left( -(T v, v) + (\Lambda f, v) \right) dx \\
+ \left[ (K v, v) + r'(t)|v|^2 \right]_{-r(t)}^{r(t)}.
\]
Indeed, $K$ is symmetric and
\[
\partial_x(Kv, v) = (K'v, v) + 2(K\partial_x v, v).
\]

But $r'(t) = -\kappa$, and by (2.4) and (1.1) we have $|(K(x)v, v)| \leq \kappa |v|^2$, thus we get
\[
E_0'(t) \leq \sup_{|x| \leq r_0} \left( \|K'(x)\| + 2\|T(x)\| \right) E_0(t) + 2 \left( \int_{-r(t)}^{r(t)} |\Lambda f(t)|^2 dx \right)^{1/2} \sqrt{E_0(t)},
\]
and, integrating in $t$,
\[
(2.8) \quad \sqrt{E_0(t)} \leq C \left( \sqrt{E_0(0)} + \int_0^t \|f(s)\|_{L^2(-r(s), r(s))} ds \right) \quad \text{for } 0 \leq t < r_0/\kappa.
\]

To get a better estimate, we differentiate (2.7) to obtain the equation
\[
(\partial_x v)_t = K\partial_x^2 v + (K' - T)\partial_x v - T'v + \partial_x(\Lambda f),
\]
or, setting $w = \partial_x v$,
\[
(2.9) \quad \partial_t w = K\partial_x w + T_1 w + Sv + \partial_x(\Lambda f), \quad \text{with } T_1 = K' - T, \quad S = -T'.
\]

Let us define
\[
I_t = [-r_0 + \kappa t, \ r_0 + \kappa t]
\]
and
\[
E_1(t) = \int_{I_t} |w(t)|^2 dx = \int_{I_t} |\partial_x v(t)|^2 dx.
\]

Proceeding as above, we derive from (2.9):
\[
E_1' \leq \sup_{x \in I_0} (\|K'(x)\| + 2\|T_1(x)\|) E_1 + \sup_{x \in I_0} \|S(x)\| \sqrt{E_0 \sqrt{E_1} + 2\|\Lambda f(t)\|_{H^1(I_t)}} \sqrt{E_1}
\]
and, by (2.8), we get the estimate
\[
\sqrt{E_1(t)} \leq C \left( \sqrt{E_1(0)} + \sqrt{E_0(0)} + \int_0^t \|f(s)\|_{H^1(I_s)} ds \right).
\]

We can go on, by putting
\[
z = \partial_x w.
\]

This verifies the equation
\[
(2.10) \quad \partial_t z = K\partial_x z + T_2 z + S_1 w + S_0 v + \partial_x^2(\Lambda f)
\]
with $T_2 = K' + T_1$, $S_1 = T'_1 + S$, $S_0 = S'$, all analytic functions of $x$. Setting

$$E_2(t) = \int_{I_t} |z(t)|^2 dx = \int_{I_t} |\partial_x w(t)|^2 dx = \int_{I_t} |\partial_x^2 v(t)|^2 dx,$$

we get from (2.10)

$$E_2' \leq C_1 E_2 + C_2 \left( \sqrt{E_1} + \sqrt{E_0} + \|f(t)\|_{H^2(I_t)} \right) \sqrt{E_2}$$

whence

$$\sqrt{E_2(t)} \leq C \left( \sqrt{E_2(0)} + \sqrt{E_1(0)} + \sqrt{E_0(0)} + \int_0^t \|f(s)\|_{H^2(I_s)} ds \right).$$

In conclusion, defining

$$E_k(t) = \|\partial_x^k v(t)\|_{L^2(I_t)}^2, \quad k = 0, 1, 2, \ldots,$$

we prove that

$$\sqrt{E_k(t)} \leq C_k \left( \sum_{h=0}^k \sqrt{E_h(0)} + \int_0^t \|f(s)\|_{H^k(I_s)} ds \right), \quad 0 \leq t < r_0/\kappa.$$  

We note that the constants $C_k$ depend on the matrix $A(x)$ and $r_0$.

Now recall $\lambda_1(x) \equiv 1$, so that

$$E_0(t) = \int_{I_t} |\Lambda(x) u(t)|^2 dx = \sum_{j=1}^N \int_{I_t} \lambda_j(x)^2 u_j(t)^2 dx \geq \int_{I_t} u_1(t)^2 dx$$

where $u = (u_1, \ldots, u_N)$, and also

$$E_k(t) = \int_{I_t} |\partial_x^k (\Lambda u(t))|^2 dx = \sum_{j=1}^N \int_{I_t} |\partial_x^k (\lambda_j u_j(t))|^2 dx \geq \int_{I_t} |\partial_x^k u_1(t)|^2 dx.$$ 

Thus (2.12) gives an estimate for the first component $u_1(t, x)$, namely

$$\|u_1(t)\|_{H^k(I_t)} \leq C_k \left( \|u(0)\|_{H^k(I_0)} + \int_0^t \|f(s)\|_{H^k(I_s)} ds \right), \quad 0 \leq t < r_0/\kappa.$$ 

Next we consider the other components $u_j$, and we define

$$\tilde{u} = \begin{pmatrix} u_2 \\ \vdots \\ u_N \end{pmatrix} \in \mathbb{R}^{N-1}.$$
We obtain the \((N - 1) \times (N - 1)\) system

\[
\partial_t \tilde{u} = \tilde{A}(x) \partial_x \tilde{u} + \tilde{f}(t, x),
\]

where \(\tilde{A}(x) = (a_{ij}(x))_{i, j = 2, \ldots, N},\)

\[
A = \begin{pmatrix}
a_{11} & a_{12} & \cdots & a_{1N} \\
a_{21} & & & \\
& & \ddots & \\
a_{N1} & & & a_N
\end{pmatrix}, \quad \tilde{f} = \begin{pmatrix}
a_{21} \partial_x u_1 + f_2 \\
a_{31} \partial_x u_1 + f_3 \\
\vdots \\
a_{N1} \partial_x u_1 + f_N
\end{pmatrix}.
\]

But \(\tilde{A}(x)\) is a pseudosymmetric matrix fulfilling the same assumptions as \(A(x)\), hence we can find \(\tilde{K}(x), \tilde{A}(x)\), with \(\tilde{A}_1(x) = 1\), satisfying (1.1), (1.2) and (2.6). By the first part of this proof we have

\[
\|u_2(t)\|_{H^k(I_t)} = \|\tilde{u}_1(t)\|_{H^k(I_t)}
\]

\[
\leq C_k \left( \|\tilde{u}(0)\|_{H^k(I_0)} + \int_0^t \|\tilde{f}(s)\|_{H^k(I_s)} \, ds \right),
\]

where the constants \(C_k\)'s may depend only on \(A(x), r_0\). On the other hand, recalling the definition of \(\tilde{f}(t, x)\), we see that, for \(j = 1, \ldots, N - 1,\)

\[
\|\tilde{f}_j(t)\|_{H^k(I_t)} = \|a_{j+1,1} \partial_x u_1(t) + f_{j+1}(t)\|_{H^k(I_t)}
\]

\[
\leq C \left( \|u_1(t)\|_{H^{k+1}(I_t)} + \|f(t)\|_{H^k(I_t)} \right).
\]

Hence it follows, by (2.13),

\[
(2.15) \quad \|\tilde{f}(t)\|_{H^k(I_t)} \leq C_k \left( \|u(0)\|_{H^{k+1}(I_0)} + \int_0^t \|f(s)\|_{H^{k+1}(I_s)} \, ds + \|f(t)\|_{H^k(I_t)} \right),
\]

and putting together (2.13), (2.14), and (2.15), we obtain

\[
\|u_1(t)\|_{H^k(I_t)} + \|u_2(t)\|_{H^k(I_t)} \leq C_k \left( \|u(0)\|_{H^{k+1}(I_0)} + \int_0^t \|f(s)\|_{H^{k+1}(I_s)} \, ds \right).
\]

Finally, going on with the remaining components, we get the a priori estimate

\[
\|u(t)\|_{H^k(I_t)} \leq C_k \left( \|u(0)\|_{H^{k+N-1}(I_0)} + \int_0^t \|f(s)\|_{H^{k+N-1}(I_s)} \, ds \right), \quad 0 \leq t < r_0/\kappa.
\]

If we differentiate in time each term of our equation (2.5), we obtain similar estimates for \(\partial_t^j u\). These estimates lead to the existence of a \(C^\infty\) solution on the cone (2.3), via a standard approximation method, e.g., by applying the Cauchy-Kowalevsky theorem.
The last part of Theorem 2.1 is a direct consequence of the last part of Proposition 1.1: if all the non-diagonal entries of \( A(x) \) are different from zero in a neighborhood \( I_0 \) of \( x_0 \), we can find an analytic matrix \( \Lambda(x) \), invertible for all \( x \in I_0 \), for which \( \Lambda(x)A(x)\Lambda^{-1}(x) \) is symmetric. Hence (2.5) results to be a smoothly symmetrizable system.

**Remark 2.3.** We have proved the local wellposedness for (2.1). In order to get the wellposedness on the whole space \( \mathbb{R}^2 \), we have to assume that (2.2) holds at each point \( x_0 \) where \( \theta(x_0) = 0 \), and moreover that the coefficients \( a_{ij}(x) \) keep bounded when \( |x| \to \infty \). Therefore, the conclusion follows from Theorem 1 by partition of the unity.

### 3. - 3 \( \times \) 3 systems

For low order systems, Theorem 2.1 can be improved. As recalled in the Introduction, we know that, for every 2 \( \times \) 2 pseudosymmetric system with analytic coefficients, there is the wellposedness even without the assumption (2.2). This is not surprising, indeed for any 2 \( \times \) 2 pseudosymmetric matrix the hyperbolicity condition \( (a_{11} - a_{22})^2 + 4a_{12}a_{21} \geq 0 \) becomes strict whenever (2.2) is violated. One can ask if the same conclusion holds true for non-analytic coefficients (depending only on \( x \)): some results in this direction have been proved, and will appear in a forthcoming paper.

In the case \( N = 3 \), we are not able to drop the assumption (2.2) completely, but we can considerably weaken it:

**Theorem 3.1.** Let \( A(x) \) be a 3 \( \times \) 3 analytic, pseudosymmetric matrix with

\[
(3.1) \quad a_{jj}(x_0) = 0, \quad j = 1, 2, 3.
\]

Then the Cauchy Problem (2.1) is well posed in \( C^\infty \) near \( x_0 \).

**Proof.** We look

\[
A(x_0) = \begin{pmatrix} 0 & a_{12}^0 & a_{13}^0 \\ a_{21}^0 & 0 & a_{23}^0 \\ a_{31}^0 & a_{32}^0 & 0 \end{pmatrix}
\]

where \( a_{ij}^0 = a_{ij}(x_0) \).

We study the characteristic polynomial

\[
(3.2) \quad P(x_0, z) = \det(A(x_0) - zI) = -z^3 + p_0z + q_0
\]

with

\[
\begin{align*}
p_0 &= a_{12}^0a_{21}^0 + a_{13}^0a_{31}^0 + a_{23}^0a_{32}^0 \\
q_0 &= a_{12}^0a_{23}^0a_{31}^0 + a_{13}^0a_{32}^0a_{21}^0 = 2a_{12}^0a_{23}^0a_{31}^0.
\end{align*}
\]
Recalling the definition of \((k_{ij})\) (see (1.1)) we have, from the pseudosymmetry,

\[
(a_{12}a_{23}a_{31})^2 = (a_{12}a_{23}a_{31}) \cdot (a_{21}a_{32}a_{31}) = k_{12}^2k_{23}^2k_{31}^2
\]

so that \(k_{12}(x)k_{23}(x)k_{31}(x) = \epsilon a_{12}(x)a_{23}(x)a_{31}(x)\) with \(\epsilon = \pm 1\). But \(k_{ij}(\bar{x})a_{ij}(\bar{x}) > 0\) at a given point \(\bar{x}\), hence \(\epsilon = 1\), that is

\[
a_{12}(x)a_{23}(x)a_{31}(x) = k_{12}(x)k_{23}(x)k_{31}(x).
\]

On the other hand we have \(a_{12}a_{21} + a_{13}a_{31} + a_{23}a_{32} = k_{12}^2 + k_{23}^2 + k_{31}^2\), hence the coefficients of the polynomial (3.2) can be expressed as

\[
\begin{align*}
p_0 &= k_{12}^2(x_0) + k_{23}^2(x_0) + k_{31}^2(x_0) \\
n_0 &= 2k_{12}(x_0)k_{23}(x_0)k_{31}(x_0).
\end{align*}
\]

We distinguish three cases:

- \(p_0 = 0\),
- \(p_0 > 0\), \(q_0 = 0\),
- \(q_0 \neq 0\).

In the first one, we have \(k_{ij}(x_0) = 0\) for all \(i \neq j\), hence also for all \((i, j)\) by our assumption (3.1). Thus, we can apply Theorem 2.1.

In the second case, we have \(k_{ij}(x_0) = 0\) for some \((i, j)\) with \(i \neq j\), and \(k_{i'j'}(x_0) \neq 0\) for some other \((i', j')\) with \(i' \neq j'\). Therefore we have

\[
P(x_0, z) = -z^3 + p_0z = -z(z^2 - p_0),
\]

so that the all eigenvalues of \(A(x_0)\), i.e., \(\{0, -\sqrt{p_0}, \sqrt{p_0}\}\), are simple since \(p_0 > 0\). That is, our system is strictly hyperbolic for \(x = x_0\), and hence in a neighborhood of \(x_0\).

In the third case, we have \(k_{ij}(x_0) \neq 0\), that is \(a_{ij}(x_0) \neq 0\) for all \((i, j)\) with \(i \neq j\). As observed at the end of Proposition 1.1, this means that the system is smoothly symmetrizable in a neighborhood of \(x_0\), and hence the result.

4. - Appendix

A proof of Lemma 1.1. We first show the uniqueness. If \(\phi, \tilde{\phi}\) satisfy (1.7), we have \((\phi - \tilde{\phi})(\phi + \tilde{\phi}) = \phi^2 - \tilde{\phi}^2 = 0\), so that by analyticity we conclude that \(\phi = \tilde{\phi}\) or \(\phi = -\tilde{\phi}\). Next we show the existence. If \(f \equiv 0\) we take \(\phi \equiv 0\); hence we may assume that \(f(x)\) has at most a countable set of isolated zeros, each of finite and even order (since \(f \geq 0\)).
We consider only the zeros of orders $4\nu + 2, \nu \in \mathbb{N}$. In the case when $f(x)$ has no zero of this type, but has only zeros of order $4\nu$, we simply take

$$\phi(x) = \sqrt[4\nu]{f(x)} \geq 0,$$

i.e., the arithmetic square root of $f(x)$. Indeed, this is an analytic function at each point $\bar{x} \in I$: this is obvious if $f(\bar{x}) \neq 0$, otherwise we write

$$f(x) = g(x)(x - \bar{x})^{4\nu_0}$$

with $g(x) > 0$ in a neighborhood $I$ of $\bar{x}$, hence $\sqrt{g} \in \mathcal{A}(I)$ and also

$$\phi(x) = \sqrt{g(x)}(x - x_0)^{2\nu_0} \in \mathcal{A}(I).$$

In the general case, let us rename the zeros $\mathcal{N} = \{x_h\}$ of $f$, where $x_h$ is a zero of order $4\nu_h + 2$, so that $x_h < x_{h+1}$. Writing $I = ]\alpha, \beta[\, \text{with} \, -\infty \leq \alpha < \beta \leq +\infty$, we have five cases:

1. there is no zero of this type ($\mathcal{N} = \emptyset$)
2. $\mathcal{N} = \{x_1, \ldots, x_k\}$ (finite)
3. $\mathcal{N} = \{x_1, x_2, \ldots\}$ (inf $\mathcal{N} > \alpha$, sup $\mathcal{N} = \beta$)
4. $\mathcal{N} = \{\ldots, x_{-2}, x_{-1}\}$ (inf $\mathcal{N} = \alpha$, sup $\mathcal{N} < \beta$)
5. $\mathcal{N} = \{\ldots, x_{-2}, x_{-1}, x_0, x_1, x_2, \ldots\}$ (inf $\mathcal{N} = \alpha$, sup $\mathcal{N} = \beta$).

In each case the intervals $I_h = [x_h, x_{h+1}]$ (with $I_0 = I$ in the first case, $I_0 = ]\alpha, x_1[$ and $I_k = [x_k, \beta[$ in the second case, etc.) form a partition of $I$ with the property that $f(x)$ has only zeros of orders $4\nu$ in the interior of $I_h$. Then, denoting by $\sqrt[4\nu]{f}$ the positive square root, we define

$$(4.1) \quad \phi(x) = (-1)^h \sqrt[4\nu]{f(x)} \quad \text{on} \quad I_h.$$  

Clearly, such a function is well defined on the whole interval $I$ and is analytic in the interior of each $I_h$. In order to prove that $\phi$ is analytic at $x_h$, let us write

$$f(x) = g_h(x)(x - x_h)^{4\nu_h + 2},$$

with $g_h(x)$ which is analytic and $> 0$ in some neighborhood $J_h$ of $x_h$. Hence, by (4.1) we have:

$$\phi(x) = \begin{cases} 
(-1)^h (x - x_h)^{2\nu_h + 1} \sqrt[4\nu]{g_h(x)} & \text{on} \quad J_h \cap I_h \\
(-1)^{h-1}(x_h - x)^{2\nu_h + 1} \sqrt[4\nu]{g_h(x)} & \text{on} \quad J_h \cap I_{h-1}
\end{cases}$$

that is, $\phi(x) = (-1)^h (x - x_h)^{2\nu_h + 1} \sqrt[4\nu]{g_h(x)} \in \mathcal{A}(J_h)$. 

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