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## One Dimensional Symmetry in the Heisenberg Group

ISABEAU BIRINDELLI – JYOTSHANA PRAJAPAT

**Abstract.** Let  $H^n$  denote the Heisenberg space and let  $u$  be a solution of  $\Delta_H u + u(1 - u^2) = 0$  in  $H^n$  satisfying  $|u| \leq 1$ . Let  $x_1$  be any variable orthogonal to the anisotropic direction  $t$ . Assume that for  $x_1$  going to plus or minus infinity  $u$  converges uniformly to 1 and  $-1$  respectively. Under these assumptions we prove that  $u$  is a function depending only on  $x_1$  and that it is monotone increasing.

This result, which is the analogue for the Heisenberg space of the weak formulation of a conjecture by De Giorgi, is obtained for a wider class of equations; it is a consequence of the invariance of the Heisenberg Laplacian with respect to Heisenberg group. The proof requires a Maximum Principle for unbounded domains which is interesting by itself. We also consider the case when  $u$  satisfies the limit condition in the  $t$  direction and we conclude that the solution is monotone in  $t$ .

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### 1. – Introduction

Let  $u$  be a classical solution of

$$(1) \quad \begin{cases} \Delta u + f(u) = 0 & \text{in } \mathbb{R}^N, \\ |u| \leq 1. \end{cases}$$

Here  $f$  is a Lipschitz continuous function, non-increasing in  $[-1, -1 + \delta]$  and in  $[1 - \delta, 1]$  for some  $\delta > 0$ , with  $f(1) = f(-1) = 0$  and we suppose that

$$(2) \quad \lim_{x_1 \rightarrow \pm\infty} u(x_1, x') = \pm 1$$

where  $x = (x_1, x') \in \mathbb{R}^N$ .

Under the additional assumption that (2) is uniform in  $x'$ , [2], [4] and [11] have proved that  $\frac{\partial u}{\partial x_1} > 0$  and there exists  $U$  such that  $u(x_1, x') = U(x_1)$ .

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Let us recall that this result is related to a conjecture of De Giorgi ([12]) where the question was raised of whether  $u$  is constant along hyperplanes without the request that the limit (2) is uniform. The conjecture has been lately solved by Ghoussoub and Gui in dimension  $N = 2$  [13] and by Ambrosio and Cabré in dimension  $N = 3$  [1].

In this paper we consider the case when the Laplacian is replaced by the Heisenberg Laplacian, precisely

$$(3) \quad \begin{cases} \Delta_H u + f(u) = 0 \text{ in } \mathbb{R}^{2n+1}, \\ |u| \leq 1. \end{cases}$$

Here  $\mathbb{R}^{2n+1}$  is endowed with the Heisenberg group action  $\circ$  and we consider the case when the limit (2) is uniform. Let us recall that  $\Delta_H$  is a degenerate elliptic operator satisfying Hormander condition and the Heisenberg space  $H^n = (\mathbb{R}^{2n+1}, \circ)$  is an anisotropic space, in particular denoting the elements of  $H^n$  by  $\xi = (x, y, t)$  with  $x \in \mathbb{R}^n$ ,  $y \in \mathbb{R}^n$  and  $t \in \mathbb{R}$ , it is easy to see that  $\Delta_H$  is homogeneous with respect to the dilation  $\delta_\lambda(x, y, t) = (\lambda x, \lambda y, \lambda^2 t)$ .

Our main result states that under the condition that

$$\lim_{s \rightarrow \pm\infty} u(x, y, t) = \pm 1 \text{ where } s = v \cdot (x, y) \text{ for some unitary vector } v \in \mathbb{R}^{2n}$$

uniformly, then there exists a function  $U : \mathbb{R} \rightarrow \mathbb{R}$  such that  $u(x, y, t) = U(s)$  and  $\frac{\partial U}{\partial s} > 0$ .

If, on the other hand,

$$(4) \quad \lim_{t \rightarrow \pm\infty} u(x, y, t) = \pm 1$$

uniformly, then we deduce only that  $\frac{\partial u}{\partial t} > 0$ .

This work has been inspired by [4] of Berestycki, Hamel and Monneau. Their proof is based on two ingredients viz., the maximum principle in unbounded domain contained in cones (see [3]) and the so called “sliding method”.

The “sliding method” adapts well to the Heisenberg space since  $\Delta_H$  is left invariant with respect to the group action  $\circ$  (see [10]). On the other hand the maximum principle in domains contained in cones is based on the construction of a comparison function, the existence of which is not known in this setting for general cones. Here we prove it for a large family of cones using some ad hoc argument.

A last remark concerns the case when condition (4) holds. It is not surprising that the situation is different in the  $t$  direction. Indeed observe that if the following implication holds true:

$$(5) \quad (4) \Rightarrow u(x, y, t) = U(t)$$

then we would deduce that there are no solutions of (3) satisfying (4), since  $U(t)$  cannot be a solution of (3). Still the question of whether (5) is true remains open.

In the next section after a basic introduction to the Heisenberg space is given, we treat the maximum principle in unbounded domains contained in cones and in Section 3 we state and prove the symmetry and monotonicity results.

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**2. – Maximum principle**

Let us recall some known facts about the Heisenberg space  $H^n$ .

We will denote by  $\xi = (x, y, t) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}$  the elements of  $H^n = (\mathbb{R}^{2n+1}, \circ)$  where the group action  $\circ$  is given by

$$(6) \quad \xi_0 \circ \xi = (x + x_0, y + y_0, t + t_0 + 2 \sum_{i=1}^n (x_i y_{0i} - y_i x_{0i})).$$

The parabolic dilation  $\delta_\lambda \xi = (\lambda x, \lambda y, \lambda^2 t)$  satisfies

$$\delta_\lambda (\xi_0 \circ \xi) = \delta_\lambda \xi \circ \delta_\lambda \xi_0,$$

and

$$|\xi|_H = \left( (x^2 + y^2)^2 + t^2 \right)^{\frac{1}{4}}$$

is a norm with respect to the parabolic dilation.

The Koranyi ball of center  $\xi_0$  and radius  $R$  is defined by

$$B_H(\xi_0, R) := \{ \xi \text{ such that } |\xi^{-1} \circ \xi_0| \leq R \}$$

and it satisfies

$$|B_H(\xi_0, R)| = |B_H(0, R)| = C R^Q$$

where  $Q = 2n + 2$  is the so called homogeneous dimension.

The Lie Algebra of left invariant vector fields is generated by

$$\begin{aligned} X_i &= \frac{\partial}{\partial x_i} + 2y_i \frac{\partial}{\partial t}, \quad \text{for } i = 1, \dots, n, \\ Y_i &= \frac{\partial}{\partial y_i} - 2x_i \frac{\partial}{\partial t}, \quad \text{for } i = 1, \dots, n, \\ T &= \frac{\partial}{\partial t}. \end{aligned}$$

Since  $[X_i, Y_i] = -4T$ , the Heisenberg Laplacian

$$\Delta_H = \sum_{i=1}^n X_i^2 + Y_i^2$$

is a second order degenerate elliptic operator of Hormander type and hence it is hypoelliptic (see e.g. [14] for more details about  $\Delta_H$ ).

Clearly the vector fields  $X_i, Y_i$  are homogeneous of degree 1 with respect to the norm  $|\cdot|_H$  while  $T$  is homogeneous of degree 2.

We now want to prove a Maximum Principle result in some unbounded domains of  $H^n$ . Precisely

PROPOSITION 2.1. *Let  $\Omega$  be an open connected subset of  $H^n$  such that one of the following conditions holds:*

1. *there exists  $\xi_0 \in H^n$  and  $\lambda \leq 0$  such that  $\overline{\xi_0 \circ \Omega} \subset \Sigma_\lambda := \{\xi \in H^n : t \geq \lambda(x^2 + y^2)\}$ .*
2. *there exists  $\xi_0 \in H^n$  such that  $\overline{\xi_0 \circ \Omega}$  lies on one side of an hyperplane parallel to the  $t$  axis i.e. there exists  $v \in \mathbb{R}^{2n}$  such that  $\overline{\xi_0 \circ \Omega} \subset \{\xi \in H^n : v \cdot (x, y) > 0\}$ .*

*Suppose that there exists  $u \in C(\overline{\Omega})$  bounded above, solution of*

$$(7) \quad \begin{cases} \Delta_H u + c(\xi)u \geq 0 & \text{in } \Omega, \text{ with } c(\xi) \leq 0, \\ u \leq 0 & \text{on } \partial\Omega, \end{cases}$$

*then  $u \leq 0$  in  $\Omega$ .*

When  $\Omega$  is bounded there is nothing to prove; when  $\Omega$  is unbounded and it satisfies the first condition the proof is quite standard and similar to the euclidean case proved by Berestycki, Caffarelli and Nirenberg in [3]. We will first give the proof in that case.

Without loss of generality we can suppose in the rest of the section that  $\xi_0 = 0$  and that  $0 \notin \overline{\Omega}$ .

Before starting the proof let us introduce some notations.

Let  $\rho := |\xi|_H$  and  $u : \partial B_H(0, 1) \rightarrow \mathbb{R}$  be a smooth function. Then

$$\begin{aligned} X_i(u(\theta)\rho^\alpha) &= (\hat{R}_i u + \alpha a_i u)\rho^{\alpha-1}, \\ Y_i(u(\theta)\rho^\alpha) &= (\hat{S}_i u + \alpha b_i u)\rho^{\alpha-1}, \\ T(u(\theta)\rho^\alpha) &= (\hat{Z}u + \alpha c u)\rho^{\alpha-1} \end{aligned}$$

where  $a_i \equiv X_i(\rho)$ ,  $b_i \equiv Y_i(\rho)$ ,  $c = \rho T\rho$  and the vector fields  $\hat{R}_i, \hat{S}_i, \hat{Z}$  are the tangential components of  $X_i, Y_i$  and  $T$  on the Koranyi unit sphere  $S_H^1 := \partial B_H(0, 1)$ .

Since  $\Delta_H$  is homogeneous, a simple computation (see [8] and [14]) shows that

$$\Delta_H(u(\theta)\rho^\alpha) = [\mathcal{L}^\alpha(u(\theta))]\rho^{\alpha-2},$$

where

$$(8) \quad \mathcal{L}^\alpha u = \sum_{i=1}^n \hat{R}_i^2 u + \hat{S}_i^2 u + (2\alpha - 1)(a_i \hat{R}_i + b_i \hat{S}_i)u + \alpha(Q - 2 + \alpha)hu,$$

here  $h = \sum_{i=1}^n (a_i^2 + b_i^2) = \frac{x^2+y^2}{\rho^2}$ . For simplicity of notations, let us also introduce the following operator

$$(9) \quad D_\alpha = \sum_{i=1}^n \left( \hat{R}_i^2 + \hat{S}_i^2 + (2\alpha - 1)(a_i \hat{R}_i + b_i \hat{S}_i) \right).$$

PROOF OF CASE 1. We will first construct an auxiliary function that plays a key role.

STEP 1. Let  $C_\lambda = \Sigma_\lambda \cap S_H^1$ . In Lemma 2.1 of [7] it is proved that for any  $\lambda_1 \leq 0$  there exists a function  $\Psi$  depending on  $\phi = \frac{t}{\rho^2}$  defined in  $C_{\lambda_1}$  and there exists  $\alpha = \alpha(\lambda_1) > 0$  such that

$$\begin{cases} \mathcal{L}^\alpha \Psi = 0 & \text{in } C_{\lambda_1}, \\ \Psi = 0 & \text{on } \partial C_{\lambda_1}, \Psi > 0 \text{ in } C_{\lambda_1}. \end{cases}$$

Let us choose  $\lambda_1 < \lambda$  such that  $\overline{\Omega \cap S_H^1} \subset\subset C_{\lambda_1}$ . Then there exists  $\delta > 0$  such that  $\Psi \geq \delta > 0$  in  $\Omega \cap S_H^1$ .

Observe that the function  $g = \rho^\alpha \Psi$  satisfies  $\Delta_H g = \rho^{\alpha-2} \mathcal{L}^\alpha \Psi$  hence:

$$(10) \quad \begin{cases} \Delta_H g + c(\xi)g = c(\xi)g \leq 0 & \text{in } \Omega, \\ g \geq \delta > 0 & \text{in } \Omega \text{ for some } \delta > 0. \end{cases}$$

STEP 2. Since  $g$  satisfies (10), the function  $\sigma = \frac{u}{g}$  is well defined in  $\Omega$ . Furthermore it satisfies the following equation

$$(11) \quad \begin{cases} \Delta_H \sigma + \frac{2}{g} \text{grad}_H \sigma \cdot \text{grad}_H g + \frac{(\Delta_H g + cg)}{g} \sigma \geq 0 & \text{in } \Omega, \\ \sigma \leq 0 & \text{on } \partial\Omega, \end{cases}$$

Observe that (10) implies that the zero order coefficient is negative and furthermore, since  $\alpha > 0$  and  $u$  is bounded above

$$(12) \quad \lim_{\rho \rightarrow \infty} \sigma = \lim_{\rho \rightarrow \infty} \frac{u}{g} \leq 0.$$

By applying the standard maximum principle we obtain that  $\sigma \leq 0$  in  $\Omega$  i.e.  $u \leq 0$  in  $\Omega$ .

This completes the proof of the Proposition 2.1 for domains  $\Omega$  satisfying condition 1. □

Before giving the proof for the domains satisfying condition 2, we need to prove a few propositions since for “cones” different from the ones of Case 1 the construction of the auxiliary function  $g$  is more involved.

Also, without loss of generality, we will suppose that the vector  $v$  of Case 2 is  $v = (1, 0, \dots, 0)$ .

Let us make a few more remarks on the operators  $\mathcal{L}^\alpha$  or  $D_\alpha$ . Following Kohn and Nirenberg [17], we will say that a point  $\xi_0$  of  $\partial\Omega'$  is a *characteristic point* for  $\mathcal{L}^\alpha$  (or for  $D_\alpha$ ) if at least one of the vector fields  $\hat{R}_i$  or  $\hat{S}_i$  is null in  $\xi_0$ .

Since  $\hat{R}_i$  and  $\hat{S}_i$  are respectively the projection on  $S_H^1$  of  $X_i$  and  $Y_i$ , it is easy to see that all the characteristic points are of the following type  $\xi_0 = (0, \dots, 1, \dots, 0)$  where 1 is in one of the first  $2n$  positions. Hence, if  $\{e_1, \dots, e_{2n+1}\}$  denotes the standard euclidean basis of  $\mathbb{R}^{2n+1}$ , then it is easy to see that  $\mathcal{L}^\alpha$  is *uniformly elliptic* in  $\Omega' \subset S_H^1$  if  $e_i \notin \overline{\Omega'}$  for  $i = 1, \dots, 2n$ .

On the other hand since  $[\hat{R}_i, \hat{S}_i] = -4\hat{Z}$ , the operator  $\mathcal{L}^\alpha$  is of Hormander type for any  $\Omega' \subset S_H^1$ .

For  $u, v \in L^2(\Omega', d\theta)$ , let  $\langle u, v \rangle = \int_{\Omega'} u(\theta)v(\theta)d\theta$  and  $\|u\|^2 = \langle u, u \rangle$ . Let us denote by

$$(13) \quad A(u, v) := \langle \hat{R}_i u, \hat{R}_i v \rangle + \langle \hat{S}_i u, \hat{S}_i v \rangle .$$

and let  $B_0$  be the closure of  $C_0^\infty(\Omega')$  with respect to the norm

$$\|u\|_{B_0} = \left( A(u, u) + \|\sqrt{h}u\|^2 \right)^{\frac{1}{2}} .$$

Let  $\Omega'$  be a subdomain of  $S_H^1$  that does not have characteristic points on the boundary. Consider the operator  $T : L^2(\Omega') \rightarrow L^2(\Omega')$  defined by  $Tf := u \in B_0$ , where  $u$  is a solution of

$$(14) \quad \begin{cases} -D_1 u = hf & \text{in } \Omega', \\ u = 0 & \text{on } \partial\Omega'. \end{cases}$$

PROPOSITION 2.2.  $T$  is well defined and it is a compact operator in  $L^2(\Omega')$ .

PROOF. Observe that we can write the operator  $-D_1$  as

$$-D_1 = -\hat{R}_i^2 - \hat{S}_i^2 - a_i \hat{R}_i - b_i \hat{S}_i .$$

A simple computation shows that

$$(15) \quad \sum_{i=1}^n (\hat{R}_i a_i + \hat{S}_i b_i) = (Q - 1) \sum_{i=1}^n (a_i^2 + b_i^2)$$

and for  $u, v \in C_0^2$

$$(16) \quad \int_{\Omega'} \hat{R}_i u v d\theta = - \int_{\Omega'} u \hat{R}_i v d\theta + (Q - 1) \int_{\Omega'} u v a_i d\theta$$

(see [8] and [14] for details). Using (15) and (16) it is easy to see that

$$\begin{aligned} \langle -D_1 u, v \rangle &= \sum_{i=1}^n \left[ \langle \hat{R}_i u, \hat{R}_i v \rangle + \langle \hat{S}_i u, \hat{S}_i v \rangle \right. \\ &\quad \left. - Q \left( \langle a_i \hat{R}_i u, v \rangle + \langle b_i \hat{S}_i u, v \rangle \right) \right] \end{aligned}$$

Let  $a(u, v)$  denote the right hand side of (17).

(16) and (15) imply furthermore that for any  $u \in B_0$

$$\sum_{i=1}^n \int_{\Omega'} a_i \hat{R}_i u u d\theta + \int_{\Omega'} b_i \hat{S}_i u u d\theta = 0.$$

Therefore, using Poincaré inequality for operators satisfying Hormander condition (see [15] and [16]), we have

$$a(u, u) = A(u, u) \geq C \|u\|_{B_0}^2$$

for some constant  $C > 0$ . Hence  $a(u, v)$  is continuous and coercive in  $B_0$  and by Lax Milgram theorem for each  $f \in L^2(\Omega')$  there exists a unique  $u \in B_0$  such that

$$a(u, v) = \langle hf, v \rangle, \quad \forall v \in B_0,$$

hence  $T$  is well defined.

By a well known result of Kohn

$$a(u, u) \geq C \|u\|_{B_0}^2 \geq C \|u\|_{\frac{1}{2}},$$

where  $\|\cdot\|_{\frac{1}{2}}$  is the norm of a Sobolev space  $H^{\frac{1}{2}}(\Omega')$  of order  $\frac{1}{2}$ . (see [14]). Hence, by standard embedding theorems, the unit ball of  $B_0$  is compact in  $L^2(\Omega')$  and therefore  $T$  is compact.  $\square$

We now want to use the Krein-Rutman theorem under the conditions given in Theorem 2.6 of [7], to prove:

**PROPOSITION 2.3.** *T has a positive eigenvalue  $\mu_0$  and the corresponding eigenfunction  $\psi$  is positive in  $\Omega'$ .*

PROOF. Let  $G$  denote the cone of positive functions in  $L^2(\Omega')$ . Clearly,  $G$  is closed, convex and  $L^2(\Omega') = \overline{G - G}$ . Furthermore the  $L^2$  norm is semi-monotone with respect to  $G$ .

Theorem 2.6 of [7] claims that if  $T$  is compact and there exists  $e$  in  $G$  and  $\gamma > 0$  such that

$$Te - \gamma e \in G$$

then  $r(T) := \lim_{k \rightarrow \infty} |T^k|^{\frac{1}{k}} := \mu_0 > 0$ . Hence, from the classical Krein-Rutman theorem,  $\mu_0$  is an eigenvalue of  $T$  and the corresponding eigenfunction  $\psi$  is positive in  $\Omega'$ .

Let us construct  $e$  and  $\gamma$  as above. Let  $\Omega'' \subset \Omega'$  such that there exists  $\Omega_1$  without characteristic boundary points, satisfying  $\Omega'' \subset \Omega_1 \subset \Omega'$ . Let  $e \in G$  bounded above such that the support of  $e$  is contained in  $\Omega''$ .

By definition of  $T$  and using the maximum principle, the function  $v := Te$  satisfies

$$\begin{cases} D_1(v) = -he \leq 0 & \text{in } \Omega_1, \\ v \geq 0 & \text{on } \partial\Omega_1. \end{cases}$$

By the strong maximum principle we know that  $Te = v > 0$  in  $\Omega'$  hence

$$\inf_D Te = \delta > 0.$$

Let us choose  $\gamma := \frac{\delta}{2\|e\|_{L^\infty}}$  then

$$\begin{cases} Te - \gamma e \geq \delta - \gamma e > 0 & \text{in } \Omega'', \\ Te - \gamma e = Te \geq 0 & \text{in } \Omega' \setminus \Omega''. \end{cases}$$

$e$  and  $\gamma$  satisfy the required conditions and this completes the proof of the Proposition 2.3. □

Observe that clearly  $\psi$  and  $\mu_0$  satisfy

$$(18) \quad \begin{cases} D_1\psi + \frac{1}{\mu_0}h\psi = 0 & \text{in } \Omega', \\ \psi = 0 & \text{on } \partial\Omega'. \end{cases}$$

We will say that  $\lambda = \frac{1}{\mu_0}$  is the weighted principal eigenvalue of  $-D_1$  in  $\Omega'$ .

We are now ready to give the

PROOF OF CASE 2 OF PROPOSITION 2.1. It is enough to construct the auxiliary function, the second step being identical to the one in the proof of Case 1 i.e. we want to construct a function  $g$  satisfying (10) such that

$$\lim_{|\xi|_H \rightarrow \infty} g(\xi) = +\infty \text{ in } \Omega.$$

Without loss of generality we can suppose that the hyperplane parallel to the  $t$  axis is  $\{x_1 = 0\}$  and we suppose that  $\overline{\Omega} \subset \{\xi; \text{ such that } x_1 > 0\} := \Pi$ .

Let  $\Sigma_0 = \Pi \cap S_H^1$ . Observe that  $\Delta_H x_1 = 0$  implies that the function  $u := \frac{x_1}{\rho}$  defined on  $S_H^1$  satisfies

$$\Delta_H x_1 = \Delta_H(\rho u) = \rho^{-1} \mathcal{L}^1(u) = 0.$$

Therefore

$$\begin{aligned} \mathcal{L}^1(u) &= D_1 u + (Q - 1)hu = 0 && \text{in } \Sigma_0, \\ u &= 0 && \text{on } \partial \Sigma_0 \\ u &> 0 && \text{in } \Sigma_0, \end{aligned}$$

i.e.  $(Q - 1)$  is the principal weighted eigenvalue of  $-D_1$  in  $\Sigma_0$ .

Let  $\Sigma_\varepsilon \supset \Sigma_0$  close enough to  $\Sigma_0$  that  $\lambda_1 = \lambda_1(\varepsilon)$  the principal weighted eigenvalue of  $-D_1$  in  $\Sigma_\varepsilon$  satisfies

$$Q - 1 - \varepsilon := \lambda_1 < Q - 1$$

for some  $\varepsilon > 0$  to be determined. We can choose  $\Sigma_\varepsilon$  such that it has no characteristic points on the boundary.

Therefore there exists  $\psi_\varepsilon > 0$  in  $\Sigma_\varepsilon$  such that

$$(19) \quad \begin{cases} D_1 \psi_\varepsilon + \lambda_1 h \psi_\varepsilon = 0 & \text{in } \Sigma_\varepsilon, \\ \psi_\varepsilon = 0 & \text{on } \partial \Sigma_\varepsilon. \end{cases}$$

The first condition required on  $\varepsilon$  is that

$$(20) \quad \frac{1}{2} \left( \frac{1}{2} + Q - 2 \right) < Q - 1 - \varepsilon.$$

In particular (20) implies that the operator  $-(D_1 + \frac{1}{2}(\frac{1}{2} + Q - 2)h)$  has a positive principal weighted eigenvalue in  $\Sigma_\varepsilon$ .

It is immediate to see that

$$\begin{aligned} \mathcal{L}^{\frac{1}{2}} &= \sum_{i=1}^n (\hat{R}_i^2 + \hat{S}_i^2) + \frac{1}{2} \left( \frac{1}{2} + Q - 2 \right) h \\ &= D_1 + \frac{1}{2} \left( \frac{1}{2} + Q - 2 \right) h - \sum_{i=1}^n (a_i \hat{R}_i + b_i \hat{S}_i), \end{aligned}$$

this leads to the following

CLAIM 1. There exists  $\varepsilon > 0$  such that there exist a function  $v > 0$  and a constant  $\mu > 0$  such that

$$(21) \quad \begin{cases} \mathcal{L}^{\frac{1}{2}} v + \mu v \leq 0 & \text{in } \Sigma_\varepsilon, \\ v = 0 & \text{on } \partial \Sigma_\varepsilon. \end{cases}$$

For some  $1 > \beta > 0$ , let us compute  $\mathcal{L}^{\frac{1}{2}}(\psi_\varepsilon^\beta)$ .  
Clearly the following equalities hold

$$\begin{aligned}\hat{R}_i(\psi_\varepsilon^\beta) &= \beta \psi_\varepsilon^{\beta-1} \hat{R}_i \psi_\varepsilon, \\ \hat{R}_i^2(\psi_\varepsilon^\beta) &= \beta(\beta-1) \psi_\varepsilon^{\beta-2} (\hat{R}_i \psi_\varepsilon)^2 + \beta \psi_\varepsilon^{\beta-1} \hat{R}_i^2 \psi_\varepsilon\end{aligned}$$

and similarly for  $\hat{S}_i$ . Hence let  $\nu = \psi_\varepsilon^\beta$ :

$$\begin{aligned}\mathcal{L}^{\frac{1}{2}}(\nu) &= \sum_{i=1}^n (\hat{R}_i^2 \psi_\varepsilon^\beta + \hat{S}_i^2 \psi_\varepsilon^\beta) + \frac{1}{2} \left( \frac{1}{2} + Q - 2 \right) h\nu \\ &= \sum_{i=1}^n (\beta \psi_\varepsilon^{\beta-1} (\hat{R}_i^2 \psi_\varepsilon + \hat{S}_i^2 \psi_\varepsilon) + \beta(\beta-1) \psi_\varepsilon^{\beta-2} ((\hat{R}_i \psi_\varepsilon)^2 + (\hat{S}_i \psi_\varepsilon)^2)) \\ &\quad + \frac{1}{2} \left( \frac{1}{2} + Q - 2 \right) h\nu.\end{aligned}$$

Using (19) we obtain

$$\begin{aligned}\mathcal{L}^{\frac{1}{2}}(\nu) &= \sum_{i=1}^n \beta(\beta-1) \psi_\varepsilon^{\beta-2} \left( (\hat{R}_i \psi_\varepsilon)^2 + (\hat{S}_i \psi_\varepsilon)^2 \right) \\ &\quad - \beta \psi_\varepsilon^{\beta-1} (a_i \hat{R}_i \psi_\varepsilon + b_i \hat{S}_i \psi_\varepsilon) \\ &\quad + \left( \frac{1}{2} \left( \frac{1}{2} + Q - 2 \right) - \beta \lambda_1 \right) h\nu.\end{aligned}$$

The Young inequality implies:

$$(-\beta \psi_\varepsilon^{\beta-1} a_i \hat{R}_i \psi_\varepsilon) \leq \beta(1-\beta) \psi_\varepsilon^{\beta-2} (\hat{R}_i \psi_\varepsilon)^2 + \beta \psi_\varepsilon^\beta \frac{a_i^2}{4(1-\beta)}.$$

Hence:

$$\mathcal{L}^{\frac{1}{2}}(\nu) \leq \left( \frac{1}{2} \left( \frac{1}{2} + Q - 2 \right) - \beta \lambda_1 + \frac{\beta}{4(1-\beta)} \right) h\nu.$$

Let

$$k(\beta) := -\frac{1}{2} \left( \frac{1}{2} + Q - 2 \right) + \beta \lambda_1 - \frac{\beta}{4(1-\beta)}.$$

If we prove that, for  $\varepsilon$  sufficiently small, there exists  $\beta_0 \in (0, 1)$  such that  $k(\beta_0) > 0$  then the claim is proved. Indeed by choosing  $\beta = \beta_0$  in the definition of  $\nu$  and  $\mu = k(\beta_0)$  we have constructed  $\nu$  and  $\mu$  with the required properties.

We define

$$h(\beta) := 4(1-\beta)k(\beta) = -4\lambda_1\beta^2 + 2\beta(2\lambda_1 - 2 + Q) - 2Q + 3,$$

hence it is enough to check that  $h(\beta_0) := \max_{\beta \in [0,1]} h(\beta) > 0$  and  $\beta_0 \in (0, 1)$ . Observe that  $\beta_0 = \frac{2\lambda_1 - 2 + Q}{4\lambda_1}$  and

$$h(\beta_0) = \frac{(2\lambda_1 - 2 + Q)^2}{4\lambda_1} + 3 - 2Q = \frac{(Q - 2)^2 + 4\epsilon^2 + 4\epsilon(1 - Q)}{4\lambda_1}.$$

We have used the fact that  $\lambda_1 = Q - 1 - \epsilon$ .

It is easy to see that  $\beta_0 \in (0, 1)$  for  $\epsilon < \frac{Q}{2}$  while  $h(\beta_0) > 0$  for  $\epsilon > 0$  sufficiently small. This completes the proof of the claim.

We will choose as auxiliary function  $g(\xi) := \rho^{\frac{1}{2}}v$ . Clearly  $g$  satisfies:

$$\begin{cases} \Delta_H g + c(\xi)g = \rho^{-\frac{3}{2}}\mathcal{L}^{\frac{1}{2}}v + c(\xi)g \leq 0 & \text{in } \Omega, \\ g \geq \delta > 0 & \text{in } \Omega, \\ \lim_{\rho \rightarrow \infty} g(\xi) = +\infty & \text{in } \Omega. \end{cases}$$

This completes the proof of Proposition 2.1. □

### 3. – One dimensional symmetry

An immediate consequence of the Maximum Principle of Proposition 2.1 is the following comparison result:

**COROLLARY 3.1.** *Let  $f$  be a Lipschitz continuous function, non-increasing on  $[-1, -1 + \delta]$  and on  $[1 - \delta, 1]$  for some  $\delta > 0$ . Assume that  $u_1, u_2$  are solutions of*

$$\Delta_H u_i + f(u_i) = 0 \text{ in } \Omega$$

*and are such that  $|u_i| \leq 1, i = 1, 2$ . Furthermore, assume that*

$$u_2 \geq u_1 \text{ on } \partial\Omega$$

*and that either*

$$u_2 \geq 1 - \delta \text{ in } \Omega$$

*or*

$$u_1 \leq 1 + \delta \text{ in } \Omega.$$

*If  $\Omega \subset H^n$  is an open connected set satisfying either of the conditions (1) or (2) of Proposition 2.1 then  $u_2 \geq u_1$  in  $\Omega$ .*

We shall use Corollary 3.1 to prove the following one dimensional symmetry results:

THEOREM 3.1. *Let  $u$  be a solution of*

$$(22) \quad \Delta_H u + f(u) = 0 \text{ in } H^n$$

*which satisfies  $|u| \leq 1$  together with asymptotic conditions*

$$(23) \quad \lim_{x_1 \rightarrow \pm\infty} u(x_1, x', y, t) = \pm 1$$

*uniformly in  $x' = (x_2, \dots, x_n) \in \mathbb{R}^{n-1}$ ,  $y = (y_1, \dots, y_n) \in \mathbb{R}^n$  and  $t \in \mathbb{R}$ . We assume that  $f$  is Lipschitz continuous in  $[-1, 1]$ ,  $f(\pm 1) = 0$  and that there exists  $\delta > 0$  such that*

$$(24) \quad f \text{ is nonincreasing on } [-1, -1 + \delta] \text{ and on } [1 - \delta, 1].$$

*Then,  $u(x_1, x', y, t) = U(x_1)$  where  $U$  is a solution of*

$$(25) \quad \left. \begin{aligned} U'' + f(U) &= 0 \text{ in } \mathbb{R}, \\ U(\pm\infty) &= \pm 1, \end{aligned} \right\}$$

*and  $u$  is increasing with respect to  $x_1$ . The existence of a solution  $u$  of (22)-(23) such that  $|u| \leq 1$  implies the existence of a solution  $U$  of (25). Furthermore, the solution  $u$  is unique up to translations of the origin.*

REMARK. The conclusion of Theorem 3.1 holds if we replace  $x_1$  by any non-anisotropic direction i.e. let  $s = a \cdot x + b \cdot y$  for some vectors  $a \in \mathbb{R}^n$  and  $b \in \mathbb{R}^n$  such that  $a^2 + b^2 = 1$ , if condition (23) is replaced by

$$(26) \quad \lim_{s \rightarrow \pm\infty} u(x, y, t) = \pm 1$$

uniformly, then there exists  $U$  such that  $u(x, y, t) = U(a \cdot x + b \cdot y)$ .

On the other hand, for the anisotropic direction we have

PROPOSITION 3.1. *Under the hypothesis of Theorem 3.1, replacing condition (23) by*

$$(27) \quad \lim_{t \rightarrow \pm\infty} u(x, y, t) = \pm 1 \text{ uniformly in } x = (x_1, \dots, x_n), y = (y_1, \dots, y_n)$$

*and further assuming that  $f$  is  $C^1$ , the function  $u$  is monotone along the  $t$ -direction, i.e.,  $\frac{\partial u}{\partial t} > 0$  in  $H^n$ .*

The equivalent of Theorem 3.1 for the classical Laplacian was obtained by Berestycki, Hamel and Monneau in [4] using the sliding method. Here we shall use the sliding method in  $H^n$  with the one parameter family of transformations defined by

$$(28) \quad \begin{aligned} \mathcal{E}_v(s)(\xi) &= (sx_0, sy_0, st_0) \circ (x, y, t) \\ &= (x + sx_0, y + sy_0, t + st_0 + 2s(y_0x - x_0y)) \end{aligned}$$

where  $\nu = (x_0, y_0, t_0)$ . In [10], we had already used the sliding method to obtain monotonicity results for solutions of semilinear equations in nilpotent, stratified Lie groups.

PROOF OF THEOREM 3.1. The proof of Theorem 3.1 is along the lines of Theorem 1 in [4]. Of course, here we use the group action  $\circ$  of  $H^N$  and we rely on the fact that the sub-Laplacian is invariant with respect to  $\circ$ . We begin by proving

CLAIM 1. For any  $\nu = (x_1^0, x^{0'}, y^0, t^0) \in \mathbb{R}^{2n+1}$  with  $x_1^0 > 0$ , we have

$$(29) \quad u_s(\xi) := u(\mathcal{E}_\nu(s)\xi) \geq u(\xi) \text{ for all } \xi \in H^n.$$

PROOF. Using the condition (23), for  $\delta > 0$  there exists  $N \in \mathbb{N}$  such that

$$(30) \quad u(x_1, x', y, t) > 1 - \delta \quad \text{for } x_1 \geq N$$

$$(31) \quad u(x_1, x', y, t) < -1 + \delta \quad \text{for } x_1 \leq -N.$$

Hence for  $s > 2N/x_1^0$ , we have

$$(32) \quad u_s(\xi) > 1 - \delta \quad \text{for } x_1 \geq -N.$$

Furthermore, the function  $u_s$  satisfies the equation (22) and

$$u_s(-N, x', y, t) > u(-N, x', y, t).$$

We now apply Corollary 3.1 to the functions  $u_s$  and  $u$  in the half spaces  $\{\xi = (x, y, t) \in H^n : x_1 \geq -N\}$  and  $\{\xi = (x, y, t) \in H^n : x_1 \leq -N\}$  to conclude that

$$u_s(x, y, t) \geq u(x, y, t) \text{ for all } (x, y, t) \in H^n.$$

Let  $\tau = \inf\{s : u_s(\xi) \geq u(\xi) \text{ for all } \xi \in H^n\}$ . We claim that  $\tau = 0$ . On the contrary, suppose that  $\tau > 0$ . We have

$$u_\tau(x, y, t) \geq u(x, y, t) \text{ for all } (x, y, t) \in H^n.$$

We consider the following two cases:

Case (i):

$$\inf_{\xi \in [-N, N] \times \mathbb{R}^{2n}} \{u_\tau(\xi) - u(\xi)\} > 0.$$

Since  $u$  is bounded and  $f$  is Lipschitz continuous, it is easy to see that using e.g. Theorem 2 of chapter XIII of [18] and Corollary IV.7.4 of [19]  $u$  is globally Lipschitz continuous.

Hence, there exists  $\varepsilon > 0$  small, such that for all  $s, \tau - \varepsilon < s < \tau$  we have

$$\inf_{\xi \in [-N, N] \times \mathbb{R}^{2n}} \{u_s(\xi) - u(\xi)\} > 0.$$

Observe that, from (30) we have

$$(33) \quad \begin{aligned} u_s(\xi) &= u(x_1 + sx_1^0, x' + sx^{0'}, y + sy^0, t + st^0 + 2s(y^0x - x^0y)) \\ &> 1 - \delta \text{ for all } s > 0 \text{ and for all } x_1 \geq N. \end{aligned}$$

Hence we can again use the comparison principle for  $u_s$  and  $u$  in the half spaces  $\{(x, y, t) \in H^n : x_1 \geq N\}$  and  $\{(x, y, t) \in H^n : x_1 \leq -N\}$ . Together with (33), we conclude that

$$u_s(\xi) \geq u(\xi) \quad \forall \xi \in H^n \text{ and for all } \tau - \varepsilon < s < \tau$$

a contradiction to the definition of  $\tau$ .

Case (ii)

$$\inf_{\xi \in [-N, N] \times \mathbb{R}^{2n}} \{u_\tau(\xi) - u(\xi)\} = 0.$$

Let  $\xi_k \in [-N, N] \times \mathbb{R}^{2n}$  such that  $u_\tau(\xi_k) - u(\xi_k) \rightarrow 0$ . Define  $v_k(\xi) = u(\xi_k \circ \xi)$  for  $\xi \in H^n$ .

By regularity estimates and embedding of the non-isotropic Sobolev spaces (see [18] and [19]) we can extract a subsequence of  $\{v_k\}$  converging uniformly to a solution  $v$  of (22). Moreover, we have  $v(0) = v_s(0)$ .

Therefore, the function  $z(\xi) = v_\tau(\xi) - v(\xi)$  satisfies

$$(34) \quad \left. \begin{aligned} \Delta_H z + c(\xi)z &= 0 \text{ in } H^n, \\ z &\geq 0 \text{ in } H^n, \\ z(0) &= 0, \end{aligned} \right\}$$

where  $c(\xi)$  is a bounded function defined by

$$(35) \quad c(\xi) = \begin{cases} \frac{f(v_\tau(\xi)) - f(v(\xi))}{v_\tau(\xi) - v(\xi)} & \text{if } v_\tau(\xi) \neq v(\xi), \\ 0 & \text{otherwise.} \end{cases}$$

The maximum principle implies that  $z \equiv 0$  i.e.,  $v(\xi) = v((\tau x^0, \tau y^0, \tau t^0) \circ \xi)$  for all  $\xi \in H^n$ .

However, this is not possible since  $v$  also satisfies the asymptotic condition (23). Hence case (ii) does not arise. Therefore, we conclude that  $\tau = 0$ ; which completes the proof of the Claim 1.

From the previous discussion we further conclude that

$$\lim_{s \rightarrow 0} \frac{u_s(\xi) - u(\xi)}{s} \geq 0$$

hence for all  $\xi \in H^n$  and for every  $\nu = (x^0, y^0, t^0) \in \mathbb{R}^{2n+1}$  with  $x_1^0 > 0$

$$(36) \quad (x^0, y^0, t^0 + 2 \sum_{i=1}^n 2(y_i^0 x_i - x_i^0 y_i) \cdot \text{grad } u \geq 0.$$

By continuity (36) holds for every  $\nu \in \mathbb{R}^{2n+1}$  with  $x_1^0 = 0$  i.e.

$$(37) \quad \sum_{i=2}^n x_i^0 \frac{\partial u(\xi)}{\partial x_i} + \sum_{i=1}^n y_i^0 \frac{\partial u(\xi)}{\partial y_i} + \left( t^0 + \sum_{i=1}^n 2(y_i^0 x_i - x_i^0 y_i) \right) \frac{\partial u(\xi)}{\partial t} \geq 0.$$

It follows from (37) that for every  $\nu \in \mathbb{R}^{2n+1}$  with  $x_1^0 = 0$  we have

$$(38) \quad \sum_{i=2}^n x_i^0 \frac{\partial u(\xi)}{\partial x_i} + \sum_{i=1}^n y_i^0 \frac{\partial u(\xi)}{\partial y_i} + \left( t^0 + \sum_{i=1}^n 2(y_i^0 x_i - x_i^0 y_i) \right) \frac{\partial u(\xi)}{\partial t} = 0.$$

Varying  $\nu$  over the standard vectors  $e_i = (0, 0, \dots, 1(i\text{-th place}), \dots, 0) \in \mathbb{R}^{2n+1}$  with  $i = 2, \dots, 2n+1$ , we conclude that all the partial derivatives  $\{\frac{\partial u}{\partial x_i}\}_{2 \leq i \leq n}$ ,  $\{\frac{\partial u}{\partial y_i}\}_{1 \leq i \leq n}$  and  $\frac{\partial u}{\partial t}$  vanish identically in  $H^n$  which implies that  $u$  is function of  $x_1$ . In particular, the second part of Theorem 3.1 holds.  $\square$

PROOF OF PROPOSITION 3.1. The proof follows the lines of the proof of Theorem 3.1 choosing  $\nu = (0, 0, t_0)$  and applying the Maximum Principle in half spaces  $\{t > k\}$ .

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