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$L^p$-spectrum of Ornstein-Uhlenbeck operators

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Abstract. We study the $L^p$-spectrum of Ornstein-Uhlenbeck operators $A = \sum_{i,j=1}^{n} q_{ij} D_{ij} + \sum_{i,j=1}^{n} b_{ij} x_j D_i$ and of the drift operators $L = \sum_{i,j=1}^{n} b_{ij} x_j D_i$. We show that the spectrum of $L$ in $L^p(\mathbb{R}^n)$ is the line $-\text{tr}(B)/p + i \mathbb{R}$, $B = (b_{ij})$, or a discrete subgroup of $i \mathbb{R}$ and that the spectrum of $A$ contains the spectrum of $L$. If $\sigma(B) \subset \mathbb{C}_-$ or $\sigma(B) \subset \mathbb{C}_+$, then the $L^p$-spectrum of $A$ is the half-plane $\{ \mu \in \mathbb{C} : \Re \mu \leq -\text{tr}(B)/p \}$. The same happens if $B = B^*$ and $QB = BQ$.

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1. Introduction

In this paper we study the $L^p$-spectrum of the Ornstein-Uhlenbeck operators

\begin{equation}
A = \sum_{i,j=1}^{n} q_{ij} D_{ij} + \sum_{i,j=1}^{n} b_{ij} x_j D_i = \text{Tr}(Q D^2) + \langle Bx, D \rangle, \quad x \in \mathbb{R}^n,
\end{equation}

where $Q = (q_{ij})$ is a real, symmetric and positive definite matrix and $B = (b_{ij})$ is a non-zero real matrix. The generated semigroup $(T(t))_{t \geq 0}$ has the following explicit representation due to Kolmogorov

\begin{equation}
(T(t)f)(x) = \frac{1}{(4\pi)^{n/2}(\det Q_t)^{1/2}} \int_{\mathbb{R}^n} e^{-\langle Q_t^{-1} y, y \rangle/4} f(e^{tB}x - y) \, dy,
\end{equation}

where

$Q_t = \int_{0}^{t} e^{sB} Q e^{sB^*} \, ds$.

The case where the spectrum of the matrix $B$ is contained in the (open) left half-plane $\mathbb{C}_-$ is the most interesting from the point of view of diffusion processes. The inclusion $\sigma(B) \subset \mathbb{C}_-$ is, in fact, necessary and sufficient for the existence of
an invariant measure of the underlying stochastic process, that is of a probability measure $\mu$ such that

$$\int_{\mathbb{R}^n} (T(t)f)(x) \, d\mu(x) = \int_{\mathbb{R}^n} f(x) \, d\mu(x)$$

for every $t \geq 0$ and $f \in BUC(\mathbb{R}^n)$. The invariant measure is unique and is given by $d\mu(x) = b(x) \, dx$ where

$$b(x) = \frac{1}{(4\pi)^{n/2}(\det Q_\infty)^{1/2}} e^{-\langle Q_\infty^{-1} x, x \rangle / 4}$$

and

$$Q_\infty = \int_0^\infty e^{sB} Q e^{sB^*} \, ds,$$

see [7, Chapter II.6].

Both the semigroup $(T(t))_{t \geq 0}$ and its generator $A$ have been extensively studied in $L^p(\mathbb{R}^n, d\mu)$, on account of their probabilistic meaning. We refer to [17] and [3] for the case $Q = I$, $B = -I$; in this situation $A$ is selfadjoint in $L^2(\mathbb{R}^n, d\mu)$ with compact resolvent and the Hermite polynomials form a complete system of eigenfunctions. Moreover, the operator $-A$ on $L^2(\mathbb{R}^n, d\mu)$ is unitarily equivalent to a Schrödinger operator $-\Delta + V$ on $L^2(\mathbb{R}^n)$, where $V$ is a quadratic potential. The domain of $A$ in $L^2(\mathbb{R}^n, d\mu)$ is described in [14] for general matrices $Q$, $B$ (with $\sigma(B) \subset \mathbb{C}_-$) whereas the analyticity of $(T(t))_{t \geq 0}$ is proved in [9].

The whole picture changes completely passing from $L^p(\mathbb{R}^n, d\mu)$ to $L^p(\mathbb{R}^n)$ (with respect to the Lebesgue measure). In fact, the unboundedness of the coefficients of $A$ is no longer balanced by the exponential decay of the measure $\mu$ and the semigroup turns out to be norm-discontinuous (see [18]). Moreover, the spectrum of $A$ is very large and $p$-dependent, as we show in this paper. Smoothing properties of $(T(t))_{t \geq 0}$ are established in [6], in spaces of continuous functions, and Schauder estimates are deduced for its generator, by means of interpolation techniques. The same approach is used in [16], [5] and [13] where similar results are proved for operators whose coefficients have linear, polynomial and exponential growth, respectively, under a dissipativity condition preventing the underlying Markov process to explode in finite time. Generation results in $L^p(\mathbb{R}^n)$ are proved in [15].

The operator $A$ is the sum of the diffusion term $\sum_{i,j=1}^n q_{ij} D_{ij}$ and of the drift term $\mathcal{L} = \sum_{i,j=1}^n b_{ij} x_j D_i$. Whereas the spectral properties of the diffusion term are quite obvious, being an elliptic operator with constant coefficients, those of the drift term are more interesting and depend both on $p$ and the matrix $B$. For example, in dimension one, the spectrum of $-xD$ on $L^p(0, \infty)$ is the line $1/p + i \mathbb{R}$. Since the inverse of $I + xD$ is Hardy’s operator

$$u \mapsto \frac{1}{x} \int_0^x u(t) \, dt,$$
every result on $-xD$ can be reformulated in terms of Hardy's operator above (see [1] and also [4]).

In Section 2 we show that the spectrum of $\mathcal{L}$ is the line $-\text{tr}(B)/p + i\mathbb{R}$ unless $B$ is (similar to) a diagonal matrix with purely imaginary eigenvalues. In this last case $\sigma_p(\mathcal{L})$ can be either $i\mathbb{R}$ or a discrete subgroup of $i\mathbb{R}$, independent of $p$. The spectrum is, therefore, $p$-dependent if and only if $\text{tr}(B) \neq 0$ and this relies on the fact that the generated semigroup has a $p$-dependent growth bound. Two different arguments are needed to achieve the results of this section. The first one is due to Arendt ([1]) and deals with the $L^p$-consistency of resolvent operators: this works if $\text{tr}(B) = 0$. In the case $\text{tr}(B) = 0$ the above argument fails and the proof uses ideas from spectral theory for bounded groups (see [11, IV.3.c]).

In Section 3 we show that the boundary spectrum of the Ornstein-Uhlenbeck operator contains the spectrum of its drift term, without any assumption on the matrices $Q$ and $B \neq 0$. This gives another proof of the norm discontinuity of $(T(t))_{t \geq 0}$.

Section 4, which contains the main results of the paper, is devoted to the computation of the spectrum of Ornstein-Uhlenbeck operators under the assumption that the spectrum of the matrix $B$ is contained in the left or in the right half-plane. In this second case it turns out that the half-plane $\{ \mu \in \mathbb{C} : \text{Re} \mu < -\text{tr}(B)/p \}$ consists of eigenvalues and that the spectrum is $\{ \mu \in \mathbb{C} : \text{Re} \mu \leq -\text{tr}(B)/p \}$. The proof of this result changes according to $p = 1$, $1 < p < 2$ and $p \geq 2$. For $p \geq 2$ we compute the Fourier transforms of the eigenfunctions and use the boundedness of the Fourier transform from $L^p'(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$ to conclude. For $p = 1$, we compute again the Fourier transforms of the eigenfunctions and then estimate their asymptotic behavior to show that they belong to $L^1$. This method gives also some partial result in the case $1 < p < 2$. To obtain the full result in this last case, we write explicitly the eigenfunctions relative to a certain range of eigenvalues as convolution integrals and then estimate them. The case where the spectrum of $B$ is contained in the left half-plane is deduced by duality from the previous one.

In Section 5 we use a tensor product argument, together with the results of Sections 3 and 4, to show that if $B$ is symmetric and $QB = BQ$ then $\sigma_p(A) = \{ \mu \in \mathbb{C} : \text{Re} \mu \leq -\text{tr}(B)/p \}$. This covers e.g. the case

$$A = \Delta + \sum_{i,j=1}^{n} b_{ij} x_j D_i$$

with $B$ symmetric.

In Section 6 we deal with the spectrum of Ornstein-Uhlenbeck operators in $\text{BUC}(\mathbb{R}^n)$. If $\sigma(B) \cap i\mathbb{R} = \emptyset$ we show that the spectrum is the left half-plane $\{ \mu \in \mathbb{C} : \text{Re} \mu \leq 0 \}$.

Most of the results of this paper hold if we only assume that the matrix $Q$ is semi-definite. In particular this is true for Theorem 3.3. Variants of Theorem 5.1 can be proved with similar arguments. Such degenerate operators
have been considered in [13] where Schauder-type estimates are proved under the hypothesis $\det Q_t > 0$ for $t > 0$. This assumption is equivalent to the fact that $\mathcal{A}$ is hypoelliptic (see [10]). If $\sigma(B) \subset \mathbb{C}_-$, then $\det Q_t > 0$ for $t > 0$ if and only if the matrix $Q_\infty$ is positive-definite. In this situation, the results of Sections 4 and 6 continue to hold with minor changes in the proofs.

**Notation.** $L^p$ stands for $L^p(\mathbb{R}^n)$, $BUC$ for $BUC(\mathbb{R}^n)$, $C_0^\infty$ for $C_0^\infty(\mathbb{R}^n)$ and $S$ for the Schwartz class. We use $L^\infty$ for $C_0(\mathbb{R}^n) = \{u \in C(\mathbb{R}^n) : \lim_{|x| \to \infty} u(x) = 0\}$. $\mathbb{C}_+ = \{\mu \in \mathbb{C} : \Re \mu > 0\}$, $\mathbb{C}_- = \{\mu \in \mathbb{C} : \Re \mu < 0\}$. The spectrum and the resolvent set of a linear operator $B$ on $L^p$ are denoted by $\sigma(B)$ and $\rho_p(B)$, respectively. The norm of a bounded operator $S$ on $L^p$ is denoted by $\|S\|_p$. The **spectral bound** of a linear operator $B$ is defined by $s(B) = \sup\{|\Re \mu : \mu \in \sigma(B)\}$ and the **boundary spectrum** is $\sigma(B) \cap \{\mu \in \mathbb{C} : \Re \mu = s(B)\}$. The approximate point spectrum $\sigma_{ap}(B)$ of $B$ is the subset of $\sigma(B)$ of all complex numbers $\mu$ for which there is a sequence $(v_n)$ contained in its domain such that $\|v_n\| = 1$ and $\|Bv_n - \mu v_n\| \to 0$ as $n \to \infty$. The sequence $(v_n)$ is called an approximate eigenvector relative to $\mu$. The topological boundary of the spectrum of $B$ is always contained in $\sigma_{ap}(B)$ (see [11, Proposition IV.1.10]).

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### 2. Spectrum of the drift

Let $B = (b_{ij})$ be a real $n \times n$ matrix and consider the drift operator

$$\mathcal{L} = \sum_{i,j=1}^n b_{ij} x_j D_i.$$

We define

$$D_p(\mathcal{L}) = \{u \in L^p : \mathcal{L}u \in L^p\}$$

for $1 \leq p \leq \infty$, where $\mathcal{L}u$ is understood in the sense of distributions.

**Lemma 2.1.** The operator $(\mathcal{L}, D_p(\mathcal{L}))$ is closed in $L^p$. 
PROOF. Suppose that \((u_n) \subset D_p(\mathcal{L})\) converges to \(u\) and that \((\mathcal{L}u_n)\) converges to \(v\) in \(L^p\). If \(\phi \in C_0^{\infty}\), denoting by \(\mathcal{L}^*\) the formal adjoint of \(\mathcal{L}\), we have
\[
\int_{\mathbb{R}^n} u\mathcal{L}^*\phi = \lim_{n \to \infty} \int_{\mathbb{R}^n} u_n\mathcal{L}^*\phi = \lim_{n \to \infty} \int_{\mathbb{R}^n} (\mathcal{L}u_n)\phi = \int_{\mathbb{R}^n} v\phi
\]
and hence \(u \in D_p(\mathcal{L})\) and \(\mathcal{L}u = v\).

PROPOSITION 2.2. The operator \((\mathcal{L}, D_p(\mathcal{L}))\) is the generator of the \(C_0\)-group \((S(t))_{t \in \mathbb{R}}\) defined by
\[
(S(t))f(x) = f(e^{tB}x)
\]
for \(f \in L^p, t \in \mathbb{R}\). \(C_0^{\infty}\) is a core of \((\mathcal{L}, D_p(\mathcal{L}))\) and
\[
\|S(t)f\|_p = e^{-\frac{t}{p}}\|f\|_p
\]
for all \(f \in L^p\).

PROOF. A simple change of variable, together with the equality \(\det e^{-tB} = e^{-t \text{tr}(B)}\), shows that (2.2) holds. Since the group law is clear, we have only to prove the strong continuity at 0. Clearly, \(S(t)f \to f\) in \(L^p\) as \(t \to 0\) if \(f\) is continuous with compact support; by density and (2.2), the same holds for every \(f \in L^p\) and hence \((S(t))_{t \in \mathbb{R}}\) is strongly continuous. Let \((L_p, D_p)\) be its generator in \(L^p(\mathbb{R}^n)\) and take \(f \in C_0^{\infty}\). A straightforward computation shows that
\[
\lim_{t \to 0} \frac{S(t)f - f}{t} = \mathcal{L}f
\]
in \(L^p\), and hence \(C_0^{\infty} \subset D_p\) and \(L_p f = \mathcal{L} f\) if \(f \in C_0^{\infty}\). Moreover, since \(C_0^{\infty}\) is dense in \(L^p\) and \(S(t)\)-invariant, it is a core for \((L_p, D_p)\). The closedness of \((\mathcal{L}, D_p(\mathcal{L}))\) implies that \(D_p \subset D_p(\mathcal{L})\) and that \(L_p f = \mathcal{L} f\) if \(f \in D_p\). Let \(\mathcal{L}^* = -\mathcal{L} - \text{tr}(B)\) be the formal adjoint of \(\mathcal{L}\) and note that \(\mathcal{L}^* = -L_p' - \text{tr}(B)\) on \(D_{p'}\), \(1/p + 1/p' = 1\). If \(u \in D_p(\mathcal{L})\), then the equality
\[
\int_{\mathbb{R}^n} \mathcal{L}u\phi = \int_{\mathbb{R}^n} u\mathcal{L}^*\phi
\]
holds for all \(\phi \in D_{p'}\), by the density of \(C_0^{\infty}\) in \(D_{p'}\) with respect to the graph norm induced by \(\mathcal{L}^*\).

For \(\lambda\) large, take \(v \in D_p\) such that \(\lambda v - L_p v = \lambda u - \mathcal{L} u\). Then \(w = v - u \in D_p(\mathcal{L})\) satisfies \(\lambda w - \mathcal{L} w = 0\) and from (2.3) we deduce that
\[
0 = \int_{\mathbb{R}^n} (\lambda w - \mathcal{L} w)\phi = \int_{\mathbb{R}^n} w(\lambda - \mathcal{L}^*)\phi,
\]
for all \(\phi \in D_{p'}\).

Since \((\lambda - \mathcal{L}^*)(D_{p'}) = (\lambda + \text{tr}(B) + L_p')(D_{p'}) = L_{p'}\) (for \(\lambda\) large), we deduce that \(w = 0\) and that \(u \in D_p\).
In the following theorem we use an argument from [1, Section 3] to compute the spectrum of $\mathcal{L}$ in the case $\text{tr}(B) \neq 0$.

**Theorem 2.3.** If $\text{tr}(B) \neq 0$ then $\sigma_p(\mathcal{L}) = -\text{tr}(B)/p + i \mathbb{R}$.

**Proof.** Suppose for example that $\text{tr}(B) < 0$ and let $1 \leq p < q \leq \infty$; then (2.2) implies that $\sigma_p(\mathcal{L}) \subset -\text{tr}(B)/p + i \mathbb{R}$ and $\sigma_q(\mathcal{L}) \subset -\text{tr}(B)/q + i \mathbb{R}$. If $\mu \in \mathbb{R}$, $-\text{tr}(B)/q < \mu < -\text{tr}(B)/p$ and $f \in C_0^\infty$, $f \geq 0$, $f \not\equiv 0$ we have

$$R(\mu, \mathcal{L}_q)f = \int_0^\infty e^{-\mu t} S(t)f \, dt > 0, \quad R(\mu, \mathcal{L}_p)f = -\int_0^\infty e^{\mu t} S(-t)f \, dt < 0,$$

so that for these values of $\mu$ the resolvent operators in $L^p$, $L^q$ do not coincide. Using [1, Proposition 2.2] we obtain that the resolvent operators do not coincide for $-\text{tr}(B)/q < \text{Re} \mu < -\text{tr}(B)/p$ and that $\sigma_p(\mathcal{L}) = -\text{tr}(B)/p + i \mathbb{R}$, $\sigma_q(\mathcal{L}) = -\text{tr}(B)/q + i \mathbb{R}$. The same argument applies if $\text{tr}(B) > 0$. $\square$

In the case $\text{tr}(B) = 0$ we need the following elementary result of linear algebra in order to construct a suitable function with compact support that will be used in the proof of Theorem 2.5.

**Theorem 2.4.** Suppose that $\text{tr}(B) = 0$ and that $B$ is not similar to a diagonal matrix with purely imaginary eigenvalues; then there exists an open subset $\Omega$ of $\mathbb{R}^n$ such that $\lim_{|t| \to \infty} |e^{tB}x| = \infty$, uniformly on compact subsets of $\Omega$.

**Proof.** Let $\lambda_1, \ldots, \lambda_k$ be the distinct eigenvalues of $B$ and define for $i = 1, \ldots, k$, $E_i = \text{Ker}(\lambda_i - B)^{k_i}$ where $k_i$ is the minimum positive integer such that $\text{Ker}(\lambda_i - B)^{k_i} = \text{Ker}(\lambda_i - B)^{k_i+1}$. The subspaces $E_i$ are invariant for $B$ and we have

$$\mathbb{C}^n = E_1 \oplus E_2 \oplus \cdots \oplus E_k.$$

Let further $P_i : \mathbb{C}^n \to E_i$ be the projections associated to the above decomposition.

On the subspace $E_i$ we can write $B = \lambda_i + B_i$ with $B_i^{k_i-1} \neq 0$, $B_i^{k_i} = 0$ so that for $x \in E_i$

$$e^{tB}x = e^{\lambda_i t} \sum_{j=0}^{k_i-1} \frac{t^j B^j x}{j!}.$$

If $\text{Re} \lambda_i = 0$ for $i = 1, \ldots, k$, then there is an integer $i$ such that $k_i > 1$ and we define $\Omega = \{x \in \mathbb{R}^n : B_i^{k_i-1} P_i(x) \neq 0\}$. If $\text{Re} \lambda_i > 0$, $\text{Re} \lambda_j < 0$ for some integers $i, j$, then we put $\Omega = \{x \in \mathbb{R}^n : B_i^{k_i-1} P_i(x) \neq 0, B_j^{k_j-1} P_j(x) \neq 0\}$. In both cases, $\Omega$ has the stated properties. $\square$

We can now compute the spectrum of $\mathcal{L}$ if $\text{tr}(B) = 0$ and $B$ is not similar to a diagonal matrix with purely imaginary eigenvalues.

**Theorem 2.5.** If $\text{tr}(B) = 0$ and $B$ is not similar to a diagonal matrix with purely imaginary eigenvalues, then $\sigma_p(\mathcal{L}) = i \mathbb{R}$. 
PROOF. The inclusion $\sigma_p(L) \subset i \mathbb{R}$ is clear because $(S(t))_{t \in \mathbb{R}}$ is a group of isometries. For $\varepsilon > 0$ and $f \in L^p$ we have

$$R(\varepsilon + ib, L)f = \int_0^\infty e^{-\varepsilon t} e^{-ibt} S(t)f \, dt$$

$$R(-\varepsilon + ib, L)f = -R(\varepsilon - ib, -L)f = -\int_0^\infty e^{-\varepsilon t} e^{ibt} S(-t)f \, dt.$$  

Put

$$V(\varepsilon + ib) f = R(\varepsilon + ib, L)f - R(\varepsilon - ib, L)f = \int_{-\infty}^\infty e^{-\varepsilon |t|} e^{-ibt} S(t)f \, dt$$

and suppose that $ib_0 \in \rho_p(L)$ for some $b_0 \in \mathbb{R}$. Then $ib \in \rho_p(L)$ if $|b - b_0| < \delta$ for a suitable $\delta > 0$, whence $\lim_{\varepsilon \to 0} V(\varepsilon + ib)f = 0$ for $|b - b_0| < \delta$ and $f \in L^p$.

Let $f \in C_0^\infty(\Omega), \ f \geq 0, f \neq 0$ where $\Omega$ is the set of Lemma 2.4. Then the function

$$g(t) = \int_{\mathbb{R}^n} f(e^{iB}x)f(x) \, dx$$

belongs to $C_0^\infty(\mathbb{R})$ since $|e^{iB}x| \to \infty$ as $|t| \to \infty$, uniformly over compact subsets of $\Omega$. From the equality

$$\int_{\mathbb{R}^n} (V(\varepsilon + ib)f)(x)f(x) \, dx = \int_{-\infty}^\infty e^{-\varepsilon |t|} e^{-ibt} g(t) \, dt,$$

letting $\varepsilon \to 0$ we obtain, by dominated convergence, $\hat{g}(b) = 0$ for $|b - b_0| < \delta$, where $\hat{g}$ is the Fourier transform of $g$. Since $\hat{g}$ is real analytic, it vanishes identically and hence $g \equiv 0$, in contrast with $g(0) > 0$. \(\square\)

Finally, we consider the case where $\text{tr}(B) = 0$ and $B$ is similar to a diagonal matrix with purely imaginary eigenvalues.

**Theorem 2.6.** Suppose that $B$ is similar to a diagonal matrix with non-zero eigenvalues $\pm i\sigma_1, \pm i\sigma_2, \ldots, \pm i\sigma_k$ and possibly $0$. Then $\sigma_p(L) = i \mathbb{R}$ if and only if there are eigenvalues $\sigma_r, \sigma_s$ such that $\sigma_r \sigma_s^{-1} \notin \mathbb{Q}$. In the other cases $\sigma_p(L)$ is a discrete subgroup of $i \mathbb{R}$ (independent of $p$).

**Proof.** The operator $L$ becomes, after a linear change of the independent variables,

$$L = \sum_{j=1}^k \sigma_j \left[ x_j \frac{\partial}{\partial y_j} - y_j \frac{\partial}{\partial x_j} \right]$$

where $2k \leq n$, the difference $n - 2k$ is the dimension of Ker $B$ and a point in $\mathbb{R}^n$ is denoted by $x = (x_1, y_1, \ldots, x_k, y_k, w_{2k+1}, \ldots, w_n)$. We introduce the angular coordinate $\theta_j$ in the plane $(x_j, y_j)$ and set $z_j = (x_j, y_j)$ so that

$$L = \sum_{j=1}^k \sigma_j \frac{\partial}{\partial \theta_j}, \quad S(t)f(x) = f(e^{it\sigma_1}z_1, \ldots, e^{it\sigma_k}z_k, w_{2k+1}, \ldots, w_n).$$
If \((n_1, \ldots, n_k) \in \mathbb{Z}^k\) and \(g \in C_0^\infty([1,2])\), the function \(f(x) = g(|x|)e^{i(n_1\theta_1 + \cdots + n_k\theta_k)} \in C_0^\infty\) is an eigenfunction relative to the eigenvalue \(i(n_1\sigma_1 + \cdots + n_k\sigma_k)\) and hence the subgroup

\[G = \{(n_1\sigma_1 + \cdots + n_k\sigma_k) : (n_1, \ldots, n_k) \in \mathbb{Z}^k\}\]

is contained in \(\sigma_p(L)\). If \(\sigma_t\sigma_s^{-1} \not\in \mathbb{Q}\) for some \(r, s\), then \(G\) is dense in \(i\mathbb{R}\) and the thesis follows since \(\sigma_p(L) \subset i\mathbb{R}\). In the other case, \(G\) is discrete, (2.4) shows that \((S(t))_{t \in \mathbb{R}}\) is periodic and hence \(\sigma_p(L) = G\) (see [11, Theorem IV.2.26]).

The computation of the spectrum of the group \((S(t))_{t \in \mathbb{R}}\) follows from that of its generator. In fact, Proposition 2.1 implies that \(\sigma_p(S(t)) \subset \{\mu \in \mathbb{C} : |\mu| = -t \text{ tr}(B)/p\}\) whereas the inclusion \(e^{\sigma_p(L)} \subset \sigma_p(S(t))\) follows from the general theory of semigroups (see [11, Section 3]). The results of this section then yield \(\sigma_p(S(t)) = \{\mu \in \mathbb{C} : |\mu| = -t \text{ tr}(B)/p\}\) when \((S(t))_{t \in \mathbb{R}}\) is not periodic and \(\sigma_p(S(t))\) equal to the unit circle \(\{\mu \in \mathbb{C} : |\mu| = 1\}\) or to a finite subgroup of it, in the periodic case.

3. Boundary spectrum of Ornstein-Uhlenbeck operators

We turn our attention to the Ornstein-Uhlenbeck operator defined in (1.1) and to the associated semigroup \((T(t))_{t \geq 0}\) given by (1.2). We start with the following lemma.

**Lemma 3.1.** The semigroup \((T(t))_{t \geq 0}\) is strongly continuous on \(L^p\), \(1 \leq p \leq \infty\), and satisfies the estimate

\[
\|T(t)\|_p \leq e^{-\frac{t}{p} \text{ tr}(B)}.
\]

**Proof.** Put

\[
g_t(y) = \frac{1}{(4\pi)^{n/2}(\det Q_t)^{1/2}}e^{-\langle Q_t^{-1}y, y \rangle/4};
\]

then \(\|g_t\|_1 = 1\) and \(T(t)f = S(t)(g_t * f)\), where \(S(t)\) is defined in (2.1).

Estimate (3.1) easily follows from (2.2) and Young’s inequality for convolutions. Since \(T(t)f \to f\) in \(L^p\), as \(t \to 0^+\), if \(f\) is continuous with compact support, by density (3.1) implies that \((T(t))_{t \geq 0}\) is strongly continuous for every \(1 \leq p \leq \infty\).

We now show that \(A\), with a suitable domain, is the generator of \((T(t))_{t \geq 0}\). For \(1 < p < \infty\) we define

\[
D_p(A) = \{u \in L^p \cap W^{2,p}_{\text{loc}}(\mathbb{R}^n) : Au \in L^p\}
\]

and for \(p = \infty\)

\[
D_\infty(A) = \{u \in L^\infty \cap W^{2,p}_{\text{loc}}(\mathbb{R}^n) \forall p > n : Au \in L^\infty\}.
\]

The following result is contained in [6] for \(p = \infty\) and partially in [15] for \(1 < p < \infty\).
PROPOSITION 3.2. If $1 < p \leq \infty$ the generator of $(T(t))_{t \geq 0}$ in $L^p$ is the operator $(A, D_p(A))$ and $C_0^\infty$ is a core of $(A, D_p(A))$. For $p = 1$ the generator is the closure of $A$ on $C_0^\infty$.

PROOF. If $1 < p \leq \infty$, then $(A, D_p(A))$ is a closed operator, by local elliptic regularity. Let $(A_p, D_p)$ be the $L^p$-generator of $(T(t))_{t \geq 0}$ and consider $f$ in the Schwartz class $S$. By Taylor's formula we can write

$$f(e^B x - y) = f(x) + \langle \nabla f(x), e^B x - x - y \rangle + \frac{1}{2} \langle D^2 f(x)(e^B x - x - y), e^B x - x - y \rangle + R(y)$$

with $|R(y)| \leq C|e^B x - x - y|^3$ and hence, using the function $g_t$ defined in (3.2), we obtain

$$T(t)f(x) - f(x) = \langle \nabla f(x), e^B x - x \rangle + \frac{1}{2} \langle D^2 f(x)(e^B x - x), e^B x - x \rangle + \frac{1}{2} \int_{\mathbb{R}^n} g_t(y) \left[ \langle D^2 f(x)y, y \rangle + R(y) \right] dy.$$ 

Since $f \in S$, we obtain

$$\frac{1}{t} \langle \nabla f(x), e^B x - x \rangle \to \langle Bx, \nabla f(x) \rangle, \quad \frac{1}{t} \langle D^2 f(x)(e^B x - x), e^B x - x \rangle \to 0$$

in $L^p$ as $t \to 0^+$. Next, note that

$$\frac{1}{t} \int_{\mathbb{R}^n} g_t(y) y_i y_j dy = \frac{1}{(4\pi)^{n/2} t} \int_{\mathbb{R}^n} e^{-|v|^2/4} (Q^{1/2}v)_i (Q^{1/2}v)_j dv$$

converges to

$$\frac{1}{(4\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-|v|^2/4} (Q^{1/2}v)_i (Q^{1/2}v)_j dv = 2q_{ij},$$

as $t \to 0^+$. From this fact one deduces that for $t \to 0^+$

$$\frac{1}{2t} \int_{\mathbb{R}^n} g_t(y) (D^2 f(x)y, y) dy \to \sum_{i,j=1}^n q_{ij} D_{ij} f(x)$$

in $L^p$. Arguing similarly for the remainder $R$ and using the estimate $|R(y)| \leq C|e^B x - x - y|^3$ it follows that $t^{-1} \int_{\mathbb{R}^n} g_t(y) R(y) dy \to 0$ in $L^p$, as $t \to 0^+$.

This shows that $S \subset D_p$ and that $A_p f = Af$ if $f \in S$. Since $S$ is dense in $L^p$ and $T(t)$-invariant by (1.2), it is a core for $(A_p, D_p)$ and hence $D_p \subset D_p(A)$ and $A_p f = Af$ for $f \in D_p$, since $(A, D_p(A))$ is closed.
If $u \in S$ and $\psi \in C_0^\infty$ is equal to 1 in a neighborhood of zero, the sequence $u_n(x) = \psi(x/n)u(x)$ converges to $u$ in $D_p$ with respect to the graph norm induced by $A_p$. This shows that $C_0^\infty$ is a core of $(A_p, D_p)$.

Finally we prove that $D_p = D_p(A)$. Let

$$A^* = \sum_{i,j=1}^n q_{ij}D_{ij} - \sum_{i,j=1}^n b_{ij}x_jD_i - \text{tr}(B)$$

be the formal adjoint of $A$ and let $D^*_p$ be the domain in $L^{p'}$ under which $A^*$ is the generator of the associated Ornstein-Uhlenbeck semigroup. If $u \in D_p(A)$, the equality

$$\int_{\mathbb{R}^n} Au\phi = \int_{\mathbb{R}^n} uA^*\phi$$

holds for all $\phi \in C_0^\infty$ and, by density, for all $\phi \in D^*_p$. At this point, the same argument as in Proposition 2.2 shows that $u \in D_p$. \hfill \Box

Even though we do not have an explicit description of the domain of $A$ in $L^1$, we shall denote by $D_1(A)$ the domain of $A$ as the $L^1$-generator of $(T(t))_{t \geq 0}$.

We can now prove the main result of this section, i.e. we compute the boundary spectrum of Ornstein-Uhlenbeck operators. In particular, the following result, together with those of Section 2, shows that $\sigma_p(A)$ contains a vertical line or a discrete subgroup of $i \mathbb{R}$ and hence that the semigroup $(T(t))_{t \geq 0}$ is not norm continuous.

**Theorem 3.3.** The boundary spectrum of $(A, D_p(A))$ contains the spectrum of the drift $(L, D_p(L))$.

**Proof.** We use an argument from [8]. For every $k \in \mathbb{N}$ let $V_k$ be the isometry of $L^p$ defined by

$$V_ku(x) = k^{-n/p}u(k^{-1}x).$$

If $u \in C_0^\infty$, then

$$V_k^{-1}AV_ku = k^{-2}\sum_{i,j=1}^n q_{ij}D_{ij}u + \sum_{i,j=1}^n b_{ij}x_jD_iu$$

and hence $V_k^{-1}AV_ku \to Lu$ in $L^p$, as $k \to \infty$, for every $u \in C_0^\infty$. Since $C_0^\infty$ is a core of $(L, D_p(L))$, by Proposition 2.2, we obtain the strong convergence, as $k \to \infty$, of the semigroups $V_k^{-1}T(t)V_k$ to $S(t)$, using Trotter-Kato theorems (see [11, III.4]). By [8, Corollary 13] we conclude that $\sigma_p(A, D_p(A))$ contains $\sigma_p(L, D_p(L))$. Since $\text{Re} \mu = -\text{tr}(B)/p$ for every $\mu \in \sigma_p(L, D_p(L))$ and $\sigma_p(A, D_p(A)) \subset \{\mu \in \mathbb{C} : \text{Re} \mu \leq -\text{tr}(B)/p\}$ by Lemma 3.1, the proof is complete. \hfill \Box
REMARK 3.4. We observe that the above theorem still holds in the case of bounded variable coefficients \((q_{ij}(x))\), as one immediately checks.

As a consequence of the above result we now compute the growth bound of the Ornstein-Uhlenbeck semigroup in \(L^p\), namely \(\omega_p = \lim_{t \to \infty} (1/t) \log \|T(t)\|_p\).

COROLLARY 3.5. The growth bound of \((T(t))_{t \geq 0}\) is given by \(\omega_p = -\text{tr}(B)/p\).

PROOF. From (3.1) we deduce that \(\omega_p \leq -\text{tr}(B)/p\). The results of Section 2 and Theorem 3.3 imply that the spectral bound of \(A\), \(s_p = \sup \text{Re} \mu : \mu \in \sigma_p(A)\) is equal to \(-\text{tr}(B)/p\). Since \(s_p \leq \omega_p\), we achieve the thesis. \(\Box\)

The equality \(s_p = \omega_p\) can be also deduced from [21], since \((T(t))_{t \geq 0}\) is a positive semigroup on \(L^p\).

In the sequel we shall need the adjoint of \(A\), namely

\[
A^* = \sum_{i,j=1}^{n} q_{ij} D_{ij} - \sum_{i,j=1}^{n} b_{ij} x_j D_i - \text{tr}(B).
\]

For \(1 < p \leq \infty\) we define the domain

\[
D^p(A^*) = \{ u \in L^p \cap W_{\text{loc}}^{2,p'}(\mathbb{R}^n) : A^* u \in L^{p'} \}
\]

and for \(p' = 1\), \(D_1(A^*)\) is defined as the domain of the \(L^1\)-generator of the Ornstein-Uhlenbeck semigroup associated to \(A^*\).

LEMMA 3.6. For \(1 < p < \infty\) the adjoint of \((A, D^p(A))\) is the operator \((A^*, D^p(A^*))\). For \(p = 1\), \((A^*, D_1(A^*))\) is the part of the adjoint of \((A, D_1(A))\) in \(C_0\). Similarly, for \(p = \infty\), \((A^*, D_1(A^*))\) is the part of the adjoint of \((A, D_\infty(A))\) in \(L^1\).

PROOF. Let \((T(t))_{t \geq 0}\) be the adjoint semigroup of \((T(t))_{t \geq 0}\). A direct computation shows that, for every \(f \in L^{p'}\)

\[
(T(t)f)(x) = \int_{\mathbb{R}^n} g_t(e^{tB} y) f(e^{-tB} x - y) \, dy,
\]

where \(g_t\) is defined in (3.2).

Observe that \(e^{-tB} Q_t e^{-tB^*} = \tilde{Q}_t\), where \(\tilde{Q}_t = \int_0^t e^{s(-B)} Q e^{s(-B^*)} \, ds\) and that

\[
\det(\tilde{Q}_t) = e^{-2t \text{tr}(B)} \det(Q_t)\text{ so that}
\]

\[
g_t(e^{tB} y) = \frac{e^{-t\text{tr}(B)}}{(4\pi)^{n/2}(\det \tilde{Q}_t)^{1/2}} e^{-<\tilde{Q}_t^{-1} y, y>/4}.
\]

By Proposition 3.2, the generator of \((T(t))_{t \geq 0}\) is \(A^*\) with domain given by (3.6). The statement then follows from the theory of adjoint semigroups (see [11, II.2.5]). \(\Box\)
4. - Spectrum of Ornstein-Uhlenbeck operators

In this section we compute the entire spectrum of Ornstein-Uhlenbeck operators under the hypothesis that the matrix $B$ satisfies $\sigma(B) \subset \mathbb{C}_+$ or $\sigma(B) \subset \mathbb{C}_-$. In the first case we shall prove that the spectrum of $A$ consists almost entirely of eigenvalues. The other case will be deduced by duality from this one, using Lemma 3.6.

The case $\sigma(B) \subset \mathbb{C}_-$ is the most important in the applications and is widely studied in the literature (see e.g. [6] and [14]).

From now on we suppose that $\sigma(B) \subset \mathbb{C}_+$. Instead of trying to compute directly the eigenvalues of $A$, we shall consider those of the associated semigroup.

Suppose that $f \in L^p$ satisfies $T(t)f = e^{itf}$ for every $t \geq 0$. This is equivalent to $[\hat{T}(t)f] = e^{it\hat{f}}$, where the Fourier transform is taken in the sense of (tempered) distributions.

However

$$ (T(t)f)(x) = (g_t * f)(e^{tB}x) $$

where $g_t$ is defined in (3.2) and belongs to $S$. Since

$$ \hat{g}_t(\xi) = e^{-iQ_t\xi^2}, $$

if we suppose that $\hat{f}$ is a function, we obtain

$$ [\hat{T}(t)f](\xi) = e^{-it\tr(B)}e^{-iQ_t^1/2 e^{-itB^*\xi^2}} \hat{f}(e^{-itB^*\xi}). $$

The equation $T(t)f = e^{itf}$, $(t \geq 0)$ is therefore equivalent to

$$ \hat{f}(e^{-itB^*\xi}) = e^{it\tr(B)}e^{-iQ_t^1/2 e^{-itB^*\xi^2}} \hat{f}(\xi), \quad t \geq 0. $$

We introduce the positive definite matrix

$$ Q_\infty = \int_0^\infty e^{-sB}Qe^{-sB^*}ds $$

and the function

$$ a(\xi) = e^{-(Q_\infty\xi,\xi)}. $$

The matrix $Q_\infty$ and the function $a$ have a probabilistic meaning in connection with the Ornstein-Uhlenbeck process $(U(t))_{t \geq 0}$ governed by the operator

$$ \sum_{i,j=1}^n q_{ij}D_{ij} - \sum_{i,j=1}^n b_{ij}x_jD_i, $$

as explained in the Introduction. In fact, $a$ is the Fourier transform of

$$ b(x) = \frac{1}{(4\pi)^n/2(\det Q_\infty)^{1/2}} e^{-<Q_\infty^{-1}x,x>/4} $$

and the measure $b(x)dx$ is the invariant measure of $(U(t))_{t \geq 0}$. To see this, we observe that $U(t)' = e^{t\tr(B)T(t)}$ (see Lemma 3.6) and that $b(x)dx$ is an invariant measure for $(U(t))_{t \geq 0}$ if and only if $U(t)'b = b$ for $t \geq 0$. Then the assertion follows from the above discussion and the following lemma.
THEOREM 4.1. The function $a$ satisfies the equality

$$a(e^{-tB^*}\xi) = e^{Q_t^{1/2}e^{-tB^*}\xi^2}a(\xi), \quad t \geq 0.$$ 

PROOF. We have

$$e^{-tB}Q_\infty e^{-tB^*} = \int_t^\infty e^{-sB}Qe^{-sB^*}ds = Q_\infty - e^{-tB}Q_t e^{-tB^*}.$$ 

It follows that

$$a(e^{-tB^*}\xi) = e^{-(e^{-tB}Q_\infty e^{-tB^*}\xi,\xi)} = e^{Q_t^{1/2}e^{-tB^*}\xi^2}a(\xi).$$ 

Since $b$ is in $L^p$ for every $1 \leq p \leq \infty$, it is an eigenfunction of $(A, D_p(A))$ and hence the point $-\text{tr}(B)$ belongs to the point spectrum of $(A, D_p(A))$.

The above lemma implies that a function $f$ satisfies (4.1) if and only if $v(\xi) = \hat{f}(\xi)/a(\xi)$ satisfies the equation

$$(4.4) \quad v(e^{-tB^*}\xi) = e^{(\mu + \text{tr}(B)t)}v(\xi), \quad t \geq 0.$$ 

The problem is therefore reduced to finding functions $v$ satisfying the above equation and then taking the inverse Fourier transform of $av$. Moreover, one can see, differentiating (4.4) with respect to $t$ and putting $t = 0$, that $v$ satisfies (4.4) if and only if it satisfies the first-order differential equation

$$(4.5) \quad \langle B^*\xi, \nabla v \rangle = -(\mu + \text{tr}(B))v.$$ 

The factorization $\hat{f} = av$ is equivalent to the equality $f = b \ast u$, where $u$ is the inverse Fourier transform of $v$ and everything is understood in the sense of distributions. Then (4.4) says that $u$ is invariant for the flow generated by the operator $\langle Bx, \nabla \rangle$, that is $u(e^{tB}x) = e^{\mu t}u(x)$, for $t \geq 0$. Even though we are looking for eigenfunctions rather than for invariant measures, this phenomenon is completely similar to that described in [7, Theorem 6.2.1].

To solve equation (4.4) we may suppose that $B^*$ is in the real canonical Jordan form. In fact, the change of variable $y = Mx$, where $M$ is a non-singular real $n \times n$ matrix, preserves the function spaces and transforms the operator $A$ into $\tilde{A} = \text{Tr}(\tilde{Q}D^2) + \langle \tilde{B}x, D \rangle$ with $\tilde{Q} = M^*QM$ and $\tilde{B} = M^{-1}BM$. Observe that only real matrices $M$ are allowed, since the differential operators are defined on functions of real variables. By a suitable choice of $M$, we can therefore assume that $B^*$ is in the real canonical Jordan form.

We shall argue for each Jordan block separately.
a) Suppose that $C$ is a Jordan block of size $k$ of $B^*$ relative to a real eigenvalue $\lambda > 0$, that is

$$
C = \begin{pmatrix}
\lambda & 1 & 0 & \cdots & 0 \\
0 & \lambda & 1 & \cdots & 0 \\
0 & 0 & \lambda & 1 & \vdots \\
\vdots & \vdots & 0 & \ddots & 1 \\
0 & \cdots & \cdots & \cdots & 0 & \lambda
\end{pmatrix}
$$

The characteristics of equation (4.5), with $C$ at the place of $B^*$, are given by the system

$$
\begin{align*}
\frac{d\xi_j}{ds} &= \lambda \xi_j + \xi_{j+1}, \quad 1 \leq j < k \\
\frac{d\xi_k}{ds} &= \lambda \xi_k \\
\frac{dv}{ds} &= -cv
\end{align*}
$$

with $c = \mu + \text{tr}(C)$. Integrating the system with $\xi_k$ as independent variable one obtains

$$
\begin{align*}
\frac{\xi_{k-r}}{\xi_k} &= \sum_{j=1}^{r} \frac{(-1)^{j-1}}{j! \lambda^j} \frac{\xi_{k-r+j}}{\xi_k} (\log |\xi_k|)^j + c_r, \quad 1 \leq r < k \\
v &= c_0 |\xi_k|^{-c/\lambda}
\end{align*}
$$

for suitable constants $c_r$, $0 \leq r < k$. We obtain therefore solutions of (4.5) of the form

$$
v(\xi) = |\xi_k|^{-c/\lambda} \Phi(c_1, \ldots, c_{k-1}),
$$

depending on an arbitrary function $\Phi$. In particular, for $\Phi(c_1, \ldots, c_{k-1}) = (|c_1| \ldots |c_{k-1}|)^{-\gamma}$, $\gamma \geq 0$, we obtain the following eigenfunctions

$$
v(\xi) = |\xi_k|^{-c/\lambda+(k-1)\gamma} \prod_{r=1}^{k-1} |\xi_{k-r} - \sum_{j=1}^{r} \frac{(-1)^{j-1}}{j! \lambda^j} \xi_{k-r+j} (\log |\xi_k|)^j|^{-\gamma}.
$$

b) Let now $D$ be a (real) Jordan block of size $2k$ of $B^*$ relative to conjugate eigenvalues $\lambda$, $\bar{\lambda}$. If $\{f_1, \ldots, f_k\}$ is a Jordan basis relative to $\lambda$, then $\{\tilde{f}_1, \ldots, \tilde{f}_k\}$ is a Jordan basis relative to $\bar{\lambda}$. Setting $g_{2h-1} = (f_h + \bar{f}_h)/2$, $g_{2h} = (f_h - \bar{f}_h)/2i$, we obtain a basis of $\mathbb{R}^{2k}$ which, as explained above, we assume to be the canonical basis. Since

$$
e^{tD} f_h = e^{t\lambda} \sum_{j=1}^{h} \frac{t^{h-j}}{(h-j)!} f_j, \quad e^{tD} \bar{f}_h = e^{t\bar{\lambda}} \sum_{j=1}^{h} \frac{t^{h-j}}{(h-j)!} \bar{f}_j,
$$
one has for $\xi = \sum_{j=1}^{2h} \xi_j g_j$

$$e^{tD} \xi = \sum_{j=1}^{k} \left( \sum_{h=j}^{t-h} \frac{t^{h-j}}{(h-j)!} \Re \left[ e^{i\lambda \eta_h} \right] \right) g_{2j-1} - \sum_{j=1}^{k} \left( \sum_{h=j}^{t-h} \frac{t^{h-j}}{(h-j)!} \Im \left[ e^{i\lambda \eta_h} \right] \right) g_{2j}$$

where $\eta_h = \xi_{2h-1} - i \xi_{2h}$. It follows that the functions

$$v(\xi) = |\eta_k|^{-c/(Re \lambda + (k-1)\gamma)} \prod_{r=1}^{k-1} |\eta_{k-r} - \sum_{j=1}^{r} \frac{(-1)^{j-1}}{j!(Re \lambda_j)^j} \eta_{k-r+j} (log |\eta_k|)^j|^{-\gamma}$$

($\gamma \geq 0$) satisfy (4.4) (with $D$ instead of $B^*$), if $c = \mu + tr(D)$.

c) The general case reduces to those considered above. Suppose that $B^*$ has Jordan blocks of length $2k_1, 2k_2 - 2k_1, \ldots, 2k_s - 2k_{s-1}$ relative to complex conjugate eigenvalues $\lambda_1, \lambda_1, \ldots, \lambda_s, \lambda_s$ and blocks of length $m_{s+1} - 2k_s, m_{s+2} - m_{s+1}, \ldots, m_{t} - m_{t-1}$ relative to real eigenvalues $\lambda_{s+1}, \ldots, \lambda_t$. Of course $m_t = n$. Setting $\eta_{kj} = \xi_{2k_j - 1} - i \xi_{2k_j}$, we define the functions

$$\psi_{j,r}(\eta_{kj-r+1}, \ldots, \eta_{kj}) = \sum_{h=1}^{r} \frac{(-1)^{h-1}}{h!(Re \lambda_j)^h} \eta_{kj-r+h} (log |\eta_{kj}|)^h$$

($1 \leq j \leq s$, $1 \leq r \leq k_j - 1$) and

$$\phi_{j,r}(\xi_{mj-r+1}, \ldots, \xi_{mj}) = \sum_{h=1}^{r} \frac{(-1)^{h-1}}{h! \lambda_j^h} \xi_{mj-r+h} (log |\xi_{mj}|)^h$$

($s+1 \leq j \leq t$, $1 \leq r \leq m_j - 1$). It follows that for every $\gamma_1, \gamma_2 \geq 0$ the function

$$v(\xi) = \prod_{j=1}^{s} \left[ |\eta_{kj}|^{-c_j/(Re \lambda_j + (kJ_j-1))\gamma_1} \prod_{r=1}^{k_j-1} |\eta_{kj-r} - \psi_{j,r}(\eta_{kj-r+1}, \ldots, \eta_{kj})|^{-\gamma_1} \right] \times \prod_{j=s+1}^{t} \left[ |\xi_{mj}|^{-c_j/(Re \lambda_j + (MJ_j-1))\gamma_2} \prod_{r=1}^{m_j-1} |\xi_{mj-r} - \phi_{j,r}(\xi_{mj-r+1}, \ldots, \xi_{mj})|^{-\gamma_2} \right]$$

(4.8) satisfies (4.4) with $\mu + tr(B) = c_1 + \cdots + c_s + \cdots + c_t$.

We define now

$$g(\xi) = a(\xi) v(\xi)$$

(4.9) and study when $g \in L^p$, where $1/p + 1/p' = 1$. Clearly $g \in L^\infty$ if and only if $\gamma_1 = \gamma_2 = 0$ and $Re c_j < 0$ for every $j = 1, \ldots, t$. For the general case we need the following easy lemma.
Lemma 4.2. Let $0 < \gamma < n$, $h \in L^1 \cap L^\infty$. Then there is $K > 0$ such that
\[
\int_{\mathbb{R}^n} |\xi - b|^{-\gamma} |h(\xi)| \, d\xi \leq K
\]
for all $b \in \mathbb{R}^n$.

Proof. In fact the above function is continuous in $b \in \mathbb{R}^n$ and tends to 0 as $|b| \to \infty$. \qed

Lemma 4.3. Let $1 \leq p < \infty$. Suppose that
\[
0 \leq \gamma_1 < 2/p', \quad 0 \leq \gamma_2 < 1/p'
\]
and that
\[
\text{Re } c_j < [2/p' + (k_j - 1)\gamma_1](\text{Re } \lambda_j), \quad j \leq s
\]
\[
\text{Re } c_j < [1/p' + (m_j - 1)\gamma_2]\lambda_j, \quad j > s.
\]
Then $g \in L^{p'}$.

Proof. Clearly $|g(\xi)|^{p'} \leq Ce^{-cp'|\xi|^2}|v(\xi)|^{p'}$ for some positive constants $C, c$. Using Fubini's theorem and the above lemma for $n = 1, 2$ repeatedly we obtain
\[
\int_{\mathbb{R}^n} |g(\xi)|^{p'} \, d\xi \leq C_1 \int_{\mathbb{R}^2s} e^{-cp'|\eta_1|^2 + \cdots + |\eta_{ks}|^2} \prod_{j=1}^{s} |\eta_{kj}|^{p'}(-c_j/\text{Re } \lambda_j + (k_j - 1)\gamma_1) \, d\eta \\
\times \int_{\mathbb{R}^t-s} e^{-cp'(|\xi_{ms+1}|^2 + \cdots + |\xi_{mt}|^2)} \prod_{j=s+1}^{t} |\xi_{mj}|^{p'}(-c_j/\lambda_j + (m_j - 1)\gamma_2) \, d\xi.
\]
The thesis then follows by noticing that the $\eta$ variables are two-dimensional whereas the $\xi$ variables are one-dimensional. \qed

We can now compute the $L^p$-spectrum of $A$ if $\sigma(B) \subset \mathbb{C}_+$ and $2 \leq p \leq \infty$.

Theorem 4.4. If $2 \leq p \leq \infty$, $\sigma(B) \subset \mathbb{C}_+$, then $\sigma_p(A) = \{ \mu \in \mathbb{C} : \text{Re } \mu \leq -\text{tr}(B)/p \}$. Moreover, every $\mu$ with $\text{Re } \mu < -\text{tr}(B)/p$ is an eigenvalue.

Proof. Since $\sigma_p(A) \subset \{ \mu \in \mathbb{C} : \text{Re } \mu \leq -\text{tr}(B)/p \}$, see Lemma 3.1, it is sufficient to prove the last statement.

Let $\gamma_1, \gamma_2$ and $c_j$ satisfy (4.10), (4.11), respectively. Then $g$ belongs to $L^{p'}$ by Lemma 4.3. Since $p' \leq 2$, its inverse Fourier transform $f$ belongs to $L^p$ and satisfies (4.1) with $c = \mu + \text{tr}(B) = \sum_{j=1}^{t} c_j$. Since $\gamma_1 < 2/p'$, $\gamma_2 < 1/p'$ are arbitrary it follows from (4.11) that $c = \sum_{j=1}^{t} c_j$ can be any complex number with real part strictly smaller than $\text{tr}(B)/p'$ and hence that $\mu = c - \text{tr}(B)$ is an arbitrary number with real part less than $-\text{tr}(B)/p$. Since $f$ is an eigenfunction relative to $\mu$, the proof is complete. \qed
We observe that the eigenspace relative to an eigenvalue $\mu$ is infinite-dimensional, if $n \geq 3$. In fact, one can choose different $c_j$ with the same sum $c$ and it is easy to verify that the corresponding eigenfunctions are linearly independent. The same happens if $n = 2$ and $B$ is diagonalizable, with real eigenvalues.

In the case $1 \leq p < 2$ we cannot argue as above since the Fourier transform does not map $L^p$ into $L^p$. We start with the case $\gamma_1 = \gamma_2 = 0$ in (4.8) and study the asymptotic behavior of the inverse Fourier transform of $g(\xi) = a(\xi)w(\xi)$, where

$$w(\xi) = \prod_{j=1}^{s} |\eta_{k_j}|^{a_j} \prod_{j=s+1}^{t} |\xi_{j_m}|^{b_j}$$

and $\Re a_j > -2, \Re b_j > -1$ (so that $g \in L^1$). This investigation will give the full result for $p = 1$ and will be a major step for the case $1 < p < 2$.

We need some properties of the Bessel functions $J_v$ for which we refer to [20]. We recall that $J_v(t) \approx t^v$, as $t \to 0$, $|J_v(t)| \leq Ct^{-1/2}$ as $t \to \infty$, and that

$$J_v(rt) = r^{-v-1} \frac{d}{dt} \left[(rt)^{v+1} J_{v+1}(rt)\right],$$

for $r > 0$.

We fix $h \in C^\infty_0([0, 1[)$ with support contained in $[0, 1/2[$, such that $h \equiv 1$ in $[0, 1/2[$.

**Lemma 4.5.** If $\Re \gamma + v > -1$ then the function

$$I(r) = \int_0^\infty h(t)t^v J_v(rt) \, dt$$

satisfies $|I(r)| = O(r^{-\Re \gamma - 1})$, $|I'(r)| = O(r^{-\Re \gamma - 2})$ as $r \to \infty$.

**Proof.** Integrating by parts and using the properties recalled above one obtains

$$I(r) = r^{-1} \int_0^\infty h_1(t)t^{v-1} J_{v+1}(rt) \, dt$$

where $h_1(t) = th'(t) + (\gamma - v - 1)h(t)$. Let $k$ be an integer greater that $\Re \gamma + 1$. Iterating the above procedure we have

$$I(r) = r^{-k} \int_0^\infty h_k(t)t^{v-k} J_{v+k}(rt) \, dt,$$

with $h_k \in C^\infty_0([0, \infty[)$, $\supp(h_k) \subset [0, 1[$ and $h_k$ constant in $[0, 1/2]$. Since $|J_{v+k}(t)| \leq Ct^{v+k}$ for $t \in [0, 1]$, we deduce

$$\left| \int_0^{1/r} h_k(t)t^{v-k} J_{v+k}(rt) \, dt \right| \leq C_1 r^{v+k} \left| \int_0^{1/r} t^{\Re \gamma + v} \, dt \right| = C_2 r^{k - \Re \gamma - 1}$$
and from $|J_{v+k}(t)| \leq C_{3}t^{-1/2}$ for $t \geq 1$,
\[ \left| \int_{1/r}^{\infty} h_{k}(t) t^{v-k} J_{v+k}(rt) \, dt \right| \leq C_{4}r^{-1/2} \left| \int_{1/r}^{\infty} t^{\Re \gamma-k-1/2} \, dt \right| = C_{5}r^{k-\Re \gamma-1}. \]
The estimate $|I(r)| = O(r^{-\Re \gamma-1})$ then follows. Since
\[ I'(r) = \int_{0}^{\infty} h(t) t^{v+1} J_{v}(rt) \, dt = r^{-1} \int_{0}^{\infty} \frac{d}{dr} [h(t) t^{v+1}] J_{v}(rt) \, dt \]
and $h' \equiv 0$ in $[0, 1/2]$, the estimate for $I'(r)$ follows from that of $I(r)$. \qed

**Lemma 4.6.** Let $\Re \gamma > -n$; then the function
\[ F(x) = \int_{\mathbb{R}^{n}} |\xi|^v e^{-c||\xi||^2} e^{i\xi \cdot x} \, d\xi \]
satisfies $|F(x)| = O(|x|^{-n-\Re \gamma})$, $|\nabla F(x)| = O(|x|^{-n-\Re \gamma-1})$ as $|x| \to \infty$.

**Proof.** If $n = 1$ an integration by parts gives the result (see [12, Chapter II (8)]). Suppose that $n \geq 2$ and let $h$ be as in the above lemma. It is sufficient to prove the statements for
\[ \int_{\mathbb{R}^{n}} h(|\xi||\xi|^v e^{-c||\xi||^2} e^{i\xi \cdot x} \, d\xi \]
since the difference between this function and the assigned one is the Fourier transform of a function in $S$. Let $h_1(t) = h(t) e^{-ct^2}$; then (see [19, Chapter IV, Theorem 3.3])
\[ \int_{\mathbb{R}^{n}} |\xi|^v h_1(|\xi||\xi|^v e^{i\xi \cdot x} \, d\xi = (2\pi)^{n/2}|x|^{-n/2} \int_{0}^{\infty} t^{n/2} h_1(t) J_{n/2-1}(|x| t) \, dt \]
and hence Lemma 4.5 gives the thesis. \qed

From the above lemma it follows that the inverse Fourier transform of $|\xi|^v e^{-c||\xi||^2}$ is in $L^p$ if $\Re \gamma > -n/p'$. Fubini’s theorem then implies that the inverse Fourier transform of $g_1(\xi) = e^{-c||\xi||^2} w(\xi)$, with $w$ defined in (4.12), belongs to $L^p$ provided that $\Re a_j > -2/p'$ and $\Re b_j > -1/p'$.

**Theorem 4.7.** If $\sigma(B) \subset \mathbb{C}_+$, then $\sigma_1(A) = \{ \mu \in \mathbb{C} : \Re \mu \leq -\tr(B) \}$. Moreover, if $\Re \mu < -\tr(B)$, then $\mu$ is an eigenvalue.

**Proof.** Let
\[ v(\xi) = \prod_{j=1}^{s} |\eta_{kj}|^{-c_{j}/\Re \lambda_{j}} \prod_{j=s+1}^{r} |\xi_{mj}|^{-c_{j}/\lambda_{j}} \]
with $\Re c_j < 0$ and set $g = av$. Choose $c > 0$ such that the quadratic form $C(\xi) = \langle Q_{\infty} \xi, \xi \rangle - c||\xi||^2$ is positive definite. The inverse Fourier transform $f$ of $g$ can be written as $f = f_1 * f_2$ where $f_1$ is the Fourier transform of $e^{-c||\xi||^2} v(\xi)$ and $f_2$ is the Fourier transform of $e^{-C(\xi)}$. Since $f_1 \in L^1$ by the above discussion and $f_2$ is clearly in $L^1$, $f$ belongs to $L^1$ as well and is an eigenfunction of $(A, D_1(A))$, relative to $\mu = \sum_{j=1}^{r} \Re c_j - \tr(B)$. Since $\Re c_j < 0$ is arbitrary, the statement follows as in Theorem 4.4. \qed
Finally, we consider the case $1 < p < 2$. It seems difficult to investigate the asymptotic behavior of the Fourier transform of $g$, defined by (4.9), (4.8), if $\gamma_1, \gamma_2 \neq 0$; therefore we try to compute the eigenfunctions directly. However, the method used for $p = 1$ already allows us to show that the half-plane \( \{ \mu \in \mathbb{C} : \text{Re} \mu \leq -\text{tr}(B) \} \) is contained in the point spectrum of $A$, as we show in the next lemma.

For a real matrix $B$, we define $c(B)$ as the sum of its eigenvalues, counted with their geometric multiplicities. If $\sigma(B) \subset \mathbb{C}_+$ then $c(B) \leq \text{tr}(B)$ and the equality $c(B) = \text{tr}(B)$ holds if and only if $B$ is diagonalizable.

**Lemma 4.8.** If $\sigma(B) \subset \mathbb{C}_+$, $1 < p < 2$, then the half-plane \( \{ \mu \in \mathbb{C} : \text{Re} \mu < c(B)/p' - \text{tr}(B) \} \) is contained in the point spectrum of $(A, D_p(A))$.

**Proof.** The proof is similar to that of Theorem 4.7. Defining $v$ as in (4.13) with $\text{Re} c_j < (2/p') \text{Re} \lambda_j$ for $j \leq s$ and $c_j < (1/p') \lambda_j$ for $j > s$, one verifies that $f$ is in $L^p$ and is an eigenfunction relative to $\mu = (1/p') \sum_{j=1}^s c_j - \text{tr}(B)$. \( \square \)

Since $c(B) > 0$, the set \( \{ \mu \in \mathbb{C} : \text{Re} \mu \leq -\text{tr}(B) \} \) is contained in the point spectrum of $A$; therefore, in the sequel, we shall confine ourselves to the case $-\text{tr}(B) < \text{Re} \mu < -\text{tr}(B)/p$.

We recall that the Fourier transform of

$$b(x) = \frac{1}{(4\pi)^{n/2}(\det Q_\infty)^{1/2}} e^{-\langle Q_\infty^{-1} x, x \rangle/4}$$

is the function $a$ defined in (4.3). If $u \in S'$, then $f = b * u$ belongs to $C^\infty \cap S'$, since $b \in S$. Suppose moreover that $u$ is a function satisfying

$$u(e^{tB}x) = e^{\mu t}u(x), \quad t \geq 0;$$

then $\hat{u}$ fulfills (4.4) in the sense of distributions and hence $\hat{f}(\xi) = a(\xi)\hat{u}(\xi)$ satisfies (4.1), again in the sense of distributions. Therefore such a $f$ is an eigenfunction of $(A, D_p(A))$ provided that it belongs to $L^p$.

To solve (4.14) we employ the same method used for (4.4) and observe that $u$ satisfies (4.14) if and only if it solves the first-order system

$$\langle Bx, \nabla u \rangle = \mu u.$$
(1 \leq j \leq s, 1 \leq r \leq k_j - 1) \) and

\[
\phi_{j,r}(x_{m_j-r+1}, \ldots, x_{m_j}) = \sum_{h=1}^{r} \frac{(-1)^{h-1}}{h!\lambda_j^h} x_{m_j-r+h} (\log|x_{m_j}|)^h
\]

\((s + 1 \leq j \leq t, 1 \leq r \leq m_j - 1),\) the functions

\[
u(x) = \prod_{j=1}^{s} \left[ \prod_{r=1}^{k_j-1} |z_{kj}|^{\lambda_j+(k_j-1)\gamma_1} \prod_{r=1}^{m_j-1} |x_{m_j-r} - \phi_{j,r}(x_{m_j-r+1}, \ldots, x_{m_j})|^{-\gamma_1} \right] \prod_{j=s+1}^{t} \left[ \prod_{r=1}^{m_j-1} |x_{m_j-r} - \phi_{j,r}(x_{m_j-r+1}, \ldots, x_{m_j})|^{-\gamma_2} \right]
\]

satisfy (4.14) with \(\mu = \mu_1 + \cdots + \mu_s + \cdots + \mu_t.\)

**Lemma 4.9.** Suppose that \(0 \leq \gamma_1 < 2, 0 \leq \gamma_2 < 1\) and that

\[
\Re \mu_j > [-2 - (k_j - 1)\gamma_1](\Re \lambda_j), j \leq s \quad \Re \mu_j > [-1 - (m_j - 1)\gamma_2]\lambda_j, j > s.
\]

Then the above function \(\nu \) belongs to \(S'.\)

**Proof.** From Lemma 4.2 it follows that if \(0 < \gamma < n, N > n\) there is a constant \(K\) such that

\[
\int_{\mathbb{R}^n} |x - b|^{-\gamma}(1 + |x|)^{-N} dx \leq K
\]

for every \(b \in \mathbb{R}^n.\) From this remark and Fubini’s theorem it follows that the function

\[
u(x) \prod_{j=i}^{s} (1 + |z_{kj}|^{-4} \prod_{j=s+1}^{t} (1 + |x_{m_j}|)^{-2}
\]

belongs to \(L^1,\) provided that the conditions in the statement hold. Then \(\nu \in S'.\) \(\Box\)

We consider now the function \(f = b * u\) and show that it is in \(L^p\) for certain values of the exponents \(\mu_j, \gamma_j.\) We need the following lemma.

**Lemma 4.10.** Let

\[
u(x) = \prod_{r=1}^{k-1} |x_{k-r} - \eta_r(x_{k-r+1}, \ldots, x_k)|^{-a_k-r},
\]

where \(x = (x_1, \ldots, x_k) \in \mathbb{R}^n, x_j \in \mathbb{R}^m\) for \(j = 1, \ldots, k, m/p < a_r < m,\) for \(1 \leq r \leq k,\) and the functions \(\eta_r : \mathbb{R}^m \to \mathbb{R}^m, r = 1, \ldots, k - 1,\) are Borel measurable. If \(c > 0,\) then the function \(u * e^{-c |x|^2}\) belongs to \(L^p.\)
PROOF. Set $\eta_0 \equiv 0$. If $0 \leq r < k$, we define

$$E_r = \{ x \in \mathbb{R}^n : |x_{k-r} - \eta_r(x_{k-r+1}, \ldots, x_k)| \leq 1 \}$$

and $F_r = \mathbb{R}^n \setminus E_r$. If $J \subset \{0, 1, \ldots, k - 1\}$ we introduce the sets

$$E_J = \bigcap_{r \in J} E_r \cap \bigcap_{r \notin J} F_r$$

and the functions

$$v_J(x) = \prod_{r \in J} |x_{k-r} - \eta_r(x_{k-r+1}, \ldots, x_k)|^{-\alpha_{k-r}},$$

$$w_J(x) = \prod_{r \notin J} |x_{k-r} - \eta_r(x_{k-r+1}, \ldots, x_k)|^{-\alpha_{k-r}}.$$

By construction,

$$u = \sum_{J \subset \{0, 1, \ldots, k-1\}} v_J w_J \chi_J,$$

where $\chi_J$ is the characteristic function of $E_J$. Let $(e_j)$ be the canonical basis of $\mathbb{R}^n$, $t = \sum_{j \in J} x_j e_j$ and $s = \sum_{j \notin J} x_j e_j$. Writing, with a little abuse of notation, $x = (t, s)$, one sees that there is $K > 0$ such that

$$\int_{R^{|J|}} v_J(t, s) \chi_J(t, s) dt \leq K$$

for all $s$. Moreover, $v_J w_J^p \chi_J$ is in $L^1$. These properties are easily verified since the change of variables $y_{k-r} = x_{k-r} - \eta_r(x_{k-r+1}, \ldots, x_k)$ is measure-preserving.

By Hölder’s inequality we obtain

$$\int_{R^{|J|}} v_J(t, s) w_J(t, s) \chi_J(t, s) e^{-c|\tau-t|^2} e^{-c|\xi-s|^2} dt$$

$$\leq e^{-c|\xi-s|^2} \left( \int_{R^{|J|}} v_J(t, s) \chi_J(t, s) dt \right)^{1/p'} \left( \int_{R^{|J|}} v_J(t, s) w_J^p(t, s) e^{-cp|\tau-t|^2} dt \right)^{1/p}.$$

Integrating with respect to $s$ and using again Hölder’s inequality we deduce

$$F_J(\tau, \xi) := \int_{\mathbb{R}^n} v_J(t, s) w_J(t, s) \chi_J(t, s) e^{-c|\tau-t|^2} e^{-c|\xi-s|^2} dt ds$$

$$\leq K_1 \left( \int_{\mathbb{R}^n} v_J(t, s) w_J^p(t, s) \chi_J(t, s) e^{-cp|\tau-t|^2} e^{-c|\xi-s|^2} dt ds \right)^{1/p},$$

with $K_1 = K^{1/p'} (\pi/c^2)^n$. Since $v_J w_J^p \chi_J$ is in $L^1$, $F_J$ belongs to $L^p$ and therefore $|u| \ast e^{-c|x|^2} = \sum_{J} F_J \in L^p$. 

$\square$
THEOREM 4.11. If $\sigma(B) \subseteq \mathbb{C}_+$ and $1 < p < 2$, then $\sigma_p(A) = \{\mu \in \mathbb{C} : \text{Re} \mu \leq -\text{tr}(B)/p\}$. Moreover, if $\text{Re} \mu < -\text{tr}(B)/p$, then $\mu$ is an eigenvalue.

PROOF. If $\text{Re} \mu \leq -\text{tr}(B)$, then Lemma 4.8 implies that $\mu$ is an eigenvalue. Suppose that $-\text{tr}(B) < \text{Re} \mu < -\text{tr}(B)/p$ and choose $2/p < \gamma_1 < 2, 1/p < \gamma_2 < 1, \mu_1, \ldots, \mu_t$ satisfying

\begin{align*}
-2 - (k_j - 1)\gamma_1 < (\text{Re} \mu_j)/(\text{Re} \lambda_j) < [-2/p - (k_j - 1)\gamma_1], & \quad j \leq s \\
-1 - (m_j - 1)\gamma_2 < (\text{Re} \mu_j)/\lambda_j < [-1/p - (m_j - 1)\gamma_2], & \quad j > s.
\end{align*}

such that $\mu = \mu_1 + \cdots + \mu_t$. Let $C, c > 0$ such that $|b(x)| \leq Ce^{-c|x|^2}$ and consider $f = b \ast u$. Clearly, $|f(x)| \leq C|u| \ast e^{-c|x|^2}$. To show that $f \in L^p$ it is therefore sufficient to argue for each Jordan block separately, as follows from (4.15).

Specializing Lemma 4.10 to the case $m = 1, 2, a_r = \gamma_1, \gamma_2$ for $r < k$ and $a_k = (\text{Re} \mu_j)/(\text{Re} \lambda_j) + (k_j - 1)\gamma_1$ or $a_k = \mu_j/\lambda_j + (m_j - 1)\gamma_2$, we obtain that $f = b \ast u \in L^p$ if $2/p < \gamma_1 < 2, 1/p < \gamma_2 < 1$ and (4.16), (4.17) hold. The fact that $f$ is an eigenfunction of $(A, D_p(A))$ relative to the eigenvalue $\mu$ follows from the discussion preceding Lemma 4.9.

As in the case $p > 2$, it follows that also for $1 \leq p \leq 2$ the eigenspace relative to an eigenvalue $\mu$ (with $\text{Re} \mu < -\text{tr}(B)/p$) is infinite-dimensional, if $n \geq 3$ or $n = 2$ and $B$ is a diagonalizable matrix with real eigenvalues.

We consider now the case $\sigma(B) \subseteq \mathbb{C}_-$.  

THEOREM 4.12. Let $1 \leq p \leq \infty$ and suppose that $\sigma(B) \subseteq \mathbb{C}_-$. Then $\sigma_p(A) = \{\mu \in \mathbb{C} : \text{Re} \mu \leq -\text{tr}(B)/p\}$.

PROOF. The proof follows immediately from Lemma 3.6, Theorems 4.4, 4.7 and 4.11 since, for $\text{Re} \mu < -\text{tr}(B)/p$, the adjoint operator is not injective.

5. – Further consequences

In this section we do not suppose that the spectrum of $B$ is contained in $\mathbb{C}_-$ or in $\mathbb{C}_+$ and show that in some cases the main results of the previous section still hold. However we shall make the (quite strong) assumptions that $B$ is symmetric and that $Q$ and $B$ commute. In this situation the spectrum can be determined by a tensor product argument, starting from the one-dimensional case. First of all, let us observe that the results of the preceding section yield $\sigma_p(A) = \{\mu \in \mathbb{C} : \text{Re} \mu \leq -b/p\}$ for every $1 \leq p \leq \infty$, for the one-dimensional operator $A = D^2 + bx D$, $b \neq 0$. Moreover, if $b > 0$, each complex number $\mu$ with $\text{Re} \mu < -b/p$ is an eigenvalue. This fact can be proved directly taking the Fourier transform of the equation $\mu u - u'' - bxu' = 0$,
instead of considering that of the semigroup, as done in Section 4 for general \( n \). One obtains \( \hat{u}(\xi) = e^{-q^2/2|\xi|^{1+\mu/b}} \) and then concludes that \( u \in L^p \) for \( \text{Re} \, \mu < -b/p \) using the one-dimensional version of Lemma 4.6.

We remark that, for \( n = 1 \), the domain \( D_p(A) \) is given by

\[
D_p(A) = \{ u \in L^p(\mathbb{R}) \cap W^{2,p}_{\text{loc}}(\mathbb{R}) : Au \in L^p(\mathbb{R}) \}
\]

also for \( p = 1, \infty \), since elliptic regularity holds in \( L^1(\mathbb{R}) \) and in \( C_0(\mathbb{R}) \).

The following result covers, e.g., the case where \( B \) is symmetric.

**Theorem 5.1.** If \( QB = BQ \) and \( B \) is symmetric, then, for \( 1 \leq p \leq \infty \), the spectrum of \( (A, D_p(A)) \) is the half-plane \( \{ \mu \in \mathbb{C} : \text{Re} \, \mu \leq -\text{tr}(B)/p \} \).

**Proof.** Let \( C \) be a real orthogonal matrix such that \( Q \) and \( B \) are diagonal. The change of variable \( y = Cx \) puts the operator \( A \) into the form

\[
A = \Delta + \sum_{i,j=1}^n b_{ij} x_j D_i
\]

with \( B \) symmetric.

**Theorem 5.1.** If \( QB = BQ \) and \( B \) is symmetric, then, for \( 1 \leq p \leq \infty \), the spectrum of \( (A, D_p(A)) \) is the half-plane \( \{ \mu \in \mathbb{C} : \text{Re} \, \mu \leq -\text{tr}(B)/p \} \).

**Proof.** Let \( C \) be a real orthogonal matrix such that \( C^{-1}QC \) and \( C^{-1}AC \) are diagonal. The change of variable \( y = Cx \) puts the operator \( A \) into the form

\[
A = \sum_{i=1}^n q_i D_{ii} + \sum_{i=1}^n b_i y_i D_i,
\]

where \( (q_i), (b_i) \) are the eigenvalues of \( Q \) and \( B \), respectively. Clearly, \( \sigma(A, D_p(A)) \subseteq \{ \mu \in \mathbb{C} : \text{Re} \, \mu \leq -\text{tr}(B)/p \} \). To prove the other inclusion we consider several cases separately.

a) \( b_i > 0 \) for every \( i = 1, \ldots, n \). Let \( \mu \in \mathbb{C} \) such that \( \text{Re} \, \mu < -\text{tr}(B)/p \) and consider \( \mu_i \in \mathbb{C} \) such that \( \text{Re} \, \mu_i < -b_i/p \) and \( \mu = \sum_{i=1}^n \mu_i \). If \( u_i \) is an eigenfunction, relative to \( \mu_i \), of the one-dimensional operator \( q_i D_i^2 + b_i y_i D_i \), it is immediate to check that \( u(y) = u_1(y_1) \cdots u_n(y_n) \) is an eigenfunction of \( A \) relative to \( \mu \).

b) \( b_i < 0 \) for every \( i = 1, \ldots, n \). In this case the result follows by duality from the previous one, as in the proof of Theorem 4.12.

c) Suppose now that at least one of the coefficients \( b_i \), say \( b_1 \) is strictly positive and set \( c = b_2 + \cdots + b_n \). We consider \( \mu \in \mathbb{C} \) such that \( \text{Re} \, \mu < -\text{tr}(B)/p \) and write it as \( \mu = \mu_1 - c/p \) with \( \text{Re} \, \mu_1 < -b_1/p \). The number \( -c/p \) is in the topological boundary of the spectrum of the \((n-1)\)-dimensional operator

\[
B = \sum_{i=2}^n q_i D_{ii} + \sum_{i=2}^n b_i y_i D_i.
\]

In fact, this is elementary if \( b_2 = b_3 = \cdots = b_n = 0 \) while, if some of the \( b_i \) is non-zero for \( i \geq 2 \), the topological boundary of the spectrum of \( B \) is the line
If \((v_n) \subset D_p(B)\) is an approximate eigenvector relative to \(-c/p\) and \(u\) is a normalized eigenfunction relative to \(\mu_1\) of the one-dimensional operator \(q_1 D^2 + b_1 y_1 D\), then the sequence \((w_n)\) defined by 
\[ w_n(y_1, \ldots, y_n) = u(y_1)v_n(y_2, \ldots, y_n) \]

is an approximate eigenvector relative to \(\mu\), as one immediately checks.

d) Suppose, finally, that \(b_i \leq 0\) for \(i = 1, \ldots, n\), that one of them, say \(b_1\), vanishes and another, say \(b_n\), is strictly negative. Define 
\[ c = b_2 + \cdots + b_n \]

and \(B\) as in (5.2). Then the line \(-c/p + i \mathbb{R}\) is in the approximate point spectrum of \(B\) while \([-\infty, 0]\) is the approximate point spectrum of the one-dimensional operator \(q_1 D^2\). We write a point \(\mu \in \mathbb{C}\), with \(\text{Re}\mu < -c/p\), in the form \(\mu = \alpha - c/p + ib\) with \(\alpha < 0\) and \(b \in \mathbb{R}\). If \((v_n), (u_n)\) are approximate eigenvectors of the operators \(B\) and \(q_1 D^2\), relative to \(-c/p + ib\) and \(\alpha\), respectively, then the sequence \((w_n)\) defined by 
\[ w_n(y_1, \ldots, y_n) = u_n(y_1)v_n(y_2, \ldots, y_n) \]
is an approximate eigenvector relative to \(\mu\). This completes the proof.

**REMARK 5.2.** In general it is not true that the spectrum of an Ornstein-Uhlenbeck operator is always a half-plane. A class of counterexamples is the following.

Let \(A = \Delta + \langle Bx, \nabla \rangle\) on \(L^p(\mathbb{R}^n)\), with \(B^* = -B\). The operators \(\Delta\) and \(\langle Bx, \nabla \rangle\) commute. Since the Laplacian generates a holomorphic semigroup, we can apply [2, Theorem 7.3] to deduce that the spectrum of \(A\) is contained in the algebraic sum \(\sigma(\Delta) + \sigma(\langle Bx, \nabla \rangle) = [-\infty, 0] + G\), with \(G\) a discrete subgroup of \(i \mathbb{R}\) (see Theorem 2.6), i.e. in a countable union of half-lines. A two-dimensional example of this situation is \(\Delta + xD_y - yD_x\).

We do not know whether the spectrum of an Ornstein-Uhlenbeck operator is always the algebraic sum of the spectra of its diffusion and drift terms.

We end this section by considering the spectrum of the semigroup \((T(t))_{t \geq 0}\). Clearly, \(\sigma_p(T(t)) \subset \{ \mu \in \mathbb{C} : |\mu| \leq -t \text{tr}(B)/p \}\), by (3.1). From Theorem 3.3 and the spectral inclusion \(e^{t\sigma_p(A)} \subset \sigma_p(T(t))\) we obtain that \(\sigma_p(T(t)) \supset \sigma_p(S(t))\) and hence that \(\sigma_p(T(t)) \supset \{ \mu \in \mathbb{C} : |\mu| = -t \text{tr}(B)/p \}\) if, for example, \(\sigma(B) \not\subset i \mathbb{R}\) (see the end of Section 2).

If we assume that \(\sigma(B) \subset \mathbb{C}_-\) or that \(\sigma(B) \subset \mathbb{C}_+\) or that \(B\) is symmetric and commutes with \(Q\), we obtain from Theorems 4.4, 4.7, 4.11, 5.1 and the above spectral inclusion that \(\sigma_p(T(t)) = \{ \mu \in \mathbb{C} : |\mu| \leq -t \text{tr}(B)/p \}\). Moreover, if \(\sigma(B) \subset \mathbb{C}_+\) then the point spectrum of \(T(t)\) in \(L^p\) contains the open ball \(\{ \mu \in \mathbb{C} : |\mu| < -t \text{tr}(B)/p \}\).

6. – Spectrum in \(BUC(\mathbb{R}^n)\)

We consider the spectrum of \(A\) in \(BUC\), the space of all bounded and uniformly continuous functions on \(\mathbb{R}^n\). The operator \(A\) and the semigroup
have been deeply studied in $BUC$ in [6]. Even though the semigroup is no longer strongly continuous on $BUC$, the operator $\mathcal{A}$ with domain

$$\mathcal{D}(\mathcal{A}) = \{u \in BUC(\mathbb{R}^n) \cap W^{2,p}_{\text{loc}}(\mathbb{R}^n) \forall p > n : Au \in BUC(\mathbb{R}^n)\}$$

can be regarded as a kind of generator of $(T(t))_{t \geq 0}$. In particular, its resolvent exists for $\Re \mu > 0$ and it is given by the Laplace transform of the semigroup.

Theorem 2.6 easily extends to the case of $BUC$. It is sufficient to note that the spectrum of the drift $\mathcal{L}$ in $C_0$ is contained in the approximate point spectrum of $(A, D_{\infty}(A))$ which, in turn, is contained in the approximate point spectrum of $(A, \mathcal{D}(A))$ since $D_{\infty}(A) \subset \mathcal{D}(A)$.

For the same reason, if $\sigma(B) \subset \mathbb{C}_+$, then every complex number with negative real part is an eigenvalue of $\mathcal{A}$ in $BUC$ and hence $\sigma(A, \mathcal{D}(A))$ is the left half-plane $\{\mu \in \mathbb{C} : \Re \mu \leq 0\}$.

However, in the case of $BUC$ we can prove a stronger result.

**Proposition 6.1.** If $\sigma(B) \cap \mathbb{C}_+ \neq \emptyset$, then $\sigma(A, \mathcal{D}(A))$ is the left half-plane $\{\mu \in \mathbb{C} : \Re \mu \leq 0\}$ and every complex number with negative real part is an eigenvalue.

**Proof.** We may suppose that $B$ is in the real Jordan form and that $W = R \times \mathbb{R}^k$, where $\mathbb{R}^n$ is the (generalized) eigenspace relative to the eigenvalues with positive real part. For $\Re \mu < 0$, let $u(x_1, \ldots, x_m)$ be an eigenfunction of the restriction of $\mathcal{A}$ to $BUC(\mathbb{R}^m)$. Then it is immediate to check that $u \in BUC(\mathbb{R}^n)$ is an eigenfunction of $(A, \mathcal{D}(A))$. \qed

A deeper argument is needed to deal with the case $\sigma(B) \subset \mathbb{C}_-$, which is the most important. Here we cannot use standard duality as in the previous sections since the operator is not densely defined.

**Theorem 6.2.** If $\sigma(B) \subset \mathbb{C}_-$ then the spectrum of $(A, \mathcal{D}(A))$ is the left half-plane $\{\mu \in \mathbb{C} : \Re \mu \leq 0\}$.

**Proof.** Let

$$\mathcal{A}^* = \sum_{i,j=1}^{n} q_{ij} D_{ij} - \sum_{i,j=1}^{n} b_{ij} x_j D_i - \text{tr } B$$

be the formal adjoint of $\mathcal{A}$. If $\Re \mu < 0$ we consider a particular $L^1$-eigenfunction $f$ of $(\mathcal{A}^*, D_1(\mathcal{A}^*))$ constructed in Theorem 4.7. Supposing, for example, that $-B$ has a non-real eigenvalue $\lambda_1$, we set (keeping the notation of Section 4)

$$f(x) = \int_{\mathbb{R}^n} |\eta_k|^{-\Re \mu/\Re \lambda_1} e^{-(Q_{\infty} \xi, \xi)} e^{ix \cdot \xi} d\xi,$$

with $Q_{\infty} = \int_0^\infty e^{sb} Q e^{sB^*} ds$. As in Theorem 4.7, we can write, for $c$ sufficiently small, $f = f_1 * f_2$ where

$$f_1(x) = \int_{\mathbb{R}^n} |\eta_k|^{-\Re \mu/\Re \lambda_1} e^{-c|\xi|^2} e^{ix \cdot \xi} d\xi$$
and

\[ f_2(x) = \int_{\mathbb{R}^n} e^{-\langle Q\xi, \xi \rangle + c|\xi|^2} e^{ix\cdot \xi} \, d\xi. \]

To simplify the notation we make a permutation of the coordinates to obtain \( \eta_{k_1} = \xi_1 - i \xi_2 \). Setting \( z = (x_1, x_2) \in \mathbb{R}^2 \) and \( x' = (x_3, \ldots, x_n) \in \mathbb{R}^{n-2} \), by Lemma 4.6 and using Fubini's theorem we obtain

\[
\begin{align*}
|f_1(x)| &\leq C_1(1 + |z|)^{-2+\text{Re}\mu/\text{Re}\lambda_1} e^{-\delta_1|x'|^2}, \\
|\nabla f_1(x)| &\leq C_1(1 + |z|)^{-2+\text{Re}\mu/\text{Re}\lambda_1} e^{-\delta_1|x'|^2},
\end{align*}
\]

for some positive \( C_1, \delta_1 \). Moreover, \( |f_2(x)| \leq C_2 e^{-\delta_2|x|^2} \) for suitable \( C_2, \delta_2 \). From these facts one deduces that \( f \) and \( \nabla f = \nabla f_1 \ast f_2 \) satisfy

\[
\begin{align*}
|f(x)| &\leq C(1 + |z|)^{-2+\text{Re}\mu/\text{Re}\lambda_1} e^{-\delta|x'|^2}, \\
|\nabla f(x)| &\leq C(1 + |z|)^{-2+\text{Re}\mu/\text{Re}\lambda_1} e^{-\delta|x'|^2},
\end{align*}
\]

for some positive \( C, \delta \).

Let \( \Omega (R_1, R_2) = B_2(R_1) \times B_{n-2}(R_2) \), where \( B_k(R) \) is the ball in \( \mathbb{R}^k \) with center 0 and radius \( R \).

If \( g \in \mathcal{D}(A) \) integrating by parts one has

\[
\int_{\Omega(R_1, R_2)} (fA g - gA^* f) \, dx = \int_{\partial\Omega(R_1, R_2)} \left( f \frac{\partial g}{\partial \nu} - g \frac{\partial f}{\partial \nu} + fgh \right) \, d\sigma,
\]

where \( h(x) = (Bx, \nu) \), \( \nu \) is the outward unit normal to \( \partial\Omega(R_1, R_2) \) and \( \nu = Qv \) is the conormal. Since \( f \) satisfies (6.1) and \( g \) and \( \nabla g \) are bounded in \( \mathbb{R}^n \) (see [6]), we obtain

\[
|f(x)g(x)h(x)| \leq C_3(1 + |z|)^{-1+\text{Re}\mu/\text{Re}\lambda_1} e^{-\delta_3|x'|^2},
\]

with \( C_3, \delta_3 > 0 \).

The surface integral is given by

\[
\begin{align*}
\int_{\partial B_2(R_1) \times B_{n-2}(R_2)} & \left( f \frac{\partial g}{\partial \nu} - g \frac{\partial f}{\partial \nu} + fgh \right) \, d\sigma \\
+ \int_{B_2(R_1) \times \partial B_{n-2}(R_2)} & \left( f \frac{\partial g}{\partial \nu} - g \frac{\partial f}{\partial \nu} + fgh \right) \, d\sigma.
\end{align*}
\]

Letting \( R_2 \to \infty \), with \( R_1 \) fixed, the second term tends to 0 because of the exponential decay in the \( x' \) variable whence

\[
\int_{B_2(R_1) \times \mathbb{R}^{n-2}} (fA g - gA^* f) \, dx = \int_{\partial B_2(R_1) \times \mathbb{R}^{n-2}} \left( f \frac{\partial g}{\partial \nu} - g \frac{\partial f}{\partial \nu} + fgh \right) \, d\sigma.
\]
Letting now $R_1 \to \infty$, the right hand side tends to 0 because of (6.1) and (6.2). Therefore
\[
\int_{\mathbb{R}^n} fA^*g \, dx = \int_{\mathbb{R}^n} gA^*f \, dx
\]
and
\[
\int_{\mathbb{R}^n} f(\mu g - A^*g) \, dx = \int_{\mathbb{R}^n} g(\mu f - A^*f) \, dx = 0.
\]
It follows that $\mu - A$ is not surjective and that $\mu$ is in the spectrum of $(A, \mathcal{D}(A))$.

If all the eigenvalues of $B$ are real, the proof is similar and simpler. □

From Proposition 6.1 and Theorem 6.2 the following more general result immediately follows.

**Corollary 6.3.** If $\sigma(B) \cap i \mathbb{R} = \emptyset$, then the spectrum of $(A, \mathcal{D}(A))$ is the left half-plane.

### REFERENCES


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