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On Hulls of Meromorphy and a Class of Stein Manifolds

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To the memory of my friend G. Lupacciolu

Abstract. If X is a Stein manifold and K a compact subset one may define the meromorphy hull of K in two different ways: with respect to principal hypersurfaces or to arbitrary hypersurfaces. It is shown that the two definitions agree for every compact subset K of X if and only if the following topological condition on X is satisfied: $\mathrm{Hom}(H_2(X,\mathbb{Z});\mathbb{Z})=0$. It is also shown that this condition is equivalent to: for every hypersurface $h\subset X$ and every relatively compact open subset $D\subset\subset X$ there exists $f\in\mathcal{O}(D)$ such that $h\cap D=\{x\in D|f(x)=0\}$.

Finally, several examples are provided, which show that the topological condition $\mathrm{Hom}(H_2(X,\mathbb{Z});\mathbb{Z})=0$ is sharp.

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0. – Introduction

Let X be a Stein manifold. Then it is known (see e.g. [10], p. 181) that the second Cousin problem on X can be solved for an arbitrary divisor if and only if $H^2(X;\mathbb{Z})=0$. On the other hand one may consider the strong Poincaré problem on X: given a meromorphic function on F on X find holomorphic functions f, g on X such that the germs f_z, g_z are relative prime at any point and $F=f/g$. As in the Cousin second problem, the strong Poincaré problem can be solved for every meromorphic function F on X if and only if $H^2(X;\mathbb{Z})=0$ (see [13], p. 250). Therefore there is a strong connection between global properties of meromorphic functions on X and the purely topological invariant $H^2(X;\mathbb{Z})$. A weaker condition than $H^2(X,\mathbb{Z})=0$ is $H^2(X,\mathbb{Z})$ is of torsion. One may easily see (Proposition 2) that this is equivalent to: every hypersurface $h\subset X$ (closed analytic subset of codimension 1) can be defined globally by one equation i.e. there exists $f\in\mathcal{O}(X)$ such that one has set-theoretically $h=\{f=0\}$.

In this paper we study a class of Stein manifolds X which satisfy a weaker condition than $H^2(X,\mathbb{Z})$ is of torsion, namely we consider the topological

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condition Hom $(H_2(X;\mathbb{Z});\mathbb{Z}) = 0$. This condition is related to the study of hulls of meromorphy of compact subsets $K \subset X$.

When $K \subset X$ is a compact subset one may define in a natural way (see [8], [9]) the following two hulls:

$$_h\widehat{K} = \{x \in X \mid \text{ every hypersurface passing through } x \text{ intersects } K\}$$
$$_H\widehat{K} = \{x \in X \mid \text{ every principal hypersurface passing through } x \text{ intersects } K\}$$

Obviously $_h\widehat{K} \subset {}_H\widehat{K}$ and it is known [8] that they are compact subsets of X. When $X = \mathbb{C}^n$ then $_h\widehat{K} = {}_H\widehat{K}$ and it is called the rational convex hull of K (see [16]).

We show (Theorem 1) that on a Stein manifold X the condition

$$(\alpha) \qquad _h\widehat{K} = {}_H\widehat{K} \text{ for every compact subset } K \subset X$$

is equivalent to the topological condition

$$(\beta) \qquad \text{Hom}\,(H_2(X;\mathbb{Z});\mathbb{Z}) = 0.$$

It is also equivalent to each of the following conditions:

γ) For every hypersurface $h \subset X$ and every relatively compact open subset $D \subset\subset X$ there exists $f \in \mathcal{O}(D)$ such that $h \cap D = \{x \in D \mid f(x) = 0\}$ (in other words the hypersurfaces on X can be defined, set-theoretically, by one equation on compact subsets).

δ) For every $\xi \in H^2(X;\mathbb{Z})$ and every relatively compact open subset $D \subset\subset X$ there exists a positive integer m, depending on ξ and D, such that $m\xi\,|_D = 0$($H^2(X;\mathbb{Z})$ is of torsion on compact subsets).

Finally we give examples showing that the statement of Theorem 1 is sharp: namely, there exists Stein manifolds X satisfying the topological condition Hom $(H_2(X;\mathbb{Z});\mathbb{Z}) = 0$ but $H^2(X;\mathbb{Z})$ is not of torsion. In particular, on these manifolds there exists hypersurfaces which cannot be defined globally by one equation but they can be defined on each compact subset of X by one equation.

AKNOWLEDGEMENT. I wish to thank G. Lupacciolu who raised me in autumn 1995, when I was visiting the University of Rome “La Sapienza”, the question wheather there exist Stein manifolds X and compact subsets $K \subset X$ such that $_h\widehat{K} \neq {}_H\widehat{K}$. This was the starting point in writing this paper. In want also to thank my friend and colleague G. Chiriacescu for helpful discussions on the algebraic results needed in the given examples, in particular for the references [1] and [12].

1. – Proof of the results

Let X be a Stein manifold of dimension n. A closed analytic subset $h \subset X$ of pure dimension $(n-1)$ is called a hypersurface. h is called a principal hypersurface iff there exists $f \in \mathcal{O}(X)$ such that one has set-theoretically $h = \{f = 0\}$.

For a compact subset $K \subset X$ we consider the following two hulls:

$$_h\widehat{K} = \{x \in X \mid \text{every hypersurface passing through } x \text{ intersects } K\}$$
$$_H\widehat{K} = \{x \in X \mid \text{every principal hypersurface passing through } x \text{ intersects } K\}$$

They are compact subsets of X [8] (p. 50 and p. 52).

Let us recall also the following result [5]

Lemma 1. *Let X be a Stein manifold, $Y \subset X$ a closed complex submanifold and denote by $N = N_{Y|X}$ the normal bundle of Y in X.Then there exists an open neighborhood U of the null section of N biholomorphic to an open neighborhood U_1 of Y in X by $\varphi : U \xrightarrow{\sim} U_1$ such that the image of the null section by φ is Y.*

Now we can prove:

Proposition 1. *Let X be a Stein manifold and assume that $_h\widehat{K} = {_H\widehat{K}}$ for every compact subset $K \subset X$. Then the Kronecker product $H^2(X;\mathbb{Z}) \times H_2(X;\mathbb{Z}) \to \mathbb{Z}$ is the null map.*

Proof. Since X is Stein it follows that $H^1(X,\mathcal{O}^*) \cong H^2(X,\mathbb{Z})$. On the other hand every line bundle L on X has a section s whose zero set is smooth (see [9], p. 883) and this section defines a positive divisor which corresponds to L. Therefore it is enough to show that the Kronecker product $< c(h), \overline{\alpha} >= 0$, where h is a smooth and connected hypersurface, $c(h) \in H^2(X;\mathbb{Z})$ denotes its Chern class and $\overline{\alpha} \in H_2(X;\mathbb{Z})$. In fact $< c(h), \overline{\alpha} >$ is the intersection number $< h, \overline{\alpha} >$ of h and $\overline{\alpha}$ and can be defined choosing a smooth 2-cycle α (representing $\overline{\alpha}$) intersecting h transversally (see [7], p. 61).

By reductio ad absurdum assume that $z =< h, \overline{\alpha} > \neq 0$. Let $N = N_{h|X}$ be the normal bundle of h in X. By Lemma 1 there exists an open neighborhood U of the null section of N biholomorphic to an open neighborhood U_1 of h in X by $\varphi : U \xrightarrow{\sim} U_1$, such that the image of the null section by φ is h. We choose a hermitian metric on N such that $\{w \in N \mid \|w\| \leq 1\} \subset U$ and define $V = \varphi(\{w \in N \mid \|w\| \leq 1\})$. Then V is a closed neighborhood of h and by Thom's isomorphism $H_2(X, X\setminus \overset{\circ}{V}; \mathbb{Z}) \cong H_0(h;\mathbb{Z}) \cong \mathbb{Z}$. In fact, if $x_0 \in h$ is any point and $B_{x_0} = B(x_0, 1) \subset V$ denotes the corresponding closed ball with center x_0 and contained in the fiber then B_{x_0} can be considered as a 2-simplex s of X with boundary contained in $X\setminus \overset{\circ}{V}$, so it defines an element $\overline{s} \in H_2(X, X\setminus \overset{\circ}{V}; \mathbb{Z}) \cong \mathbb{Z}$ and $\overline{s}$ is a generator. In what follows we fix some point $x_0 \in h$.

Consider the exact sequence:

$$H_2(X\setminus \overset{\circ}{\mathrm{V}};\mathbb{Z}) \to H_2(X;\mathbb{Z}) \overset{i}{\to} H_2(X, X\setminus \overset{\circ}{\mathrm{V}};\mathbb{Z}) \cong \mathbb{Z}$$

and let us remark that $i(\overline{\alpha}) \neq 0$ (here i denotes the natural homomorphism at homology induced by the map $(X, \emptyset)\to(X, X\setminus \overset{\circ}{\mathrm{V}})$ which is the identity on X). Otherwise $\alpha = u + v$ with $u =$ boundary in X, $v =$cycle in $X\setminus \overset{\circ}{\mathrm{V}}$ and it would follow that $z =< h, \overline{\alpha} >= 0$ since v does not meet h. Therefore $i(\overline{\alpha}) = \lambda\overline{s}$ with $\lambda \in \mathbb{Z}\setminus\{0\}$. We see that

$$(*) \qquad \alpha - \lambda s = \alpha_1 + b$$

with $\alpha_1 =$ chain in $X\setminus \overset{\circ}{\mathrm{V}}$ and $b =$boundary in X (and in fact $\lambda = z$).
Let us define the compact set $K =$ supp $(\alpha_1) \cup \partial B_{x_0}$. By our assumption $_h\widehat{K} = {}_H\widehat{K}$. But $x_0 \in h$ and $h \cap K = \emptyset$ (empty set), so there exists a principal hypersurface $H = \{f = 0\}$, $f \in \mathcal{O}(X)$ with $x_0 \in H$ and $H \cap K = \emptyset$. We have $< H, \alpha - \lambda s >=< H, \alpha_1 + b >$ where $<,>$ denotes the intersection number. On the other hand $< H, \alpha >= 0$ since H is principal, $< H, b >= 0$ since b is a boundary, $< H, \alpha_1 >= 0$ since $H \cap$supp $(\alpha_1) = \emptyset$. But $< H, s >\neq 0$ because $H \cap \partial B_{x_0} = \emptyset$, $H(x_0) = 0$ and on B_{x_0} we have a complex structure (see [7] p. 63, [9] Lemme 5.3). We get $\lambda = 0$ which is a contradiction. Therefore $< h, \overline{\alpha} >= 0$ and the proof of Proposition 1 is complete.

THEOREM 1. *Let X be a Stein manifold. Then the following conditions are equivalent:*

1) $_h\widehat{K} = {}_H\widehat{K}$ *for every compact subset K of X*

2) Hom $(H_2(X;\mathbb{Z});\mathbb{Z}) = 0$

3) *For every $\xi \in H^2(X;\mathbb{Z})$ and every relatively compact open subset $D \subset\subset X$ there exists a positive integer $m = m(D, \xi)$ such that $m\xi\ |_D = 0$.*

4) *For every hypersurface $h \subset X$ and every relatively compact open subset $D \subset\subset X$ there exists a holomorphic function $f \in \mathcal{O}(D)$ such that one has set-theoretically $h \cap D = \{f = 0\}$.*

PROOF. 1) $\Longrightarrow$ 2)

It is known ([6], p. 132) that the natural morphism (induced by the Kronecker product) $H^2(X;\mathbb{Z}) \to$ Hom $(H_2(X;\mathbb{Z});\mathbb{Z})$ is surjective. Therefore every $u \in$ Hom$(H_2(X;\mathbb{Z});\mathbb{Z})$ is of the form $u(\overline{\alpha}) =< \xi, \overline{\alpha} >$ for some $\xi \in H^2(X;\mathbb{Z})$. By Proposition 1 it follows that Hom $(H_2(X;\mathbb{Z});\mathbb{Z}) = 0$.

2) $\Longrightarrow$ 3)

Let $\xi \in H^2(X;\mathbb{Z}), D \subset\subset X$ a relatively compact open subset, and we have to find a positive integer m such that $m\xi\ |_D = 0$. We may assume that the boundary of D is smooth, therefore the homology groups of D are finitely generated. We define $\xi_1 = \xi\ |_D \in H^2(D;\mathbb{Z})$. By our assumption $< \xi_1, \overline{\alpha} >= 0$ for every $\overline{\alpha} \in H_2(D;\mathbb{Z})$. Since the homology groups of D are finitely generated it follows from ([6], p. 136) that ξ_1 is a torsion element of $H^2(D;\mathbb{Z})$.

3) $\Longrightarrow$ 4)

Let $h \subset X$ be a hypersurface and $D \subset\subset X$ a relatively compact open subset. We may assume that D is Stein, therefore $H^2(D;\mathbb{Z}) \cong H^1(D,\mathcal{O}^*) \cdot h$ defines a line bundle $L \in H^1(X,\mathcal{O}^*)$ which has a canonical section $s \in \Gamma(X,L)$ with $h = \{s = 0\}$. Define $\xi = c(L) \in H^2(X;\mathbb{Z})$ the Chern class of L. By our assumption there exists a positive integer m with $m\xi = 0$ on D. Therefore $L^m \mid_D$ is trivial. Also $s^m \in \Gamma(X, L^m)$ and $f = s^m \mid_D$ is a holomorphic function on D with $h \cap D = \{f = 0\}$.

4) $\Longrightarrow$ 1)

Let $K \subset X$ be a compact subset and $x_0 \in X$ such that there exists a hypersurface $h \subset X$ with $x_0 \in h$ and $h \cap K = \emptyset$. We have to find a principal hypersurface $H \subset X$ with $x_0 \in H$ and $H \cap K = \emptyset$. Let $D \subset\subset X$ be a Runge domain with $K \cup \{x_0\} \subset D$. By our assumption there is a holomorphic function $H_1 \in \mathcal{O}(D)$ with $\{H_1 = 0\} = h \cap D$ (set-theoretically).
Let $\varepsilon_0 = \inf\{|H_1(x)| \mid x \in K\} > 0$. Since $D \subset X$ is Runge we can approximate H_1 on $K \cup \{x_0\} \subset D$ by $\widetilde{H}_1 \in \mathcal{O}(X)$ such that $|\widetilde{H}_1(x) - H_1(x)| \leq \varepsilon_0/4$ if $x \in K$ and $|\widetilde{H}_1(x_0)| = |\widetilde{H}_1(x_0) - H_1(x_0)| \leq \varepsilon_0/4$. Define $H(x) = \widetilde{H}_1(x) - \widetilde{H}_1(x_0)$. Then obviously $H(x_0) = 0$ and $H(x) \neq 0$ if $x \in K$.

Thus our theorem is completely proved.

Corollary 1. *Let X be a Stein manifold such that $H_1(X;\mathbb{Z})$, $H_2(X;\mathbb{Z})$ are finitely generated (e.g. X is affine algebraic).*
Then the following conditions are equivalent:

1) ${}_h\widehat{K} = {}_H\widehat{K}$ *for every compact subset* $K \subset X$.
2) $H^2(X;\mathbb{Z})$ *is of torsion*
3) *For every hypersurface $h \subset X$ there exists $f \in \mathcal{O}(X)$ such that one has set-theoretically $h = \{f = 0\}$.*

Proof. Since $H_1(X;\mathbb{Z})$, $H_2(X;\mathbb{Z})$ are finitely generated it follows from ([6], p. 136) that we have a (non-canonical) isomorphism:

$$(*) \qquad H^2(X;\mathbb{Z}) \cong \mathrm{Hom}\,(H_2(X;\mathbb{Z});\mathbb{Z}) \oplus T_1$$

where T_1 denotes the torsion part of $H_1(X;\mathbb{Z})$.
Now the corollary follows immediately from $(*)$ and Theorem 1.

Proposition 2. *Let X be a connected Stein manifold. Then the following two conditions are equivalent:*

1) $H^2(X;\mathbb{Z})$ *is of torsion*
2) *For every hypersurface $h \subset X$ there exists $f \in \mathcal{O}(X)$ such that one has set-theoretically $h = \{f = 0\}$.*

We first show that 1) $\Longrightarrow$ 2).

Let $h \subset X$ be a hypersurface and let $L \in H^1(X,\mathcal{O}^*)$ be the corresponding line bundle, therefore there is a canonical section $s \in \Gamma(X,L)$ with $h = \{s = 0\}$. Since X is Stein $H^2(X;\mathbb{Z}) \cong H^1(X,\mathcal{O}^*)$, hence there is a positive integer m

such that L^m is trivial. s^m is a section in $\Gamma(X, L^m)$ and if we set $f = s^m$ then f is a holomorphic function on X such that $h = \{f = 0\}$.

We prove now that 2) $\Longrightarrow$ 1).

We recall the following result (see [3]): If L is a line bundle over a connected Stein manifold X then there is a section $s \in \Gamma(X, L)$ such that $\{s = 0\}$ is irreducible (in fact the set of sections $s \in \Gamma(X, L)$ with $\{s = 0\}$ irreducible is dense in $\Gamma(X, L)$).

Let now $\xi \in H^2(X;\mathbb{Z}) \cong H^1(X, \mathcal{O}^*)$ and let $L \in H^1(X, \mathcal{O}^*)$ be the corresponding line bundle. We choose $s \in \Gamma(X, L)$ such that $h = \{s = 0\}$ is irreducible. If we consider (h) as a divisor there is a positive integer n such that $L = n(h)$ (n is the order of s along h, which is well defined because h is irreducible). On the hand there exists $f \in \mathcal{O}(X)$ with $h = \{f = 0\}$ (set-theoretically). If m is the order of f along h then $m(h) = 0$. Therefore L^m is the trivial line bundle and consequently $m\xi = 0$. So we have showed that $H^2(X;\mathbb{Z})$ is of torsion, and the proof of Proposition 2 is complete.

Remark 1. There is a surjective homomorphism group (see [6], p. 132)

$$H^2(X;\mathbb{Z}) \to \mathrm{Hom}(H_2(X;Z);\mathbb{Z})$$

from which it follows that:

$$H^2(X;\mathbb{Z}) \text{ is of torsion } \Longrightarrow \mathrm{Hom}(H_2(X;\mathbb{Z});\mathbb{Z}) = 0\,.$$

We shall give examples of Stein manifolds X such that $\mathrm{Hom}(H_2(X;\mathbb{Z});\mathbb{Z}) = 0$ but $H^2(X;\mathbb{Z})$ contains nontorsion elements. Of course, for such Stein manifolds X, every $\xi \in H^2(X;\mathbb{Z})$ is of torsion on compact subsets, i.e. for every $D \subset\subset X$ there is a positive integer $m = m(D, \xi)$ with $m\xi = 0$ on D. But it is possible that $m \to \infty$ as it will be shown by our next examples.

Example 1. In [11] it is given an example of a Stein domain $X \subset \mathbb{C}^2$ with $H_1(X;\mathbb{Z}) = \mathbb{Q}$ (rational numbers) and $H_2(X;\mathbb{Z}) = 0$.
Let us study $H^2X;\mathbb{Z})$. There is an exact sequence ([4], p. 153):

$$0 \to \mathrm{Ext}(H_1(X;\mathbb{Z});\mathbb{Z}) \to H^2(X;\mathbb{Z}) \to \mathrm{Hom}(H_2(X;\mathbb{Z});\mathbb{Z}) \to 0$$

Therefore we get $H^2(X;\mathbb{Z}) = \mathrm{Ext}(\mathbb{Q};\mathbb{Z})$. Clearly $\mathrm{Ext}(\mathbb{Q};\mathbb{Z})$ is a $\mathbb{Q}$ vector space, so every $\xi \in \mathrm{Ext}(\mathbb{Q};\mathbb{Z})\backslash\{0\}$ is a nontorsion element. We shall prove that $\dim_{\mathbb{Q}} \mathrm{Ext}(\mathbb{Q};\mathbb{Z}) = \infty$.

If p is a prime we denote by P =the additive group of those rational numbers whose denominators are powers of p and by $\mathbb{Z}(p^\infty)$ the quotient $\mathbb{Z}/P$. There is a group isomorphism (see [12], p. 6): $\mathbb{Q}/\mathbb{Z} \cong \oplus\mathbb{Z}(p^\infty)$. It follows:

$$\mathrm{Hom}(\mathbb{Q};\mathbb{Q}/\mathbb{Z}) = \mathrm{Hom}(\mathbb{Q};\oplus\mathbb{Z}(p^\infty)) = \oplus\mathrm{Hom}(\mathbb{Q};\mathbb{Z}(p^\infty))\,.$$

Now for every prime p $\mathrm{Hom}(\mathbb{Q}; \mathbb{Z}(p^\infty)) \neq 0$ since we have a surjective homomorphism $\mathbb{Q} \to \mathbb{Z}(p^\infty)$ obtained from the composition of two surjective homomorphisms $\mathbb{Q} \to \oplus\mathbb{Z}(p^\infty) \xrightarrow{\mathrm{pr}} \mathbb{Z}(p^\infty)$. If follows that $\dim_{\mathbb{Q}} \mathrm{Hom}(\mathbb{Q}; \mathbb{Q}/\mathbb{Z}) = \infty$.

From the exact sequence $0 \to \mathbb{Z} \to \mathbb{Q} \to \mathbb{Q}/\mathbb{Z} \to 0$ applying $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Q}; \cdot)$ we get the exact sequence of $\mathbb{Q}$ vector spaces:

$$0 \to \mathrm{Hom}(\mathbb{Q}; \mathbb{Z}) \to \mathrm{Hom}(\mathbb{Q}; \mathbb{Q}) \to \mathrm{Hom}(\mathbb{Q}; \mathbb{Q}/\mathbb{Z}) \to \mathrm{Ext}(\mathbb{Q}; \mathbb{Z}) \to \mathrm{Ext}(\mathbb{Q}; \mathbb{Q}) = 0$$

Since $\mathrm{Hom}(\mathbb{Q}; \mathbb{Z}) = 0, \mathrm{Hom}(\mathbb{Q}; \mathbb{Q})$ has dimension 1 as a $\mathbb{Q}$ vector space and $\dim_{\mathbb{Q}} \mathrm{Hom}(\mathbb{Q}; \mathbb{Q}/\mathbb{Z}) = \infty$ it follows that $\dim_{\mathbb{Q}} \mathrm{Ext}(\mathbb{Q}; \mathbb{Z}) = \infty$.

EXAMPLE 2. For each integer $m \geq 2$ consider the map $\varphi : \overline{D} \to \mathbb{R}^4(\overline{D}$ is the closed unit disc, i.e. $\overline{D} = \{z \in \mathbb{C} \mid |z| \leq 1)$ given by $\varphi(z) = (z^m, (1 - |z|)z)$ where $\mathbb{R}^4$ is identified with $\mathbb{C}^2$ in the usual way. Then $\varphi\mid_D$ si injective and $\varphi\mid_{\partial D}$ has degree m.
Since $\partial D = S^1$, if we set $K_m = \varphi(\overline{D})$, then K_m is obtained from S^1 by adding a two cell by a map of degree m (see [6], p. 83). It follows from ([6], p. 89) that $H_1(K_m; \mathbb{Z}) = \mathbb{Z}/m\mathbb{Z}$ and $H_2(K_m; \mathbb{Z}) = 0$.
Consider in $\mathbb{R}^4$ an infinite real line d, and on d we add the compacts K_m such that $K_m \cap d = a$ point $P_m, K_m \cap K_n = \emptyset$ if $m \neq n$ and $\{K_m\}$ is locally finite. Thus we get a locally finite cellular complex $M \subset \mathbb{R}^4$. One may easily see that $H_1(M; \mathbb{Z}) = \oplus_{m\geq 2}\mathbb{Z}/m\mathbb{Z}$ and $H_2(M; \mathbb{Z}) = 0$.
From the exact sequence:

$$0 \to \mathrm{Ext}(H_1(M; \mathbb{Z}); \mathbb{Z}) \to H^2(M; \mathbb{Z}) \to \mathrm{Hom}(H_2(M; \mathbb{Z}); \mathbb{Z}) \to 0$$

we get $H^2(M; \mathbb{Z}) = \mathrm{Ext}(\bigoplus_{m\geq 2}\mathbb{Z}/m\mathbb{Z}; \mathbb{Z})$. From ([1], § 5, Prop. 7, p. 89) $\mathrm{Ext}(\oplus G_m; \mathbb{Z}) \cong \prod \mathrm{Ext}(G_m; \mathbb{Z})$ and by ([4], p. 148) $\mathrm{Ext}(\mathbb{Z}/m\mathbb{Z}; \mathbb{Z}) = \mathbb{Z}/m\mathbb{Z}$. We deduce that $H^2(M; \mathbb{Z}) = \prod_{m\geq 2}\mathbb{Z}/m\mathbb{Z}$. We take now and open neighborhood U of M in R^4 such that M is a deformation retract of U. Considering the inclusion $R^4 \subset \mathbb{C}^4$ given by $y_1 = \ldots = y_4 = 0$ where $z_k = x_k + iy_k$ are the complex coordinates on $\mathbb{C}^4$, there exists by [14] a Stein domain $X \subset \mathbb{C}^4$ such that $X \cap \mathbb{R}^4 = U$ and U is a deformation retract of X. We have $H^2(X; \mathbb{Z}) = \prod_{m\geq 2}\mathbb{Z}/m\mathbb{Z}$ and $H_2(X; \mathbb{Z}) = 0$. The element $(\widehat{1}, \widehat{1}, \ldots \widehat{1}, \ldots)$ (taking $\widehat{1}$ on all factors of the infinite product) is a nontorsion element of $H^2(X; \mathbb{Z})$.

EXAMPLE 3. In examples 1) and 2) we have $H_2(X; \mathbb{Z}) = 0$. But it is possible to find X with $H_2(X; \mathbb{Z}) \neq 0$, $\mathrm{Hom}(H_2(X; \mathbb{Z}); \mathbb{Z}) = 0$ and $H^2(X; \mathbb{Z})$ has nontorsion elements. To see this we replace in example 1) X by $X_1 = X \times \{z \in \mathbb{C} \mid 0 < |z| < 1\}$. Then by Künneth formula $H_2(X_1; \mathbb{Z}) = \mathbb{Q}$ and $H_1(X_1; \mathbb{Z}) = \mathbb{Z} \oplus \mathbb{Q}$. If follows that $\mathrm{Hom}(H_2(X_1; \mathbb{Z}); \mathbb{Z}) = 0$ and $H^2(X_1; \mathbb{Z}) = \mathrm{Ext}(H_1(X_1; \mathbb{Z}); \mathbb{Z}) = \mathrm{Ext}(\mathbb{Z} \oplus \mathbb{Q}; Z) = \mathrm{Ext}(\mathbb{Q}; \mathbb{Z}) \neq 0$.
Similarly we may replace X in example 2) by its product with $\{z \in \mathbb{C} \mid 0 < |z| < 1\}$.

EXAMPLE 4. By [15] it is possible to construct, for every countable torsion free abelian group G, a compact connected subset $K \subset \mathbb{R}^3$ (in fact a curve) such that $H_1(\mathbb{R}^3 \backslash K; \mathbb{Z}) \cong G$ and $H_2(\mathbb{R}^3 \backslash K; \mathbb{Z}) = 0$. Taking G such that $\mathrm{Ext}(G; \mathbb{Z})$ contains nontorsion elements and $X \subset \mathbb{C}^3$ a Stein open subset such that $X \cap \mathbb{R}^3 = \mathbb{R}^3 \backslash K$ and $\mathbb{R}^3 \backslash K$ is a deformation retract of X, one gets as above examples of Stein manifolds X with $\mathrm{Hom}(H_2(X; \mathbb{Z}); \mathbb{Z}) = 0$ but $H^2(X; \mathbb{Z})$ has nontorsion elements.

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