ANNALI DELLA
SCUOLA NORMALE SUPERIORE DI PISA
Classe di Scienze

CLAUDIO AREZZO

Minimal surfaces and deformations of holomorphic curves in Kähler-Einstein manifolds

Annali della Scuola Normale Superiore di Pisa, Classe di Scienze 4e série, tome 29, nº 2 (2000), p. 473-481

<http://www.numdam.org/item?id=ASNSP_2000_4_29_2_473_0>

Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)
Vol. XXIX (2000), pp. 473-481

Minimal Surfaces and Deformations of Holomorphic Curves in Kähler-Einstein Manifolds

CLAUDIO AREZZO

To the memory of my friend Giorgio Valli

Abstract. In this note we give explicit examples of stable nonholomorphic minimal surfaces representing classes of type $(1, 1)$ in Kähler-Einstein 4-manifolds of negative scalar curvature. We use the implicit function theorem for the area functional to deform some holomorphic rational curves as minimal surfaces for nearby Kähler-Einstein metrics, and some results in the theory of deformations of Hodge structures to prove that the generic of these deformations cannot be holomorphic for the deformed complex structure. We also show that this strategy cannot work in the Ricci-flat case, by getting a riemannian proof of the fact that a rigid nodal curve in a K3 surface can be deformed into a holomorphic curve in any direction which keeps the class of type $(1, 1)$.

Mathematics Subject Classification (1991): 58E12 (primary), 53A10 (secondary).

1. – Introduction

A classical problem in the theory of volume minimizing submanifolds in Kähler-Einstein manifolds (K-E, from now on) is to determine their relationship with holomorphic submanifolds.

We know that only classes of pure type in the Hodge decomposition can be represented by holomorphic submanifolds. If we ignore this restriction many examples are by now known of area minimizing surfaces in classes not of type $(1, 1)$, and these are therefore not holomorphic. Most of the examples known are in fact lagrangian surfaces in K-E 4-manifolds of negative scalar curvature ([13], [14], [17], [21], [24]). When restricting to $(1, 1)$ classes, the problem becomes more delicate, because these classes can be represented by a divisor (in general not effective) by Lefschetz $(1, 1)$-Theorem. We therefore end up studying also the problem of whether a non-effective divisor is a volume minimizer.

Pervenuto alla Redazione il 5 luglio 1999 e in forma definitiva il 23 febbraio 2000.

Atiyah and Hitchin ([4]) found an example of an area minimizing (as proven by Micallef-Wolfson in [16]) two-sphere in a Ricci-flat noncompact 4-manifold which cannot be holomorphic w.r.t. any compatible complex structure. For other results in the noncompact case see [1], [3] and [15].

From now on we focus our attention to *compact* K-E manifolds, M, and to its *compact* submanifolds.

Our problem depends on two parameters, the real dimension m of M, and the sign of $c_1(M)$ (equivalently, the sign of the scalar curvature of M). In the following list we summarize some of the most important results, so to highlight the missing spots, some of which will be filled in this note.

1. [12] If $M = P^m$ with the Fubini-Study metric, then every stable (i.e. minimizing up to second order) minimal surface is an algebraic cycle.
2. [19] A stable minimal two-sphere in a Kähler manifold with positive holomorphic bisectional curvature is plus or minus holomorphic.
3. [16] If $m = 4$, $c_1(M) > 0$, and the normal bundle of the stable minimal surface admits a holomorphic section, then the surface is holomorphic w.r.t. a complex structure compatible with the metric.
4. [16] If $m = 4$, $c_1(M) > 0$, then a stable totally real (i.e. without complex and anticomplex points) minimal surface has genus zero.
5. [7] If $m = 4$, $c_1(M) = 0$, and the Euler characteristic of the normal bundle of the stable minimal surface is at least -3 , then the surface is holomorphic w.r.t. a complex structure compatible with the metric.
6. [15] If M is a flat 4-torus then every stable minimal surface is holomorphic w.r.t. a complex structure compatible with the metric.
7. [2] For $k \geq 4$, every flat $m = 2k$-torus contains a stable minimal surface of genus k, which is not holomorphic w.r.t. any complex structure compatible with the metric.

In this note we will provide a general method to construct stable minimal surfaces which are not holomorphic but which still represent classes of type $(1, 1)$. We will show that the method works in a satisfying manner when $c_1(M) < 0$, does not for $c_1(M) = 0$ (which is one of the problems listed by Yau in his famous Open Problems [25]), and we will explain the difficulties left in the case of positive scalar curvature. What seems to us to enhance interest in this construction is that when the method does not work to give stable nonholomorphic minimal surfaces, it provides an application to the Infinitesimal Hodge Conjecture, which is a problem of considerable interest in Algebraic Geometry.

The main idea is to start with a holomorphic curve C_0 in a fixed K-E manifold (M, g_0) and to apply the implicit function theorem (IFT) to the area functional to construct surfaces arbitrarily close to C_0 which are minimal for some K-E metric near g_0. To use the IFT we need to start with a C_0 with no Jacobi fields, which is equivalent by a theorem of Simons ([18]) to the nonexistence of global sections of the normal bundle to C_0 in M. Moreover when this happens we can be sure that the deformed minimal immersion is in fact stable, because of the continuity of the eigenvalues of the second variation of the area operator (if they are all positive at time 0, they will stay positive at least for small variations, see [2]).

Because of the existence and uniqueness result in the solution of the Calabi conjecture for $c_1(M) < 0$, we can think of the change of K-E metric as a change in the complex structure on the underlying differentiable manifold and viceversa. We then want to find a K-E manifold with a class α in the second homology group, which is of type $(1, 1)$ for some complex structure J_0, and in fact represented by a holomorphic curve C_0, and for which there are deformations J_t in the moduli of complex structures for which α stays of type $(1, 1)$ but for which the curve C_0 cannot be deformed in a family of holomorphic representatives C_t. The existence of such classes on certain algebraic manifolds is a classical problem in Algebraic geometry, known as (Grothendieck's) Infinitesimal Hodge Conjecture. Some examples are known to exist, but, as one expects, they are very special. Indeed, if M is an algebraic hypersurface of P^3 and $C_0 \subset M$ is a complete intersection, Steenbrink ([20]) proved that such a phenomenon cannot occur. More generally, Bloch in [5] found an equivalent condition for the existence of the family C_t.

What this strategy suggests is to look at the following subspaces of the space of infinitesimal deformations of complex structures on M at J_0, $\mathcal{T}$:

1. $\mathcal{T}_\alpha = \{$infinitesimal deformations of complex structures for which α stays of type $(1, 1)\}$;
2. $\mathcal{T}_C = \{$ infinitesimal deformations of complex structures for which C_0 can be deformed in a family of holomorphic curves $\}$

Of course $\mathcal{T}_C \subset \mathcal{T}_\alpha$.

The moral is then the following: every time we find a holomorphic curve $C_0 \subset M$, whose normal bundle has no global sections and for which, having set $\alpha = [C_0]$, $\mathcal{T}_\alpha$ is nonempty and different from $\mathcal{T}_C$, then we will have for the generic nearby K-E metrics a stable non holomorphic minimal surface representing α. We show that such a curve exists at least in one example:

THEOREM 1.1. *There exists an open subset of the* 40*-dimensional moduli space of smooth quintic hypersurfaces of* P^3 *whose elements contain a nonholomorphic embedded two-sphere which is symplectic, stable, minimal for the associated K-E metrics, and of type* $(1, 1)$ *for a* 36*-dimensional family of complex structures.*

In the proof of the above result we will describe more precisely this open set in the moduli space.

Of course the case of quintic surfaces should be seen as the simplest situation when the described phenomenon occurs. In particular, by taking complex surfaces of higher degree one can construct also stable minimal surfaces with all the same properties and of higher genus.

We then go on to analyze what happens when $c_1(M) = 0$, i.e. for Calabi-Yau manifolds. If $m = 4$, the afore mentioned result of Micallef-Wolfson proves that if the normal bundle to a stable minimal surface has a global holomorphic section ($h^0(\nu\Sigma) \neq 0$), then the surface is in fact holomorphic w.r.t. some complex structure compatible with the metric. On the other hand, we know that any small deformation of a holomorphic curve satisfies the "adjunction formula", in the sense of Chen-Tian ([6], see formula (2) on page 479), and therefore use their results (formula (1) on page 479) to relate its number of complex and anticomplex points to topological quantities. In a Calabi-Yau this implies that all possible nonholomorphic minimal (not a priori necessarily stable) deformations of a holomorphic curve (regardless of how big $h^0(\nu\Sigma)$ is) are totally real, and therefore holomorphic w.r.t. a different complex structure (by a result of Wolfson, [23]), thus getting an application of minimal surface theory to the infinitesimal Hodge conjecture.

THEOREM 1.2. *Let M be a K3 surface equipped with the Ricci-flat Kähler metric, and let C_0 be an immersed holomorphic curve in M such that $h^0(\nu\Sigma) = 0$ (therefore C_0 has to be a "nodal" 2-sphere). Then for any smooth deformation of complex structure J_t on M for which $\alpha = [C_0]$ is of type $(1, 1)$, there is a smooth family C_t of curves holomorphic w.r.t. J_t.*

For $m > 4$ we do not have at our disposal anything similar to Chen-Tian and Wolfson's results, which were crucial to prove the above results. Nevertheless for a two-sphere with $h^0(\nu) = 0$ in a Calabi-Yau threefold the adjunction formula gives directly that the normal bundle has $h^1 = 0$, which implies Bloch's semi-regularity and therefore the deformability of holomorphic curves in every direction.

The last case to study is when $c_1(M) > 0$. The following is essentially contained in Micallef-Wolfson ([16]):

PROPOSITION 1.1. *Let M be a K-E 4-manifold with positive scalar curvature. If $\Sigma \subset M$ is a stable minimal immersion which satisfies the adjunction formula, then Σ is a holomorphic (or antiholomorphic) two-sphere. In particular all stable minimal deformations of holomorphic curves are holomorphic.*

This shows that small stable minimal deformations of holomorphic curves in this case have to remain holomorphic. It also follows from Chen-Tian's work, via a maximum principle argument, that a symplectic minimal surface in a K-E 4-manifold of positive scalar curvature has to be holomorphic (I owe this remark to G. Tian), which implies that all stable nonholomorphic minimal surfaces have to have lagrangian points (if they exist).

We do not know what happens when $m > 4$. As for the negative curvature case, it is possible to construct examples of holomorphic curves in K-E manifolds of positive scalar curvature, for which there are deformations of complex structures which keep the class of type $(1, 1)$, but for which a holomorphic representative does not exist. Unfortunately it is intrinsic of the Fano case that $h^0(\nu\Sigma) \neq 0$ which raises the problem of whether the deformed minimal map given by the Implicit Function Theorem is stable (now 0 being in the spectrum, some negative direction for the second variation of area operator could arise). A further difficulty arises from the possible existence of continuous families of automorphisms of Fano manifolds, which causes a great deal of problems in proving the existence of nearby Kähler-Einstein metrics. We believe this to be a very interesting and challenging problem.

Another aspect worth pointing out is that the strategy used in this note can be applied also to higher dimensional subvarieties, while, with the exception of [12], all known techniques to study this kind of problems is based on the identification of minimal immersions with conformal harmonic maps, which is true only for two dimensional domains.

We leave these questions for further investigations.

This project has been carried out while the author was at the Massachusetts Institute of Technology. It is a pleasure to thank Gang Tian for his exceptional help, mathematical and otherwise. I wish to thank also Gabriele La Nave for helping me in some of the algebraic details of this work.

2. – The implicit function theorem for the area functional

There are several versions of the IFT for minimal surfaces and harmonic maps into riemannian manifolds, depending on the spaces of maps and metrics one wants to allow. For harmonic maps it has been proved by Eells-Lemaire in [14] and extended to the area functional near a strictly stable (possibly branched) minimal immersion of a two-dimensional domain by Lee in [14], exploiting the fact that such a minimal immersion is a conformal harmonic map and thus relying on Eells-Lemaire's theorem.

The version we want to use in the following part of the paper is due to B.White ([22]). In this case the class of maps allowed are only immersions, which is enough for our applications, but might not be when looking at the problem with $m > 4$ and $c_1(M) > 0$, where allowing branched immersions might be useful.

THEOREM 2.1. *Let N and M be smooth riemannian manifolds with N compact and* $\dim N < \dim M$. *Let Γ be an open set of C^q riemannian metrics on M, and let $\mathcal{M}$ be the set of ordered pairs $(g, [u])$ where $g \in \Gamma$ and $u \in C^{j,\alpha}(N, M)$ is a g-minimal immersion (here we denote by $[u]$ the set of all maps obtained by composing a fixed one with all diffeomorphism of N).*

Then $\mathcal{M}$ is a separable C^{q-j}- Banach manifold modelled on Γ, and the map

$$\Pi : \mathcal{M} \to \Gamma$$

$\Pi(g, [u]) = g$ *is a C^{q-j} Fredholm map with Fredholm index* 0. *The kernel of $D\Pi_{(g,[u])}$ has dimension equal to the nullity of $[u]$ with respect to g (i.e. the maximum number of linearly independent normal g-Jacobi fields of u).*

3. – Deformations of Hodge structures and minimal surfaces

The IFT gives a rather clear picture of the local structure of the moduli space of minimal immersions. In this section we want to use some notions of the theory of infinitesimal variations of Hodge structures to understand the local theory of deformations of holomorphic curves, for which a standard reference is [9]. In particular we want to describe the spaces $\mathcal{T}_\alpha$ and $\mathcal{T}_C$ defined in the introduction. Let us first consider the case of M a hypersurface of P^3 and let C be a holomorphic curve in M. From now on we indicate by $N_{X/Y}$ the normal bundle of X in Y, and by α the class represented by C.

The standard inclusions then give the following sequences:

1. $0 \to \mathcal{O}_M \to \mathcal{O}_M(C) \to N_{C,M} \to 0$,
2. $0 \to N_{C,M} \to N_{C,P^3} \to N_{M,P^3} \otimes \mathcal{O}_C \to 0$,
3. $0 \to N_{M,P^3}(-C) \to N_{M,P^3} \to N_{M,P^3} \otimes \mathcal{O}_C \to 0$.

which in cohomology induce

$$H^0(N_{M,P^3}) \xrightarrow{\beta} H^0(N_{M,P^3} \otimes \mathcal{O}_C) \xrightarrow{\gamma} H^1(N_{C,M}) \xrightarrow{\delta} H^2(\mathcal{O}_M)\,.$$

It was proved by Bloch ([5]) that

1. $\mathcal{T}_\alpha = \ker(\delta \circ \gamma \circ \beta)$,
2. $\mathcal{T}_C = \ker(\gamma \circ \beta)$.

More generally (i.e. not requiring $m = 4$) we can think of $\mathcal{T}_\alpha$ as the kernel of the map $H^1(M, TM) \to H^{(n,n-2)}(M, \mathbb{C})$ obtained by contraction with the element in $H^{(n-1,n-1)}(M, \mathbb{C})$ Poincaré dual to α.

Proof of Theorem 1.1. First let us recall that the space of quintic hypersurfaces in P^3 has complex dimension $56-1 = 55$ and that of the space of quartic rational curves is 16. Moreover $\dim H^{(2,0)}(M, \mathbb{C}) = 4$ and $\dim H^0(N_{C/M}) = 0$ by adjunction. This implies that the codimension of $\mathcal{T}_\alpha$ in $H^1(M, TM)$ is at most 4.

We now want to know the dimension of the space of quintics which contain some quartic rational curve. This can be done by estimating the dimension of the space of quintic which contain a fixed (generic) quartic rational curve. This is a classic case of postulation problem, which is solved in [10] by proving that the sequence

$$H^0(\mathcal{O}_{P^3}(5)) \to H^0(\mathcal{O}_C(5)) \to 0$$

is exact. This implies directly that the generic fibre of the projection from the incidence variety

$$\{(C, M) \mid C \subset M, C \text{ rational of degree } 4\}$$

to the space of rational curves of degree 4 has dimension 30 and therefore that the incidence variety has dimension 50. Being the projection complex analytic and each C rigid in any M, the codimension of $\mathcal{T}_C$ in $H^1(M, TM)$ is 5. Recall also that $\dim H^1(M, TM) = 40$, which gives the number of effective moduli of quintic surfaces (see e.g. Kodaira ([11]) pag. 240).

Now, since $\dim H^0(N_{C/M}) = 0$, the IFT tells us that for any deformation of complex structure, there is a corresponding deformation of maps ϕ_t from S^2 to M which is minimal for the K-E metric associated to the complex structure by Yau's solution of Calabi's Conjecture. The stability is ensured by the continuity of the eigenvalues of the second variation of area operator. Moreover these two spheres are symplectic because they are smooth deformations of holomorphic curves and therefore if ω_t is the Kähler form for J_t, $\omega_{t|_{\phi_t(S^2)}}$ has to be nondegenerate everywhere for small $| t |$.

Such a map cannot be holomorphic precisely when the deformation is outside the locus $\mathcal{T}_C$ and it will keep the class of type $(1,1)$ if it is in $\mathcal{T}_\alpha$. The dimensional counting above shows that these two space do not coincide. Therefore, if $(\mathcal{M}, B, \pi)$ is a sufficiently small complete effectively parametrized complex analytic family (as in th classical theory of Kodaira, [11] pag. 215) of smooth quintic hypersurfaces with central fibre $\pi^{-1}(0) = M_0$, s.t. M_0 contains a rational curve of degree 4, then all the fibres in $\mathcal{M}$ contain a stable minimal symplectic embedded 2-sphere which cannot be holomorphic away from a 35 dimensional subfamily, but which represent a class of type $(1,1)$ in a subfamily of dimension 36. □

Proof of Theorem 1.2. Also in this case we associate to any complex structure a unique Ricci-flat K-E metric. The difference with the situation above is that given any 2-homology class and any hyperkähler metric, there exists a unique complex structure compatible with the metric for which the class is of type $(1,1)$.

Now, if we take a family of complex structures J_t for which α stays of type $(1,1)$, and we look at the corresponding minimal surfaces given by the IFT, these are either holomorphic or not. If not, Wolfson proved that under our assumptions, having called P the number of complex points and Q the number of anti-complex points of the minimal surface Σ representing α,

$$-P - Q = \chi(T\Sigma) + \chi(\nu\Sigma) Q - P = c_1(M)(\alpha). \tag{1}$$

Moreover, since α satisfies the adjunction formula,

$$\chi(T\Sigma) + \chi(\nu\Sigma) = c_1(M)(\alpha) \tag{2}$$

Therefore $P = Q = 0$, i.e. the minimal surface has to be totally real, and then holomorphic w.r.t. another complex structure on M ([23]). But this is impossible since α can be of type $(1,1)$ for a unique compatible complex structure, therefore Σ had to be holomorphic w.r.t J_t. □

PROOF OF PROPOSITION 1.1. Once again, the adjunction formula with Chen-Tian formulae (which for immersed surfaces were proved by Webster and Wolfson [23]) with $c_1(M) > 0$ imply $P = Q = 0$, so the surface has to be totally real. Micallef-Wolfson ([16], Corollary 6.2) proved that such a surface has to be a two-sphere. But then the normal bundle $\nu\Sigma$ has to be of the form $\mathcal{O}(a)$, $a \in \mathbb{Z}$. If $a \geq 0$, $h^0(\nu\Sigma) \neq 0$ and therefore Σ is holomorphic by [16].

If $a < 0$ we have $0 < c_1(M)([\Sigma]) = 2 + a - 2DP$, where DP is the number of double points of the immersion. So the only possibility is $a = -1$ and $DP = 0$. If Σ were not holomorphic then one can construct explicitely a holomorphic section of $K \otimes \nu\Sigma$ as in [1] and [16], which is impossible in our case. □

REFERENCES

[1] C. AREZZO, *Complete minimal surfaces in hyperkähler manifolds*, Compositio Math. **112** (1998), 33-40.

[2] C. AREZZO – M. J. MICALLEF, *Minimal surfaces in flat tori*, to appear in Geom. Funct. Anal.

[3] C. AREZZO – M. J. MICALLEF – G. P. PIROLA, *Stable complete minimal surfaces of finite total curvature*, preprint.

[4] M. ATIYAH – N. HITCHIN, "The Geometry and Dynamics of Magnetic Monopoles", Princeton University Press, Princeton, 1988.

[5] S. BLOCH, *Semi-regularity and deRham cohomology*, Invent. Math. **17** (1972), 51-66.

[6] J. CHEN – G. TIAN, *Minimal surfaces in Riemannian 4-manifolds*, Geom. Funct. Anal. **7** (1997), 873-916.

[7] S. K. DONALDSON, *Moment maps and diffeomorphisms*, Asian J. Math. **3** (1999), 1-16.

[8] J. EELLS – L. LEMAIRE, *Deformations of metrics and associated harmonic maps*, Proc. Indian Academy of Science **90** (1981), 33-45.

[9] P. GRIFFITHS *et al.*, *Infinitesimal variations of Hodge structure, I, II and III*, Compositio Math. **50** (1983), 109-324.

[10] A. HIRSCHOWITZ, *Sur la postulation generique des courbes rationnelles*, Acta Math. **146** (1981), 209-230.

[11] K. KODAIRA, "Complex Manifolds and Deformations of Complex Structures", Grundlehren der Mathematischen Wissenschaften n. 283, Springer-Verlag, 1986.

[12] B. LAWSON – J. SIMONS, *On stable currents and their applications to global problems in real and complex geometry*, Ann. of Math. **98** (1973), 427-450.

[13] Y.-I. LEE, *Lagrangian minimal surfaces in Kähler-Einstein surfaces of negative scalar curvature*, Comm. Anal. Geom. **2** (1994), 579-592.

[14] Y.-I. LEE, *The Deformation of Lagrangian Minimal Surfaces in Kahler-Einstein Surfaces*, J. Differential Geom. **50** (1998), 299-330.

[15] M. J. Micallef, *Stable minimal surfaces in Euclidean space*, J. Differential Geom. **19** (1984), 57-84.

[16] M. J. Micallef – J. G. Wolfson, *The second variation of area of minimal surfaces in four-manifolds*, Math. Ann. **295** (1993), 245-267.

[17] R. Schoen, *The role of harmonic mappings in rigidity and deformation problems*, Proc. of the Osaka International Conf. on Complex Geometry, G. Komatsu and Y. Sakane (eds.) (1993), 179-200.

[18] J. Simons, *Minimal varieties in riemannian manifolds*, Ann. of Math. **88** (1968), 62-105.

[19] Y. T. Siu – S. T. Yau, *Compact Kähler manifolds of postive bisectional curvature*, Invent. Math. **59** (1980), 189-204.

[20] J. Steenbrink, *Some remarks about the Hodge conjecture*, In: "Hodge Theory", Lecture Notes in Math., 1246, Springer, Berlin-New York, 1987, 165-175.

[21] S. Trapani – G. Valli, *One-harmonic maps on Riemann surfaces*, Comm. Anal. Geom. **3** (1995), 645-681.

[22] B. White, *Minimal submanifolds for varying riemannian metrics*, Indiana Univ. Math. J. **40** (1991), 161-200.

[23] J. G. Wolfson, *Minimal surfaces in Kähler surfaces and Ricci curvature*, J. Differential Geom. **29** (1989), 281-294.

[24] J. G. Wolfson, *Minimal Lagrangian diffeomorphisms and the Monge-Ampere equation*, J. Differential Geom. **46** (1997), 335-373.

[25] S.-T. Yau, *Open problems in geometry*, Proc. Symp. Pure Math. A.M.S. **54** (1993), 1-28.

Dipartimento di Matematica
Università di Genova
Via Dodecaneso 35
16146, Genova, Italy
arezzoc@dima.unige.it