ANNALI DELLA
SCUOLA NORMALE SUPERIORE DI PISA
Classe di Scienze

CIRO CILIBERTO
KLAUS HULEK

A series of smooth irregular varieties in projective space

Annali della Scuola Normale Superiore di Pisa, Classe di Scienze 4^e série, tome 28, nº 2 (1999), p. 357-380

<http://www.numdam.org/item?id=ASNSP_1999_4_28_2_357_0>

© Scuola Normale Superiore, Pisa, 1999, tous droits réservés.

L'accès aux archives de la revue « Annali della Scuola Normale Superiore di Pisa, Classe di Scienze » (http://www.sns.it/it/edizioni/riviste/annaliscienze/) implique l'accord avec les conditions générales d'utilisation (http://www.numdam.org/conditions). Toute utilisation commerciale ou impression systématique est constitutive d'une infraction pénale. Toute copie ou impression de ce fichier doit contenir la présente mention de copyright.

NUMDAM
Article numérisé dans le cadre du programme
Numérisation de documents anciens mathématiques
http://www.numdam.org/

Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)
Vol. XXVIII (1999), pp. 357-380

A Series of Smooth Irregular Varieties in Projective Space

CIRO CILIBERTO - KLAUS HULEK

Abstract. One of the simplest examples of a smooth, non degenerate surface in $\mathbb{P}^4$ is the quintic elliptic scroll. It can be constructed from an elliptic normal curve E by joining every point on E with the translation of this point by a non-zero 2-torsion point. The same construction can be applied when E is replaced by a (linearly normally embedded) abelian variety A. In this paper we ask the question when the resulting scroll Y is smooth. If A is an abelian surface embedded by a line bundle L of type (d_1, d_2) and $r = d_1 d_2$, then we prove that for general A the scroll Y is smooth if r is at least 7 with the one exception where $r = 8$ and the 2-torsion point is in the kernel $K(L)$ of L. In this case Y is singular. The case $r = 7$ is particularly interesting, since then Y is a smooth threefold in $\mathbb{P}^6$ with irregularity 2. The existence of this variety seems not to have been noticed before. One can also show that the case of the quintic elliptic scroll and the above case are the only possibilities where Y is smooth and the codimension of Y is at most half the dimension of the surrounding projective space.

Mathematics Subject Classification (1991): 14M07 (primary), 14N05, 14K99 (secondary).

0. - Introduction

One of the simplest examples of a smooth, non degenerate surface in $\mathbb{P}^4$ is the quintic elliptic scroll Y. Its construction goes as follows. Let A be an elliptic normal curve of degree 5 in $\mathbb{P}^4$ and let ϵ be a non zero point of order two on A. Then the union of all the lines joining pairs of points of type x and $x + \epsilon$ on A is an elliptic quintic scroll.

Exactly the same construction can be repeated starting from any abelian variety A of dimension n, with A linearly normally embedded in a projective space $\mathbb{P}^N$ via a very ample line bundle $\mathcal{L}$, and from any non trivial point $\epsilon \in A$ of order two. We investigate this construction in the present paper. In this way we get a scroll Y of dimension $n + 1$ in $\mathbb{P}^N$ related to the above data

The present collaboration took place in the framework of the HCM contract AGE (Algebraic Geometry in Europe), no. ERBCHRXCT940557.

Pervenuto alla Redazione il 20 ottobre 1998 e in forma definitiva il 25 febbraio 1999.

$(A, \epsilon, \mathcal{L})$ and the first interesting question is: when is Y smooth? It is well known that this is the case if $n = 1$ and $N \geq 4$. So the next interesting case is that of surfaces, i.e. $n = 2$, embedded in $\mathbb{P}^{r-1}$ via a (d_1, d_2)-polarization, with $r = d_1 \cdot d_2$. If $r \leq 6$ there is no hope for Y to be smooth because of Lefschetz's hyperplane section theorem. So the question becomes relevant as soon as $r \geq 7$. In fact the main part of this paper is devoted to proving that if A is general in its moduli space (it is enough to assume that $\mathrm{End}(A) \simeq \mathbb{Z}$ or $NS(A) \simeq \mathbb{Z}$ depending on the case under consideration), and if $r \geq 7$ and $r \neq 8$, then Y is smooth. This is particularly remarkable in the case $r = 7$, since Y is then an irregular, codimension three manifold in $\mathbb{P}^6$, whose existence does not seem to have been previously noticed. As we remark at the end of Section 2, for no other dimension of A, but 1 and 2, and $N = 4$ and $N = 6$ respectively, Y can be smooth of codimension $c \leq \frac{N}{2}$ in $\mathbb{P}^N$. The case $d = 8$ is also interesting. If A is a general abelian surface with a polarization of type $(1, 8)$, then Y is smooth, unless the translation by the point ϵ of order two fixes the polarization, in which case Y is singular. If the polarization is of type $(2, 4)$, then the translation by ϵ automatically fixes the polarization and Y is again singular.

The paper is organised as follows. In Section 1 we present the construction of a suitable projective bundle X over $\bar{A} = A/\epsilon$ which maps to Y via its tautological line bundle. In Section 2 we prove that this map is finite and we compute the double point cycle of the composite map of $X \to Y$ with a general projection in a $\mathbb{P}^l$, with $n + 1 \leq l \leq N$. From Section 3 on we restrict our attention to the case of abelian surfaces. In particular in Section 3 we prove that Y is smooth as soon as $r \geq 10$. This comes as a consequence of the fact that, in this situation, if A is general enough, then it has no quadrisecant plane. A property which, in turn, follows as an application of Reider's method. Finally in Section 4 we prove that Y is smooth if $r = 7$, in Section 5 we analyse the case $r = 8$ and in Section 6 the case $r = 9$. The idea for the proof that Y is smooth and the tools we use in the cases $r = 7$, $r = 8$ and the polarization is of type $(1, 8)$ with ϵ not fixing it, and $r = 9$ and the polarization is the triple of a principal polarization (which is the only critical case for $r = 9$) are the same: we first bound dimension and degree of the possible singular locus of Y by using geometric arguments and the double point formula, then we use the action of the Heisenberg group to give a lower bound for the degree of the singular locus, finally contradicting the previous estimate.

ACKNOWLEDGMENTS. We are very grareful to the referee who has not only made a number of suggestions which have improved the presentation of the paper, but who has also pointed out some inaccuracies such as that the formula given in Proposition 2.3 was incorrect as it was stated in the first version of this paper. The referee's comments also lead to a shorter proof of Theorem 5.1.

1. – Some projective bundles over abelian varieties

Let A be an abelian variety of dimension n with a polarization $\Theta \in NS(A)$ of type $(d_1, \dots, d_n)$ with $d_1 | \dots | d_n$ (our general reference for the theory of abelian varieties will be [LB]). Let us take a non trivial point $\epsilon \in A$ of order two.

Let $K(\Theta)$ be the kernel of the isogeny $\lambda_\Theta : A \to \hat{A} = \mathrm{Pic}^0(A)$ determined by the polarization. Recall that $K(\Theta) \simeq (\mathbb{Z}_{d_1} \times \dots \times \mathbb{Z}_{d_n})^2$ and that, if d_1 is even, then Θ is divisible by two in $NS(A)$ and every point of order two of A is an element of $K(\Theta)$.

Let $\mathcal{L}$ be a line bundle on A representing Θ. Then we have:

$$t_\epsilon^* \mathcal{L} \simeq \mathcal{L} \otimes \mathcal{L}_0$$

where t_x is the translation by a point $x \in A$ and $\mathcal{L}_0 \in \mathrm{Pic}^0(A)$ is the point of order two given by $\lambda_\Theta(\epsilon)$. Hence $\mathcal{L}_0$ is trivial if and only if $\epsilon \in K(\Theta)$.

Let $\bar{A}$ be the quotient A/ϵ and let $\pi : A \to \bar{A}$ be the quotient map, which is an isogeny of degree 2. If $\epsilon \in K(\Theta)$, then there is a line bundle $\bar{\mathcal{L}}$ on $\bar{A}$, such that $\pi^*(\bar{\mathcal{L}}) = \mathcal{L}$. The line bundle $\bar{\mathcal{L}}$ represents a polarization $\bar{\Theta}$ on $\bar{A}$, of type $(\bar{d_1}, \dots, \bar{d_n})$, such that:

$$2 \cdot \bar{d_1} \cdot \dots \cdot \bar{d_n} = d_1 \cdot \dots \cdot d_n\,,$$

a relation which is obtained from $\Theta^n = \pi^*(\bar{\Theta})^n = 2\bar{\Theta}^n$. In particular, if $d_1 = \dots = d_{n-1} = 1$, $d_n = d$, then d is even and $\bar{\Theta}$ is of type $(1, \dots, 1, \frac{d}{2})$.

One has:

$$\pi_* \mathcal{O}_A = \mathcal{O}_{\bar{A}} \oplus \mathcal{M}_1$$

where $\mathcal{M}_1$ is a non trivial 2-torsion point in $\mathrm{Pic}^0(\bar{A})$. The induced map $\pi^* : \mathrm{Pic}^0(\bar{A}) \to \mathrm{Pic}^0(A)$ is also an isogeny of degree 2, whose kernel is generated by $\mathcal{M}_1$. Therefore we have two line bundles $\mathcal{M}_2, \mathcal{M}_3 \in \mathrm{Pic}^0(\bar{A})$ such that:

$$\pi^*(\mathcal{M}_2) = \pi^*(\mathcal{M}_3) = \mathcal{L}_0$$

and one has $\mathcal{M}_2 = \mathcal{M}_3 \otimes \mathcal{M}_1$. The elements $\mathcal{M}_i$, $i = 1, 2, 3$, and the trivial bundle form a subgroup G of $\mathrm{Pic}^0(\bar{A})$, which is the inverse image via π^* of the subgroup generated by $\mathcal{L}_0$.

We have the:

LEMMA 1.1. *If $\epsilon \in K(\Theta)$ then G is the group of order two generated by $\mathcal{M}_1$. Otherwise G is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$.*

PROOF. The first assertion is clear. To prove the second assertion let L be a lattice which defines $A = \mathbb{C}^g/L$. Then the point ϵ is represented by an element $e \in \frac{1}{2}L$. The fact that ϵ is not in $K(\Theta)$ is equivalent to the existence of some element $f \in L$ such that for the pairing defined by the polarization $(e, f) = \frac{1}{2} \bmod \mathbb{Z}$. The lattice $\bar{L}$ which defines the quotient $\bar{A}$ is the lattice generated by L and e. We denote by $L^\vee$ the dual lattice of L. This defines the dual variety $\mathrm{Pic}^0(A) = \mathbb{C}^g/L^\vee$ of A. The element e is not contained in $L^\vee$ and represents the line bundle $\mathcal{L}_0$ in $\mathrm{Pic}^0(A)$. Similarly $f \notin \overline{L}^\vee$, but $2f \in \overline{L}^\vee$. The element f corresponds to the line bundle $\mathcal{M}_1$ in $\mathrm{Pic}^0(\bar{A})$ which is 2-torsion. The element e also defines a line bundle in $\mathrm{Pic}^0(\bar{A})$, whose pullback to A is $\mathcal{L}_0$ and this corresponds to $\mathcal{M}_2$ or $\mathcal{M}_3$. The claim follows if we can show that $\mathcal{M}_2$ or $\mathcal{M}_3$ is 2-torsion. But this follows since $2e \in \overline{L}^\vee$. (A different proof will follow from Proposition 1.5 below (see Remark 1.6)). □

Let us set:

$$\mathcal{E} = \mathcal{E}(A, \epsilon, \mathcal{L}) = \pi_* \mathcal{L}\,.$$

This is a rank 2 vector bundle on $\bar{A}$, and we can consider the associated projective bundle:

$$X = X(A, \epsilon, \mathcal{L}) = \mathbb{P}(\mathcal{E}) = \mathbf{Proj}(\oplus_{i=0}^{\infty} \mathrm{Sym}^i(\mathcal{E}))$$

with its tautological line bundle $\mathcal{O}_X(1)$ and its structure map $p : X \to \bar{A}$. We will denote by F a fibre of p and by H a divisor in $|\mathcal{O}_X(1)|$. We will use the same notation to denote their classes in the homology ring of X.

If $\epsilon \in K(\Theta)$ then, since $\pi^*(\bar{\mathcal{L}}) = \mathcal{L}$, the projection formula tells us that:

$$\mathcal{E} = \bar{\mathcal{L}} \oplus (\bar{\mathcal{L}} \otimes \mathcal{M}_1)\,.$$

By contrast, as we shall see in a moment, if $\epsilon \notin K(\Theta)$, then the bundle $\mathcal{E}$ in general does not split.

The natural map $\pi^*\mathcal{E} \to \mathcal{L}$ defines an inclusion i of A into X such that H restricts to $\mathcal{L}$ on $i(A)$. The image $i(A)$ is a 2-section over $\bar{A}$. If there is no danger of confusion we shall denote $i(A)$ also by A. Let

$$\widetilde{X} = \mathbb{P}(\pi^*\mathcal{E})\,.$$

We have a natural étale map $f : \widetilde{X} \to X$ of degree 2. The inverse image of the 2-section A in X under the map f consists of 2 sections of $\widetilde{X}$ corresponding to the 2 projections $\pi^*\mathcal{E} \to \mathcal{L}$ and $\pi^*\mathcal{E} \to t_\epsilon^*\mathcal{L}$ whose existence follows from the construction of $\mathcal{E}$. This shows that $\pi^*\mathcal{E}$ splits, more precisely

$$\pi^*\mathcal{E} \simeq \mathcal{L} \oplus t_\epsilon^*\mathcal{L} = \mathcal{L} \oplus (\mathcal{L} \otimes \mathcal{L}_0)\,. \tag{1}$$

Notice that, if $\epsilon \in K(\Theta)$, then $\widetilde{X} = A \times \mathbb{P}^1$ is trivial.

LEMMA 1.2. *One has:*

(i) $\mathcal{O}_X(1)_{|A} \simeq \mathcal{L}$;

(ii) $\mathcal{O}_A(A) \simeq \mathcal{L}_0$;

(iii) $\mathcal{O}_X(-2A) \simeq \mathcal{O}_X(2K_X)$.

Moreover, if $\epsilon \notin K(\Theta)$ *then:*

(iv) *there is no section* A' *of* X *over* $\bar{A}$ *which is disjoint from* A.

Hence if $\epsilon \notin K(\Theta)$ *and* A *does not contain elliptic curves, then* $\mathcal{E}$ *does not split.*

PROOF. (i) follows by the definition of the tautological bundle.

(ii) There are two sections $\tilde{A}_1$ and $\tilde{A}_2$ of $\tilde{X}$ over A, which map both isomorphically to A via f. These sections correspond to the splitting of $\pi^*(\mathcal{E}) = \mathcal{L} \oplus (\mathcal{L} \otimes \mathcal{L}_0)$. Since the normal bundle of both these sections in $\tilde{X}$ is given by $\mathcal{L}_0$, and since f is étale, we have the assertion.

(iii) Since $A \cdot F = -K_X \cdot F = 2$, there is a line bundle $\mathcal{M}$ on $\bar{A}$ such that $\mathcal{O}_X(-A) \simeq \mathcal{O}_X(K_X) \otimes \pi^*(\mathcal{M})$. On the other hand, by adjunction, one has $\mathcal{O}_A(-A) \simeq \mathcal{O}_A(K_X)$. This implies that either $\mathcal{M} \simeq \mathcal{O}_{\bar{A}}$ or $\mathcal{M} \simeq \mathcal{M}_1$. This immediately yields the assertion.

(iv) Suppose A' is disjoint from A. Then A' would pull back to a section $\tilde{A}'$ of $\tilde{X}$, disjoint from $\tilde{A}_1$ and $\tilde{A}_2$, which would give another way of splitting $\pi^*(\mathcal{E} \otimes \mathcal{L}^*)$. This is impossible under the assumption $\epsilon \notin K(\Theta)$.

If $\mathcal{E}$ splits, we have two sections A' and A'' of X which do not meet. However they both meet A and they must cut out two divisors C' and C'' on A which do not meet each other. Hence C' and C'' are pull-backs from an elliptic curve and so A contains an elliptic curve. □

Notice that the map $p : X \to \bar{A}$ induces an isomorphism $p^* : \mathrm{Pic}^0(\bar{A}) \to \mathrm{Pic}^0(X)$. We will identify $\mathrm{Pic}^0(\bar{A})$ and $\mathrm{Pic}^0(X)$ using p^*. We have the:

PROPOSITION 1.3. *Let η be an element of* $\mathrm{Pic}^0(X)$. *One has:*

(i) *if $\epsilon \in K(\Theta)$ then $h^0(X, \mathcal{O}_X(A) \otimes \eta) = 0$ unless $\eta = \mathcal{O}_X$, in which case $h^0(X, \mathcal{O}_X(A)) = 2$, and $\eta = \mathcal{M}_1$, in which case $h^0(X, \mathcal{O}_X(A) \otimes \mathcal{M}_1) = 1$;*

(ii) *if $\epsilon \notin K(\Theta)$ then $h^0(X, \mathcal{O}_X(A) \otimes \eta) = 0$ unless $\eta = \mathcal{O}_X, \mathcal{M}_2, \mathcal{M}_3$, in which cases $h^0(X, \mathcal{O}_X(A) \otimes \eta) = 1$.*

PROOF. We have:

$$p_*\mathcal{O}_X(1) \simeq \mathcal{E}, \quad p_*\mathcal{O}_X(2) \simeq \mathrm{Sym}^2 \mathcal{E}.$$

Therefore, by using (1) we have:

$$\pi^* p_* \mathcal{O}_X(2) \simeq \mathrm{Sym}^2 \pi^* \mathcal{E} \simeq \mathcal{L}^{\otimes 2} \oplus \mathcal{L}^{\otimes 2} \oplus (\mathcal{L}^{\otimes 2} \otimes \mathcal{L}_0). \tag{2}$$

Moreover, since $A \cdot F = 2$, there is a line bundle $\mathcal{N}$ on $\bar{A}$ such that:

$$\mathcal{O}_X(A) \simeq \mathcal{O}_X(2) \otimes p^*\mathcal{N}. \tag{3}$$

Hence, by the projection formula, one has:

$$\pi^* p_*(\mathcal{O}_X(A) \otimes \eta) \simeq \pi^*(p_*\mathcal{O}_X(2) \otimes (\mathcal{N} \otimes \eta)) \simeq \pi^* p_* \mathcal{O}_X(2) \otimes \pi^*(\mathcal{N} \otimes \eta). \tag{4}$$

From Lemma 1.2, (i), (ii) and restricting (3) to A, we obtain:

$$\pi^* \mathcal{N}^* \simeq \mathcal{L}^{\otimes 2} \otimes \mathcal{L}_0. \tag{5}$$

Now, by (2), (4) and (5), we get:

$$\pi^* p_*(\mathcal{O}_X(A) \otimes \eta) \simeq (\mathcal{L}_0 \oplus \mathcal{L}_0 \oplus \mathcal{O}_A) \otimes \pi^* \eta$$

and therefore if $\epsilon \in K(\Theta)$:

$$(6) \qquad h^0(A, \pi^* p_* \mathcal{O}_X(A \otimes \eta)) = \begin{cases} 0 & \text{if } \pi^*\eta \neq \mathcal{O}_A \\ 3 & \text{if } \pi^*\eta = \mathcal{O}_A, \text{ i.e. } \eta \simeq \mathcal{O}_{\bar{A}}, \mathcal{M}_1 \end{cases}$$

whereas:

$$(6') \qquad h^0(A, \pi^* p_* \mathcal{O}_X(A \otimes \eta)) = \begin{cases} 0 & \text{if } \pi^*\eta \neq \mathcal{O}_A, \mathcal{L}_0 \\ 1 & \text{if } \pi^*\eta = \mathcal{O}_A, \text{ i.e. } \eta \simeq \mathcal{O}_{\bar{A}}, \mathcal{M}_1 \\ 2 & \text{if } \pi^*\eta = \mathcal{L}_0, \text{ i.e. } \eta \simeq \mathcal{M}_2, \mathcal{M}_3 \end{cases}$$

otherwise. Notice that:

$$\begin{aligned} h^0(A, \pi^* p_* \mathcal{O}_X(A \otimes \eta)) &= h^0(\bar{A}, \pi_* \pi^* p_* \mathcal{O}_X(A \otimes \eta)) \\ &= h^0(\bar{A}, p_* \mathcal{O}_X(A \otimes \eta)) + h^0(\bar{A}, (p_* \mathcal{O}_X(A \otimes \eta)) \otimes \mathcal{M}_1) \\ &= h^0(X, \mathcal{O}_X(A \otimes \eta)) + h^0(X, \mathcal{O}_X(A \otimes \eta) \otimes \mathcal{M}_1) . \end{aligned}$$

Then, by (6), resp. (6′) if $\eta \neq \mathcal{O}_{\bar{A}}, \mathcal{M}_i$, $i = 1, 2, 3$, we have:

$$h^0(X, \mathcal{O}_X(A \otimes \eta)) + h^0(X, \mathcal{O}_X(A \otimes \eta) \otimes \mathcal{M}_1) = 0 ,$$

in particular $h^0(X, \mathcal{O}_X(A \otimes \eta)) = 0$.

Let $\epsilon \in K(\Theta)$. If $\eta = \mathcal{O}_{\bar{A}}, \mathcal{M}_1$, we find:

$$h^0(X, \mathcal{O}_X(A)) + h^0(X, \mathcal{O}_X(A) \otimes \mathcal{M}_1) = 3 .$$

We claim that $h^0(X, \mathcal{O}_X(A)) = 2$ and $h^0(X, \mathcal{O}_X(A) \otimes \mathcal{M}_1) = 1$. Let $\tilde{A}$ be a trivial section of $\widetilde{X} = A \times \mathbb{P}^1$ over A. Of course $|\tilde{A}|$ is a base point free pencil on $\widetilde{X}$. The image of this pencil under the map $f : \widetilde{X} \to X$ is a system of divisors on X which is contained in a linear system. Since A is in this system, we see that $h^0(X, \mathcal{O}_X(A)) \geq 2$. on the other hand we cannot have $h^0(X, \mathcal{O}_X(A)) \geq 3$ because of Lemma 1.2, (ii). Hence the assertion follows, proving (i).

Let now $\epsilon \notin K(\Theta)$. If $\eta = \mathcal{O}_{\bar{A}}, \mathcal{M}_1$, we find:

$$h^0(X, \mathcal{O}_X(A)) + h^0(X, \mathcal{O}_X(A) \otimes \mathcal{M}_1) = 1 ,$$

and since $h^0(X, \mathcal{O}_X(A)) \geq 1$, we have $h^0(X, \mathcal{O}_X(A)) = 1$ and $h^0(X, \mathcal{O}_X(A) \otimes \mathcal{M}_1) = 0$. Finally, if $\eta = \mathcal{M}_i$, $i = 2, 3$, we have:

$$h^0(X, \mathcal{O}_X(A) \otimes \mathcal{M}_2) + h^0(X, \mathcal{O}_X(A) \otimes \mathcal{M}_3) = 2 .$$

We claim that both summands are smaller than 2. Otherwise the linear system, say, $|\mathcal{O}_X(A) \otimes \mathcal{M}_2|$ would be a pencil, and therefore we would find an element of it meeting A. But by Lemma 1.2, (ii), the restriction of $\mathcal{O}_X(A) \otimes \mathcal{M}_2$ to A is trivial. This would yield that A itself is an element of the pencil, implying $h^0(X, \mathcal{M}_2) > 0$ and hence that $\mathcal{M}_2$ is trivial on X, a contradiction. In conclusion we have:

$$h^0(X, \mathcal{O}_X(A) \otimes \mathcal{M}_2) = h^0(X, \mathcal{O}_X(A) \otimes \mathcal{M}_3) = 1$$

which finishes our proof. □

REMARKS 1.4. (i) First we consider the case $\epsilon \in K(\Theta)$. We reconsider the relation between the pencils $|\tilde{A}|$ on $\tilde{X}$ and $|A|$ on X. The map f sends each element of $|\tilde{A}|$ to an element of $|A|$. As we saw in the proof of Lemma 1.2, (ii), we have $f^*(A) = \tilde{A}_1 + \tilde{A}_2$. Hence f is two-to-one between $|\tilde{A}|$ and $|A|$. This means that all, but two, elements of $|A|$ are smooth, irreducible, isomorphic to A, and that there are two elements of $|A|$ of type $2\bar{A}^+$, $2\bar{A}^-$ with $\bar{A}^{\pm}$ sections of X over $\bar{A}$. One moment of reflection shows that these two sections, which do not meet, correspond to the splitting of $\mathcal{E}$. Of course $\bar{A}^{\pm}$ are isomorphic to $\bar{A}$ and one has $\mathcal{O}_{\bar{A}^{\pm}}(\bar{A}^{\pm}) \simeq \mathcal{M}_1$. In addition $2\bar{A}^{\pm} \equiv A$ but of course $\bar{A}^+ \not\equiv \bar{A}^-$. Hence $\bar{A}^+ - \bar{A}^-$ gives a non trivial point of order two in $\mathrm{Pic}^0(X) \simeq \mathrm{Pic}^0(\bar{A})$. By restricting to $\bar{A}^{\pm}$, we see that this point of order two is $\mathcal{M}_1$. Hence $\mathcal{O}_X(\bar{A}^+) \simeq \mathcal{O}_X(\bar{A}^-)\otimes\mathcal{M}_1$ and therefore $\mathcal{O}_X(A) \simeq \mathcal{O}_X(2\bar{A}^+) \simeq \mathcal{O}_X(\bar{A}^+ + \bar{A}^-)\otimes\mathcal{M}_1$, whence $\mathcal{O}_X(A)\otimes\mathcal{M}_1 \simeq \mathcal{O}_X(\bar{A}^+ + \bar{A}^-)$, which fully explains the meaning of part (i) of Proposition 1.3.

Notice that all the smooth abelian varieties in $|A|$ play a symmetric role in the construction of X and of its tautological line bundle.

One more obvious remark. Let $\bar{A}^-$ correspond to the quotient $\mathcal{E} \to \bar{\mathcal{L}}$ and $\bar{A}^+$ to the quotient $\mathcal{E} \to \bar{\mathcal{L}}\otimes\mathcal{M}_1$. Then $\mathcal{O}_{\bar{A}^-}(1) \simeq \bar{\mathcal{L}}$ and $\mathcal{O}_{\bar{A}^+}(1) \simeq \bar{\mathcal{L}}\otimes\mathcal{M}_1$.

(ii) Now we take up the case $\epsilon \notin K(\Theta)$. Consider the varieties A_2, A_3 which are the unique divisors in the linear systems $|\mathcal{O}_X(A)\otimes\mathcal{M}_3|$, $|\mathcal{O}_X(A)\otimes\mathcal{M}_2|$, respectively. As we saw in the proof of Proposition 1.3, we have $A \cap A_2 = A \cap A_3 = \emptyset$. Then, by Lemma 1.2, (iv), A_2 and A_3 are irreducible. We shall see in Proposition 1.5 that these varieties are smooth abelian. We also set $A_1 = A$.

PROPOSITION 1.5. *One has:*

(i) *if* $\epsilon \in K(\Theta)$, *then* $h^0(X, \mathcal{O}_X(2A)) = 3$;

(ii) *if* $\epsilon \notin K(\Theta)$, *then* $h^0(X, \mathcal{O}_X(2A)) = 2$. *Moreover the pencil* $|2A|$ *has exactly* 3 *singular elements namely* $2A_i$ *for* $i = 1, 2, 3$. *All other elements D in* $|2A|$ *are smooth abelian. The reduced varieties* A_i *are also smooth abelian.*

PROOF. In case (i) the linear system is composed with the pencil $|A|$, hence the assertion.

Let us consider case (ii). Since $2A \equiv 2A_2 \equiv 2A_3$, it is clear that $h^0(X, \mathcal{O}_X(2A)) \geq 2$. Suppose $h^0(X, \mathcal{O}_X(2A)) = r + 1 \geq 3$. Then the linear system $|2A|$ would have dimension $r \geq 2$. Moreover $\mathcal{O}_A(2A)$ is trivial. Therefore the linear system $|A|$ would have dimension at least $r - 1 \geq 1$, contradicting Proposition 1.3. We now look at the pencil $|2A|$. We have already seen that the A_i are irreducible. The same is true for the elements D. In fact D does not meet A and by Lemma 1.2, (iv) D cannot have a component which is a section. In addition, there cannot be a non trivial component which is a 2-section either, because such a 2-section would be numerically equivalent to A, hence would be equal to A_2 or A_3, which is not possible since $A_2 + A_3$ is not equivalent to $2A$. It follows from adjunction that the square of the dualizing sheaf ω_{A_i} is trivial and that ω_D is trivial. Our assertion follows if we can show that the projection onto $\bar{A}$ defines an étale $2:1$ cover from A_i to $\bar{A}$, resp. an étale $4:1$ cover from D to $\bar{A}$. To see this we look at the pencil of degree 4 cut out by $|2A|$ on each ruling. This is base point free and has at least 3 singular elements consisting of 2 double points each, corresponding to the 2-sections A_i. By the Hurwitz formula there can be no worse singularities and this gives the claim. □

REMARKS 1.6. (i) Assume that $\epsilon \notin K(\Theta)$. Then we have just seen that A_2 and A_3 are smooth abelian varieties isogenous to $\bar{A}$ via the degree 2 maps π_2, π_3 induced by p. In addition we have $A_2 \cap A_3 = \emptyset$. Hence $\pi_2^*\mathcal{M}_2 \simeq \mathcal{O}_{A_2}$ and $\pi_3^*\mathcal{M}_3 \simeq \mathcal{O}_{A_3}$, which gives another proof of Lemma 1.1 in the present case.

(ii) In this situation the projection p induces on every smooth element $D \in |2A|$ an isogeny $\delta : D \to \bar{A}$ of degree 4. We have just seen that $\pi_2^*\mathcal{M}_2 \simeq \mathcal{O}_{A_2}$ and $\pi_3^*\mathcal{M}_3 \simeq \mathcal{O}_{A_3}$. It follows from this that $\delta^*(\mathcal{M}_i)$ is trivial for every $i = 1, 2, 3$. Hence D is constant in moduli and it is the unique degree 4 cover of $\bar{A}$ with this property. We also remark that, in view of the above description, the isogeny δ factors through degree 2 isogenies $\delta_i : D \to A_i$, $i = 1, 2, 3$.

(iii) We consider the line bundles $\mathcal{L}_i := \mathcal{O}_X(1)_{|A_i}$, and the corresponding polarizations Θ_i, $i = 1, 2, 3$. Of course $\mathcal{E} \simeq \pi_{i*}\mathcal{L}_i$, $i = 1, 2, 3$, and the abelian varieties A_i play a symmetric role in the construction of X and of its tautological line bundle $\mathcal{O}_X(1)$.

PROPOSITION 1.7. *One has:*

(i) *if* $\epsilon \in K(\Theta)$, *then* $\mathcal{O}_X(K_X) \simeq \mathcal{O}_X(-A)$;

(ii) *if* $\epsilon \notin K(\Theta)$, *then* $\mathcal{O}_X(K_X) \simeq \mathcal{O}_X(-A_i) \otimes \mathcal{M}_i$, *for* $i = 1, 2, 3$.

PROOF. (i) By the proof of Lemma 1.2, we know that either $\mathcal{O}_X(K_X) \simeq \mathcal{O}_X(-A) \otimes \mathcal{M}_1$ or $\mathcal{O}_X(K_X) \simeq \mathcal{O}_X(-A)$. The assertion follows by restricting to $\bar{A}^{\pm}$.

(ii) As above the proof of Lemma 1.2 tells us that either $\mathcal{O}_X(K_X) \simeq \mathcal{O}_X(-A_i) \otimes \mathcal{M}_i$ or $\mathcal{O}_X(K_X) \simeq \mathcal{O}_X(-A_i)$. Suppose that $\mathcal{O}_X(K_X) \simeq \mathcal{O}_X(-A)$. Then the adjunction formula tells us that $\mathcal{O}_{A_2} \simeq \mathcal{O}_X(K_X + A_2) \otimes \mathcal{O}_{A_2} \simeq \mathcal{O}_X(A_2 - A_1) \otimes \mathcal{O}_{A_2} \simeq \mathcal{M}_3 \otimes \mathcal{O}_{A_2}$, which is a contradiction, since only $\mathcal{M}_2 \otimes \mathcal{O}_{A_2}$ is trivial on A_2. □

Let us now consider the action of $K(\Theta)$ on A. We will assume Θ is a primitive polarization, i.e. it is an indivisible element of $NS(A)$, of type $(d_1, d_2, \ldots, d_n)$. This is equivalent to $d_1 = 1$.

LEMMA 1.8. *Let us suppose that the Neron-Severi group of A is generated by* Θ. *Let* γ *be an effective divisor on A fixed by* $K(\Theta)$. *Then there is a positive integer a such that* $\gamma = a \cdot d_n \cdot \Theta$ *in the Neron-Severi group of A.*

PROOF. Let $\hat{A}$ be the polarized dual variety of A. The primitive dual polarization $\widehat{\Theta}$ is of type $(1, \frac{d_n}{d_{n-1}}, \ldots, \frac{d_n}{d_2}, d_n)$ and it generates the Neron-Severi group of $\hat{A}$. We have the map $\lambda_\Theta : A \to \hat{A}$. Then an easy computation using self-intersection numbers shows that:

$$\lambda_\Theta^*(\widehat{\Theta}) = d_n \Theta \,.$$

On the other hand we have $\gamma = \lambda_{\Theta}^*(\hat{\gamma})$, where $\hat{\gamma}$ is an effective divisor on $\hat{A}$. Therefore we have $\hat{\gamma} = a\widehat{\Theta}$ for some positive integer a. By pulling this back to A via λ_{Θ}, we get the assertion. □

LEMMA 1.9. *Assume that $\epsilon \notin K(\Theta)$ and that the Neron-Severi group of A is generated by Θ. Let D be any irreducible element of the pencil $|-2K_X|$ and let H_D be the element of the Neron-Severi group of D given by the restriction of H to D. Then $H_D = \delta_i^*(\Theta_i)$, $i = 1, 2, 3$ and H_D is at most divisible by 2 in the Neron-Severi group of D.*

PROOF. The assumption that $NS(A) \simeq \mathbb{Z}$ implies that also $NS(D) \simeq \mathbb{Z}$. The assertion $H_D = \delta_i^*(\Theta_i)$ is then purely numerical and follows from the fact that $D = 2A$ in $NS(X)$. Since the maps δ_i are $2:1$ covers the maps $\delta_i{}^* : NS(A_i) \simeq \mathbb{Z} \to NS(D) \simeq \mathbb{Z}$ have a cokernel which is torsion of order at most 2. □

In the case of abelian surfaces we can extend Lemma 1.8 above in the following way:

LEMMA 1.10. *Let A be an abelian surface with a polarization Θ of type $(1, 2n)$, resp. $(2, n)$ and assume that $NS(A) \simeq \mathbb{Z}$. If C is a curve invariant under a group $G \simeq \mathbb{Z}_n \times \mathbb{Z}_n$ which acts on A by translation, then $C = a \cdot \Theta$ where a is a multiple of n, resp. $n/2$.*

PROOF. By the assumption $NS(A) \simeq \mathbb{Z}$ the curve C is a multiple of Θ, resp. $\Theta/2$. Since C is invariant under the group G the associated line bundle descends to A/G. But this implies that G is a totally isotropic subgroup with respect to the Weil pairing of $K(a\Theta)$ (cf. [LB, Corollary 6.3.5]). This is only possible if a is divisible by n, resp. $n/2$ (cf. the description of the Weil pairing given in [LB, Example 7.7.4]). □

2. – Some scrolls of secant lines to abelian varieties

Let us consider a linearly normal abelian variety $A \subset \mathbb{P}^{r-1}$ of dimension n, embedded via a very ample line bundle $\mathcal{L}$ belonging to a polarization Θ of type $(d_1, d_2, \dots, d_n)$. Then $r = d_1 \cdot \ldots \cdot d_n$ and the degree of A equals $n! \cdot d_1 \cdot \ldots \cdot d_n$. Let ϵ be a non trivial 2-torsion point on A. We are interested in the $n+1$-dimensional scroll:

$$Y = Y(A, \epsilon, \mathcal{L}) = \cup_{x \in A} L(x, x + \epsilon)$$

where $L(a, b)$ is the line joining two distinct points a, b in projective space. We notice that, unless $n = 1$ and $r = d = 3$, Y is a proper subvariety in $\mathbb{P}^{r-1}$.

As we saw in Section 1, from which we keep the notation, we can associate to this situation a $\mathbb{P}^1$-bundle X on $\bar{A} = A/\epsilon$. The relation between X and Y is described in the following lemma:

LEMMA 2.1. *One has the following commutative diagram:*

$$\begin{tikzcd}
A \arrow[r] \arrow[d, "\phi_{\mathcal{L}}"'] & X \arrow[d, "\phi:=\phi_{\mathcal{O}_X(1)}"] \\
\mathbb{P}^{r-1}=\mathbb{P}(H^0(A,\mathcal{L})^*) \arrow[r, phantom, "\simeq"] & \mathbb{P}(H^0(X,\mathcal{O}_X(1))^*)
\end{tikzcd}$$

Moreover the map ϕ is a morphism and its image is Y.

PROOF. Since $p_*\mathcal{O}_X(1) \simeq \mathcal{E} \simeq \pi_*\mathcal{L}$ we have a canonical isomorphism of vector spaces $H^0(A,\mathcal{L}) \simeq H^0(X,\mathcal{O}_X(1))$. Moreover, by Lemma 1.2, (ii) we have $\mathcal{O}_X(1)_{|A} \simeq \mathcal{L}$. This shows the existence of the commutative diagram above, which in turn, implies $\phi(X) = Y$.

Let F be any ruling of X and let $x, x+\epsilon$ be the two points where F intersects A. Since $\mathcal{L}$ is very ample on A, the linear system $|\mathcal{O}_X(1)|$ separates these two points and hence $|\mathcal{O}_X(1)|$ has no base points on X, i.e. ϕ is a morphism. □

We can now prove the following propositions:

PROPOSITION 2.2. *The map ϕ is finite, i.e. $\mathcal{O}_X(1)$ is ample.*

PROOF. Assume that there is an irreducible curve C which is contracted under ϕ. The curve C can only meet a ruling F once, i.e. the projection of C to $\bar{A}$ is birational onto its image. After possibly replacing C by its normalization we obtain a smooth curve $\tilde{C}$ and a morphism $f : \tilde{C} \to \bar{A}$ which is birational onto its image such that

$$f^*\mathcal{E} = \mathcal{O}_{\tilde{C}} \oplus \xi$$

for some suitable line bundle ξ on $\tilde{C}$. The $\mathbb{P}^1$-bundle $V = \mathbb{P}(f^*\mathcal{E})$ over $\tilde{C}$ is mapped to a cone W in Y. Let C_0 be the section of V which is mapped to the vertex of W and let C_1 be the 2-section which is the pullback of the 2-section A of X. We denote by f the class of a fibre in V. Then $C_1 = 2C_0 + af$ in the Neron-Severi group of V for some integer a. Since C_0 is contracted it follows that $e = C_0^2 < 0$. On the other hand it follows from Lemma 1.2 (ii) that $C_1^2 = 0$ and hence $a = -e$. But then $C_0.C_1 = e < 0$, i.e. C_0 and C_1 have a common component. This contradicts the fact that the abelian variety A is embedded by the map ϕ. □

PROPOSITION 2.3. *One has:*

$$\deg(\phi) \cdot \deg(Y) = \frac{1}{2}(n+1)! \cdot r\,.$$

If $n = 1$ and $r \geq 4$, then ϕ is birational onto its image and it is an embedding as soon as $r \geq 5$.

PROOF. The result is well known for $n = 1$ (see e.g. [CH, Proposition 1.1 and Proposition 1.2]). So we assume $n \geq 2$. We have:

$$\deg(\phi) \cdot \deg(Y) = \mathcal{O}_X(1)^{n+1} = \frac{1}{2}(f^*\mathcal{O}_X(1))^{n+1} = \frac{1}{2}(\mathcal{O}_{\pi^*(\mathcal{E})}(1))^{n+1}.$$

We can then deduce from formula (1) of Section 1 that:

$$\mathcal{O}_X(1)^{n+1} = \frac{1}{2}(n+1)\Theta^n = \frac{1}{2}(n+1)! \cdot r$$

and hence the assertion. □

In what follows we will need a formula for the double point cycle $\mathbb{D}(X, l)$ (see [F, p. 166]) of a map $f : X \to \mathbb{P}^l$, where $l \leq r - 1$ and f is the composition of ϕ with a general projection $\mathbb{P}^{r-1} \to \mathbb{P}^l$. We assume $l \geq n + 1$. In this situation the map f is finite.

THEOREM 2.4. *One has:*

$$\mathbb{D}(X, l) = \left(\frac{1}{2}(n+1)! \cdot r - \binom{l+1}{n+2}\right) H^{l-n-1} + \binom{l+1}{n+3} H^{l-n-2} \cdot A$$

in the homology ring of X.

PROOF. By applying Theorem 9.3 from [F, p. 166], one has:

$$\mathbb{D}(X, l) = f^*f_*[X] - c_{l-n-1}(N_f) \cap [X]$$

where N_f is the normal sheaf to the map f which is defined by the exact sequence:

$$0 \to T_X \to f^*T_{\mathbb{P}^l} \to N_f \to 0.$$

By Proposition 2.3 and finiteness of f, we have:

$$f^*f_*[X] = \left(\frac{1}{2}(n+1)! \cdot r\right) H^{l-n-1}.$$

Moreover from the Euler sequence, we see that $c(f^*T_{\mathbb{P}^l}) = (1 + H)^{l+1}$. Also by the exact sequence:

$$0 \to p^*T_{\bar{A}} \to T_X \to T_{X|\bar{A}} \to 0$$

we deduce that $c(T_X) = 1 + A$. Since $A^2 = 0$ in homology, the assertion follows by an easy computation. □

Corollary 2.5. *Let* $n \geq 2$ *and* $r = 2n+3$. *Then:*

$$\mathbb{D}(X, r-1) = n! \cdot (2n+3) \times \left[\frac{1}{2}(n+1)\left(\frac{1}{2}(n+1)! \cdot (2n+3) - \binom{2n+3}{n+2}\right) + \binom{2n+3}{n+3}\right]$$

which is equal to 0 *if and only if* $n = 2$.

Proof. The formula for $\mathbb{D}(X, r-1)$ follows right away from the above theorem. For $n = 2$ we have $\mathbb{D}(X, 6) = 0$ and for $n = 3$ one computes that $\mathbb{D}(X, 8) = 6 \cdot 9 \cdot 48$. In order to prove the second assertion, it is sufficient to show that:

$$F(n) = \frac{1}{2}(n+1)! \cdot (2n+3) - \binom{2n+3}{n+2} \geq 0$$

for all $n \geq 4$. Since

$$F(n) > \frac{1}{2}(n+1)! \cdot (2n+3) - 2^{2n+2}$$

one can easily verify this by induction on n. □

Remark 2.6. The above corollary suggests that the secant scroll to an abelian surface of type $(1,7)$ in $\mathbb{P}^6$ should be smooth. This we are going to prove in Section 4. Moreover this is the only case, apart from the elliptic scroll of degree 5 in $\mathbb{P}^4$, in which an $(n+1)$-dimensional scroll obtained as above from an n-dimensional abelian variety, which is linearly normal in $\mathbb{P}^{2n+2}$, can be smooth. This is what makes the consideration of the surface case, to which the main part of this paper is devoted, particularly interesting.

We want to finish this section with a remark which is specific to the case $\epsilon \in K(\Theta)$.

Remark 2.7. If $\epsilon \in K(\Theta)$, then ϵ acts as an involution on:

$$H^0(A, \mathcal{L}) \simeq H^0(X, \mathcal{O}_X(1)) \simeq H^0(\bar{A}, \mathcal{E}) \simeq H^0(\bar{A}, \overline{\mathcal{L}}) \oplus H^0(\bar{A}, \overline{\mathcal{L}} \otimes \mathcal{M}_1),$$

the invariant and anti-invariant eigenspaces being $H^0(\bar{A}, \overline{\mathcal{L}})$ and $H^0(\bar{A}, \overline{\mathcal{L}} \otimes \mathcal{M}_1)$ respectively. Recall that $\overline{\mathcal{L}}$ represents a polarization of type $(\overline{d_1}, \dots, \overline{d_n})$ on $\bar{A}$, such that $2 \cdot \overline{d_1} \cdot \ldots \cdot \overline{d_n} = d_1 \cdot \ldots \cdot d_n$. Set $\bar{r} = \overline{d_1} \cdot \ldots \cdot \overline{d_n}$. Then $h^0(\bar{A}, \overline{\mathcal{L}}) = h^0(\bar{A}, \overline{\mathcal{L}} \otimes \mathcal{M}_1) = \bar{r}$ and $2\bar{r} = r$.

Accordingly ϵ acts as an involution on $\mathbb{P}^{r-1} = \mathbb{P}(H^0(A, \mathcal{L})^*)$. The invariant and anti-invariant subspaces both have dimension $\bar{r}$ and are $\mathbb{P}^+ = \mathbb{P}(H^0(\bar{A}, \overline{\mathcal{L}})^*)$ and $\mathbb{P}^- = \mathbb{P}(H^0(\bar{A}, \overline{\mathcal{L}} \otimes \mathcal{M}_1)^*)$. One has the morphisms $\phi^+ = \phi_{\overline{\mathcal{L}}} : \bar{A} \to \mathbb{P}^+$ and $\phi^- = \phi_{\overline{\mathcal{L}} \otimes \mathcal{M}_1} : \bar{A} \to \mathbb{P}^-$. The images Y^+ and Y^- of these maps are nothing but the images via ϕ of the two sections $\bar{A}^+$ and $\bar{A}^-$ respectively of X (see Remark 1.4, (i)).

Finally we have a different description of Y which will be useful to take into account. Take any point $x \in \bar{A}$ and consider the corresponding points $x_{\pm} \in \bar{A}^{\pm}$. Set $y^{\pm} = \phi(x^{\pm})$. Then:

$$Y = \bigcup_{x \in \bar{A}} L(y^{+}, y^{-}).$$

3. – Secant scrolls related to abelian surfaces

From now on we will consider the case where the abelian variety A is a surface, i.e. $n = 2$, and the polarization Θ is very ample of type (d_1, d_2). Hence $r = d_1 \cdot d_2$ and the surface A is embedded into $\mathbb{P}^{r-1}$, via a line bundle $\mathcal{L}$ representing Θ, as a surface of degree $2 \cdot d_1 \cdot d_2$. We will also denote the image by A.

We are interested in characterizing the cases in which the map $\phi : X \to Y$ introduced in general in Section 2 is an embedding, if A is a general polarized abelian surface of type (d_1, d_2). Notice that, by Lefschetz's hyperplane section theorem, there is no chance that ϕ is ever an embedding if $r < 7$. So we will assume from now on $r \geq 7$.

In order to study the map $\phi : X \to Y$, we need some information about the embedding of A in $\mathbb{P}^{r-1}$. We will use a well known result of Reider, in the following form due to Beltrametti and Sommese [BS, Theorem 3.2.1]:

THEOREM 3.1. *Let L be a numerically effective (nef) divisor on a surface S. Assume that $L^2 \geq 4k + 1$. Given any 0-dimensional scheme Z of length k on S, then either the natural restriction map*

$$H^0(S, \mathcal{O}_S(K + L)) \to H^0(S, \mathcal{O}_Z(K + L))$$

is surjective, or there exist an effective divisor C on S and a non-empty subscheme Z' of Z of length $k' \leq k$, such that:

(i) *the map*

$$H^0(S, \mathcal{O}_S(K + L)) \to H^0(S, \mathcal{O}_{Z'}(K + L))$$

is not surjective;

(ii) *Z' is contained in C and there is an integer m such that $m(L - 2C)$ is effective;*

(iii) *one has*

$$L \cdot C - k' \leq C^2 < \frac{L \cdot C}{2} < k'.$$

As a consequence we have the following:

PROPOSITION 3.2. *Let A be a polarized abelian surface of type (d_1, d_2) and let $\mathcal{L}$ be a line bundle on A representing the given polarization Θ. Set $r = d_1 \cdot d_2$. One has:*

(i) *if $r \geq 5$, then $\mathcal{L}$ is very ample, unless there is a curve C on A such that $C^2 = 0$ and $\Theta \cdot C \leq 2$;*

(ii) *assume that $\mathcal{L}$ is very ample and that $r \geq 7$, then $\phi_{\mathcal{L}}(A)$ has no 3-secant lines, unless there is a curve C on A such that either $C^2 = 0$ and $\Theta \cdot C = 3$ (a plane cubic) or $C^2 = 2$ and $\Theta \cdot C = 5$ (a genus 2 quintic);*

(iii) *assume that $\mathcal{L}$ is very ample, that $\phi_{\mathcal{L}}(A)$ has no 3-secant lines and that $r \geq 9$, then $\phi_{\mathcal{L}}(A)$ has no 4-secant planes, unless there is a curve C on A such that either $C^2 = 0$ and $\Theta \cdot C = 4$ (a genus 1 quartic) or $C^2 = 2$ and $\Theta \cdot C = 6$ (a genus 2 sextic).*

In particular if $NS(A) \simeq \mathbb{Z}$ then:

(i′) *if $r \geq 5$, then $\mathcal{L}$ is very ample;*

(ii′) *if $r \geq 7$, then $\phi_{\mathcal{L}}(A)$ has no 3-secant lines;*

(iii′) *if $r \geq 9$, then $\phi_{\mathcal{L}}(A)$ has no 4-secant planes unless Θ is the triple of a principal polarization.*

PROOF. The first part is an immediate application of Theorem 3.1, by taking into account that for any effective divisor C on A, the integer C^2 is even and non-negative.

If $NS(A) \simeq \mathbb{Z}$, then there is no curve on A with $C^2 = 0$. In addition, if C is an effective divisor such that $C^2 = 2$, then C is irreducible and its class θ in $NS(A)$ is a principal polarization which is indivisible, hence it generates $NS(A)$. Thus there is a positive integer a such that $\Theta = a\theta$. If $\Theta \cdot \theta = 2a \leq 6$, then $a \leq 3$, and Θ is a polarization of type (a, a). Since we are assuming $r = a^2 \geq 5$, we have $a = 3$, a case which indeed gives rise to 4-secant planes (see Section 6). □

As a further consequence we can prove the:

THEOREM 3.3. *Let $A \subset \mathbb{P}^{r-1}$ be a linearly normal, smooth abelian surface such that $\mathcal{L} = \mathcal{O}_A(1)$ determines a polarization Θ of type (d_1, d_2) with $r = d_1 \cdot d_2$. Let $\epsilon \in A$ be a non trivial point of order two. Suppose that $NS(A) \simeq \mathbb{Z}$. Consider the map $\phi : X \dashrightarrow Y \subset \mathbb{P}^{r-1}$. Then:*

(i) *distinct rulings of X are sent by ϕ to distinct lines in $\mathbb{P}^{r-1}$, as soon as $r \geq 7$;*

(ii) *the differential of the map $\phi : X \dashrightarrow Y$ is injective along A and Y is smooth along A, as soon as $r \geq 7$;*

(iii) *the map $\phi : X \dashrightarrow Y$ is an isomorphism as soon as $r \geq 9$, unless Θ is the triple of a principal polarization, i.e. $d_1 = d_2 = 3$ and $r = 9$.*

PROOF. (i) If two distinct rulings of X were mapped to the same line L in $\mathbb{P}^{r-1}$, then L would be a 4-secant line to A, contradicting Corollary 3.2, (ii′).

(ii) Let x be a point of A. Then the tangent space to X at x is spanned by $T_{A,x}$ and by the tangent space to the ruling F through x. Since $L = \phi(F)$

cannot be tangent to A by (3.2, ii′), it follows that $d\phi$ is injective at x. The smoothness of Y along A is then a consequence of (3.2, ii′).

(iii) If ϕ were not injective, we would have two distinct secant lines to A meeting at a point, and therefore a 4-secant plane to A, contradicting (3.2, iii′).

Suppose $d\phi$ is not injective at a point $z \in X$, which we may assume not to be on A by (ii). The Gaussian map $\gamma : X \dashrightarrow \mathbf{G}(3, r-1)$ is a rational map whose restriction to each ruling is defined by (ii). By following the argument in [R, p. 215], we see that such a restriction is given by quadratic forms. If F is the ruling of X through z, the forms defining $\gamma_{|F}$ all vanish at z. Hence by [R, Lemma 25], the union of the tangent spaces to Y along the image L of F is a $\mathbb{P}^4$. In particular the two tangent planes to A at the points x and $x+\epsilon$ where L intersects A meet at a point. This either yields the existence of a tangent line to A which meets A once more, which is impossible, or of two tangent lines r, r' to A at x and $x' = x+\epsilon$ which meet, hence the existence of a 4-secant plane to A, a contradiction to (3,2, ii′, iii′), unless $d_1 = d_2 = 3$, $r = 9$ and $\Theta = 3\theta$, with θ a principal polarization on A. □

We also have the following consequence:

Proposition 3.4. *Let A be a polarized abelian surface of type (d_1, d_2) with $r = d_1 \cdot d_2 \geq 7$ such that $NS(A) \simeq \mathbb{Z}$. Then the map $\phi : X \dashrightarrow Y$ is birational.*

Proof. Let δ be the degree of ϕ. Let F be a general ruling of X and let L be its image under ϕ. Let x, $x+\epsilon$ be the two points of A on L. Let $z \in L$ be a general point. Then by (3.2, ii′) there are $\delta - 1$ images of rulings of X through z different from L. This situation produces a $(\delta-1)$-cover of L and again by (3.2, ii′) this cover is totally ramified at x and $x+\epsilon$, i.e. $\phi^{-1}(\phi(x)) = x$ and $\phi^{-1}(\phi(x+\epsilon)) = x+\epsilon$. Since z is general this implies that ϕ itself should be ramified along the points of A, contradicting (3.3, ii). □

In what follows we will need some information about the hyperplane sections of Y. Let $A \subset \mathbb{P}^{r-1}$ be a linearly normal, smooth abelian surface such that $\mathcal{L} = \mathcal{O}_A(1)$ determines a polarization Θ of type (d_1, d_2) with $r = d_1 \cdot d_2$. Let $\epsilon \in A$ be a non trivial point of order two. Let H be a divisor in $|\mathcal{O}_X(1)|$. We abuse notation and we denote by $p : H \dashrightarrow \bar{A}$ the restriction to H of the projection $p : X \dashrightarrow \bar{A}$.

Lemma 3.5. *In the above setting, if $NS(A) \simeq \mathbb{Z}$ is generated by Θ and if H is a general divisor in $|\mathcal{O}_X(1)|$, then:*

(i) *the map $p : H \dashrightarrow \bar{A}$ is the blow-up of $\bar{A}$ at r distinct points $p_1, \ldots, p_r$ and the exceptional divisors are r rulings $F_1, \ldots, F_r$ of X contained in H;*

(ii) *if $\epsilon \in K(\Theta)$, then $\mathcal{O}_X(1)_{|H} = \mathcal{O}_X(-K_X) \otimes \mathcal{O}_H(F_1 + \ldots + F_r) = \mathcal{O}_X(A) \otimes \mathcal{O}_H(F_1 + \ldots + F_r)$, whereas if $\epsilon \notin K(\Theta)$, then $\mathcal{O}_X(1)_{|H} = \mathcal{O}_X(-K_X) \otimes \mathcal{O}_H(F_1 + \ldots + F_r) = \mathcal{O}_X(A) \otimes \mathcal{M}_1 \otimes \mathcal{O}_H(F_1 + \ldots + F_r)$.*

Proof. Let H be the zero locus of the section $s \in H^0(X, \mathcal{O}_X(1))$. One has $H^0(X, \mathcal{O}_X(1)) \simeq H^0(A, \mathcal{L})$ and $H^0(X, \mathcal{O}_X(1)) \simeq H^0(\bar{A}, \mathcal{E})$, thus we may interpret s as a non-zero section of $\mathcal{L}$ on A and of $\mathcal{E}$ on $\bar{A}$.

We let C_H be the zero-divisor of $s \in H^0(A, \mathcal{L})$ on A, i.e. the intersection of H with A. According to our assumption on A, the curve C_H is irreducible and reduced. Then we consider the curve $C_{H,\epsilon} = C_{H+\epsilon}$, which is the zero locus of the section $s_{H,\epsilon} = t_\epsilon^*(s) \in H^0(A, t_\epsilon^*\mathcal{L})$. The curve $C_{H,\epsilon}$ is also irreducible and reduced. Furthermore, it is distinct from C_H. This is clear if $\epsilon \notin K(\Theta)$, whereas, if $\epsilon \in K(\Theta)$ it follows from the considerations in Remark 2.7. Hence we can consider the zero-cycle Z of lenght $2r$ given by the intersection of C_H with $C_{H,\epsilon}$.

Let $\overline{Z}$ be the zero locus of $s \in H^0(\bar{A}, \mathcal{E})$ on $\bar{A}$. It is clear that $\pi^*(\overline{Z}) = Z$. Hence the general section of $\mathcal{E}$ vanishes in codimension two. Notice that this fits with the fact that $c_2(\mathcal{E}) = \frac{\Theta^2}{2} = r$.

Now we claim that, by the generality assumption we are making on H, the cycle $\overline{Z}$ consists of r distinct points. In view of the above considerations, this amounts to proving that a general section $s \in H^0(\bar{A}, \mathcal{E})$ vanishes at r distinct points of $\bar{A}$. This is a consequence of the following claim, which is a Bertini type theorem for vector bundles:

Claim. Let V be a smooth irreducible variety and let $\mathcal{E}$ be a vector bundle of rank two on V. Assume that:

(i) $NS(V) \simeq \mathbb{Z}$ is generated by $c_1(\mathcal{E})$,
(ii) $\mathcal{E}$ is generated by global sections away from a subvariety of V of codimension ≥ 3.

Then the zero locus of the general section of $\mathcal{E}$ is reduced.

As for the proof, let σ, τ be general sections of $\mathcal{E}$. Then $\sigma \wedge \tau$ is not identically zero by (ii). Let D be the zero locus of $\sigma \wedge \tau$. By (i), the subvariety D is an irreducible, reduced divisor on V. In particular this implies that the general section of $\mathcal{E}$ vanishes in codimension 2 and, again by (ii), the zero loci of two general sections, like σ, τ of $\mathcal{E}$ have no common component.

Set $s(\lambda, \mu) = \lambda\sigma + \mu\tau$, with $[\lambda, \mu] \in \mathbb{P}^1$. The zero locus $W(\lambda, \mu)$ of $s(\lambda, \mu)$ describes, as $[\lambda, \mu]$ varies in $\mathbb{P}^1$, a linear system of divisors on D, which, by the above considerations, has no fixed divisor. Hence Bertini's theorem ensures that for $[\lambda, \mu]$ general in $\mathbb{P}^1$, the scheme $W(\lambda, \mu)$ is reduced.

Let us now return to $\overline{Z}$. By the above claim (Note that condition (ii) is fulfilled by Lemma 2.1) we can write $\overline{Z} = p_1 + \ldots + p_r$, with $p_1, \ldots, p_r$ distinct points of $\bar{A}$. Then H contains the rulings $F_1, \ldots, F_r$ over the points $p_1, \ldots, p_r$. The map $p : H - \sum_{i=1}^r F_i \to \bar{A} - Z$ is clearly an isomorphism and since the F_i's are contracted by p to the smooth points p_i, it follows that $p : H \to \bar{A}$ is the blow-up in $\overline{Z}$. This proves part (i).

As for part (ii), we apply the adjunction formula and obtain:

$$\mathcal{O}_H(F_1 + \ldots + F_r) = \mathcal{O}_H(K_H) = \mathcal{O}_H \otimes \mathcal{O}_X(1) \otimes \mathcal{O}_X(K_X)$$

which, by Proposition 1.6, concludes the proof. □

4. – The case $r = 7$

In this section we will prove the:

THEOREM 4.1. *Let A be a linearly normal abelian surface embedded in $\mathbb{P}^6$ via a line bundle $\mathcal{L}$ which determines a polarization Θ of type* $(1, 7)$. *Assume that* $\mathrm{End}(A) \simeq \mathbb{Z}$. *Then the map $\phi : X \to Y$ is an embedding.*

Fist of all we notice that $\mathrm{End}(A) \simeq \mathbb{Z}$ implies $NS(A) \simeq \mathbb{Z}$ (see [LB, p. 122]). Since Θ is indivisible, it generates $NS(A)$. Next we remark that no non-trivial ϵ is an element of $K(\Theta)$.

The proof will require several steps. Let us start by denoting by Σ the singular locus of Y and let $S = \phi^{-1}(\Sigma)$. Recall that $|2A|$ is a base point free pencil on X containing three double fibres $A = A_1$, A_2, and A_3, which are the only reducible, singular members of the pencil (see Section 1, from which we keep the notation). Notice also that $\mathrm{End}(A_i) \simeq \mathbb{Z}$, for $i = 1, 2, 3$, since we have made the assumption that $\mathrm{End}(A) \simeq \mathbb{Z}$. In particular $\mathcal{L}_i$ is very ample on A_i, $i = 1, 2, 3$, and A_1, A_2, A_3 play a symmetric role with respect to X and Y.

LEMMA 4.2. *One has:*

$$S \cap A_i = \emptyset, \quad i = 1, 2, 3 \tag{1}$$

and for every irreducible component Z of S, there is a unique irreducible, smooth surface $D \in |2A|$ on X containing Z.

PROOF. As we noticed, A_1, A_2, A_3 play a symmetric role with respect to X and Y. Therefore (1) follows by (3.3, ii). Moreover, $Z \cdot A = 0$ in the homology ring and $|2A|$ is a base point free pencil, whose elements sweep out X. Therefore there is an element $D \in |2A|$ containing Z. By Proposition 1.5 the surface D is smooth and irreducible. □

We want to prove that $\Sigma = \emptyset$. First we prove that:

LEMMA 4.3. *One has $dim(\Sigma) \leq 1$.*

PROOF. We argue by contradiction and therefore, according to Lemma 4.2, we may assume that one of the following happens:

(i) there is a surface $D \in |2A|$ on which $d\phi$ is not injective;
(ii) there are surfaces $D_1, D_2 \in |2A|$ which have the same image via ϕ;
(iii) there is a surface $D \in |2A|$ on which ϕ is not injective.

In case (i) the differential $d\phi$ would not be injective at 4 distinct points of a general ruling F of X. By the same argument we made in the proof of part (iii) of Theorem 3.3, $d\phi$ would not be injective along the whole of F, a contradiction.

In case (ii) we may assume that both D_1 and D_2 map birationally, via ϕ, to some irreducible component of Σ, otherwise we are in case (iii). Hence we have a birational map $a : D_1 \to D_2$, which is an isomorphism, since D_1 and D_2 are both abelian surfaces. Recall that D_1 and D_2 are isomorphic as double covers of A and that this isomorphism is compatible with the projection onto $\bar{A}$. Since $\mathrm{End}(A) \simeq \mathbb{Z}$ the same is true for D_1 and D_2. Hence the map a is of type $a : z \in D_1 \to \pm z + k \in D_2$, where k is a fixed element in D_2. But then this would imply that the 4 points of intersection of D_1 with any ruling F of X are mapped, via a, to the 4 points of intersection of D_2 with another ruling F'. Hence ϕ would map F and F' to the same line in $\mathbb{P}^6$, contradicting Theorem 3.3, (i).

In case (iii), let μ be the degree of $\phi_{|D}$. Since $H^2 \cdot D = 28$, we can only have the possibilities $\mu = 2, 4, 7, 14, 28$. If $\mu \geq 7$, then $\Sigma' = \phi(D)$ would be degenerate, implying that the linear system $|H - D| = |H - 2A|$ is effective, a contradiction, since $F \cdot (H - 2A) = -3$.

If $\mu = 2$, then we have an involution a on D and we can argue as we did in case (ii) to arrive at a contradiction.

Let us consider the last case $\mu = 4$, in which the degree of Σ' is 7, and we may assume that Σ' is non-degenerate in $\mathbb{P}^6$. Let z be a general point of D and let F be the ruling of X through z. Since ϕ maps F isomorphically to a line L in $\mathbb{P}^6$, every such ruling is a proper 4-secant to the surface Σ', i.e. it intersects the surface in 4 different points. We see that the fibres containing z of the maps $D \to \bar{A}$ and $D \to \Sigma'$ intersect only at z. This implies that through the general point $p = \phi(z) \in \Sigma'$ there are at least four 4-secant lines to Σ'. Hence, by projecting Σ' from p down to $\mathbb{P}^5$, we have a surface Σ'' of degree 6 with at least 4 distinct triple points p_1, p_2, p_3, p_4.

First we claim that p_1, p_2, p_3, p_4 are not collinear for a general projection. Suppose in fact they lie on a line L. Then a simple application of Bezout's theorem shows that the general hyperplane section Γ of Σ'' through L consists of $2L$ plus a residual curve Γ' of degree 4 containing p_1, p_2, p_3, p_4. Hence, again by Bezout's theorem, Γ' must be reducible into degenerate components. This implies that the projection of Σ'' from L to $\mathbb{P}^3$ is a non-degenerate curve Λ. Let $\alpha \geq 2$ be the multiplicity of Σ'' at a general point of L, let $\lambda \geq 3$ be the degree of Λ and let β be the degree of the general fibre of the projection $\Sigma'' \to \Lambda$. Then we have:

$$6 = \alpha + \lambda\beta \geq 2 + \lambda\beta\,.$$

This gives us the possibilities: $\alpha = 2$, $\lambda = 4$ and $\beta = 1$ or $\alpha = 3$, $\lambda = 3$ and $\beta = 1$. In either case Σ'' would be a scroll, and Γ' would consist of 3 or 4 rulings. We shall first treat the case where $\alpha = 2$. Since Σ'' has a double line we have two possibilities. Either Σ' has a double line or Σ' has a pencil of plane curves (recall that the general projection of Σ' contains a double line by our assumption). In the second case the surface Σ' cannot span $\mathbb{P}^6$. This follows since in this case every ruling meets each of these plane curves in one point (recall that $\beta = 1$) and this shows that all rulings are incident to two planes, i.e. contained in a fixed $\mathbb{P}^5$. Finally assume that Σ' has a double line. Recall that there are 4 proper 4-secant lines through a general point of Σ'. Hence the multiplicity of the corresponding singular points is greater than the multiplicity of a general point of the line L and we see that when Γ' moves, there are infinitely many rulings through each of the points p_1, p_2, p_3, p_4. Hence Σ' would be a cone with vertex each one of the points p_1, p_2, p_3, p_4, a contradiction. It remains to treat the case where $\alpha = 3$. If Σ' does not have a multiple line then we can argue as above that it cannot span $\mathbb{P}^6$. Finally assume that Σ' has a multiple line. Again, since there are 4 proper 4-secant lines through a general point of Σ', the multiplicity of the corresponding singular points is greater than the multiplicity of a general point of the line L and our above argument goes through unchanged.

Let us now project Σ'' to $\mathbb{P}^4$ from one of the points p_1, p_2, p_3, p_4. If the projection is not a surface, then Σ'' is a cone with vertex the point we are projecting from. This cannot happen for all the points under consideration, therefore we can assume that the projection from, say, p_1 is a surface, which is an irreducible, non-degenerate surface of degree 3 in $\mathbb{P}^4$, with at least two points of multiplicity at least three, a contradiction. □

The next step is to further bound the dimension of the singular locus Σ of X.

LEMMA 4.4. *One has* $dim(\Sigma) \leq 0$.

PROOF. Again we argue by contradiction and we assume that the locus Σ has components of dimension 1. Hence $S = \phi^{-1}(\Sigma)$ has components of dimension 1. Let S_1 be their union. According to Lemma 4.2, the curve S_1 is contained in the union of finitely many irreducible divisors of $|2A|$. Let D be one of these and let γ be the part of S_1 contained in D. By Lemma 1.9 the polarization H_D is of type $(1, 14)$ and hence there is a positive integer a such that $\gamma = aH_D$ in $NS(D)$.

Since the group $K(\Theta)$ fixes $\mathcal{L}$, it also acts on the vector bundle $\mathcal{E}$ and therefore it acts on X. Furthermore it acts on $H^0(X, \mathcal{O}_X(1)) \simeq H^0(A, \mathcal{L})$. This action is trivial on the pencil $|-K_X| = |2A|$, since the divisors $A_i, i = 1, 2, 3$, are fixed. Notice that S is fixed by this action, hence S_1 is and therefore γ is.

Consider now a general map $f : X \to \mathbb{P}^5$ as in Section 2. A straightforward parameter count shows that the singular locus of $f(X)$ has still dimension 1. Then by applying Theorem 2.4, we see that $\mathbb{D}(X, 5) = 6H \cdot (H + A)$. Hence $6H \cdot (H + A) - \gamma$ is represented by an effective cycle. By Lemma 1.10 we have $\gamma = 7aH_D$ in $NS(D)$, with a a positive integer. Hence we have $\gamma = 14aH \cdot A$. Therefore $6H^2 + (6 - 14a)H \cdot A$ is represented by an effective cycle. By Lemma 3.5, (ii), we have $H^2 = H \cdot A + 7F$. Hence $(12 - 14a)H \cdot A + 42F$ is represented by an effective cycle, whose intersection with the pull-back W on X of an ample divisor on $\bar{A}$ is non negative. This forces $a \leq 0$, a contradiction. □

Finally we can finish the:

PROOF OF THEOREM 4.1. Since we know now that $S = \phi^{-1}(\Sigma)$ is finite, we can use the double point formula from Corollary 2.5, which tells us that $\mathbb{D}(X, 6) = 0$, implying that S is empty. □

5. – The case $r = 8$

In the case $r = 8$ we have two possibilities, namely the abelian surface A is either embedded in $\mathbb{P}^7$ via a line bundle $\mathcal{L}$ belonging to a polarization Θ of type $(1, 8)$ or to a polarization Θ of type $(2, 4)$. In the former case there are three non trivial points $\epsilon \in A$ of order two which are contained in $K(\Theta) \simeq \mathbb{Z}/8 \times \mathbb{Z}/8$, in the latter case every such point is an element of $K(\Theta)$.

As we will see in a moment, the two cases $\epsilon \notin K(\Theta)$ and $\epsilon \in K(\Theta)$ give rise to a completely different behaviour of the map $\phi : X \to Y$.

The main result of this section is the following:

THEOREM 5.1. *Let A be an abelian surface such that* $\mathrm{End}(A) \simeq \mathbb{Z}$, *linearly normally embedded in $\mathbb{P}^7$ via a line bundle $\mathcal{L}$ giving a polarization Θ in $NS(A)$. Let ϵ be a non-trivial point of order two on A. Then:*

(i) *if $\epsilon \in K(\Theta)$, the map $\phi : X \to Y$ fails to be an embedding along the sections $\bar{A}^{\pm}$ of X (see Remark* 1.4, (i)), *whereas it is an embedding on the open subset which is the complement of these two sections;*

(ii) *if $\epsilon \notin K(\Theta)$, which implies Θ to be of type* $(1, 8)$, *the map $\phi : X \to Y$ is an embedding.*

PROOF. (i) We use the notation of Remark 2.7. The spaces $\mathbb{P}^{\pm}$ are both of dimension 3 and the maps $\phi^{\pm} : \bar{A}^{\pm} \to Y^{\pm}$ cannot be embeddings. On the other hand ϕ is an embedding on $X - (\bar{A}^{+} \cup \bar{A}^{-})$. This follows from Theorem 3.3, (ii): the linear system $|\bar{A}|$ is a base point free pencil whose elements, with the exception of $2\bar{A}^{+}$ and $2\bar{A}^{-}$, are smooth and isomorphic to A. The pencil $|A|$ sweeps out X and its smooth elements play a symmetric role in the description of X and ϕ.

(ii) The structure of the proof in this case is somewhat similar to the one of the case $d_1 = 1$, $d_2 = 7$, $r = 7$. We still denote by Σ the singular locus of Y and we let $S = \phi^{-1}(\Sigma)$. Lemma 4.2 still holds. The proof of Lemma 4.3 can be adapted with minor changes to the present situation, showing again that the dimension of Σ is at most 1. We leave these details to the reader.

Let now S_1 be the union of the one-dimensional components of S. Again S_1 is contained in the union of finitely many irreducible divisors D of $|2A|$. By Theorem 2.4 we find for the double locus $\mathbb{D}(X, 5) = 9H^2 + 6H \cdot A = 15H \cdot A + 72F$ where the last equality follows from Theorem 3.5. By Lemma 1.9 the polarization H_D is of type $(1, 16)$ or $(2, 8)$. By Lemma 1.10 there is an integer a such that $S_1 = 4aH \cdot D = 8aH \cdot A$ in homology. Since $\mathbb{D}(X, 5) - S_1 = (15 - 8a)H \cdot A + 72F$ is effective it follows that $a = 1$ and that S_1 is contained in a unique surface $D \in |2A|$.

We first remark that S_1 is reduced by Lemma 1.10. It then follows that the map ϕ restricted to S_1 is generically two-to-one onto the curve $\Gamma = \phi(S_1)$. (Here we use the fact that if a point P is simple for the double point cycle, then the differential at P is injective and there is only one other point Q mapped to the same point as P. This can be deduced from the construction of the double point locus (cf. [F, p. 166]).) But then the degree of Γ is 64. Let C be a general tangent hyperplane section of $\phi(D)$. This is an irreducible reduced curve of degree 32 with 65 nodes. Its pullback on C has self-intersection 32 and one node, hence geometric genus 16. Thus the arithmetic genus of C is 81 which is equal to the Castelnuovo bound for non-degenerate curves in $\mathbb{P}^6$ (cf. [EH, p. 87]). Since this bound can only be achieved by smooth curves we obtain a contradiction.

We may therefore assume that the singular locus Σ of Y is finite. Assume Σ is not empty. Then $S = \phi^{-1}(\Sigma)$ consists of orbits of $K(\Theta) \simeq \mathbb{Z}_8^2$. From Theorem 2.4 we see that $\mathbb{D}(X, 6) = 72$. This implies that S consists of a unique orbit formed by 64 distinct points that are pairwise coupled by ϕ, which sends them to 32 distinct points of Y where two branches of Y meet transversally. Let Z be this set of 32 points in $\mathbb{P}^7$.

Since every surface $D \in |2A|$ is $K(\Theta)$-invariant, S must lie on a unique surface $D \in |2A|$, which is therefore mapped by ϕ to a surface Δ with at least 32 double points at Z. Since Z is also $K(\Theta)$-invariant, there is for each $z \in Z$ an element h_z of order 2 of $K(\Theta)$ fixing it. The element h_z does not depend on z. This follows since the stabilizers of elements in the same orbit for a group action are conjugated, resp. equal if the group is abelian.

Now there are three elements of order two in $K(\Theta) \simeq \mathbb{Z}_8^2$. Remember how $K(\Theta)$ acts on $\mathbb{P}^7$ where the coordinates are $[x_0, \dots, x_7]$. We have the two generators σ and τ of $K(\Theta)$ acting as follows (cf. [LB, p. 169]):

$$\sigma : [x_0, \dots, x_7] \to [x_7, x_0 \dots, x_6,]$$

$$\tau : [x_0, \dots, x_7] \to [x_0, \xi^{-1}x_1, \dots, \xi^{-i}x_i, \dots, \xi^{-7}x_7]$$

where $\xi = \exp(\frac{2\pi\sqrt{-1}}{8})$. The elements of order two are σ^4, τ^4 and their product. Suppose $h = \tau^4$ (the discussion is similar in the other cases). Then:

$$h : [x_0, \dots, x_7] \to [x_0, -x_1, \dots, (-1)^i x_i, \dots, -x_7]$$

and therefore its eigenspaces are both of dimension 3. Since Z is contained in their union, we deduce that at least 16 points of Z lie in a $\mathbb{P}^3$ which we denote by P. Notice that $P \cap \Delta$ is finite. Otherwise we could deduce by applying σ that both eigenspaces of h would intersect Δ in a curve. But then D would contain 2 curves which do not intersect which contradicts our assumption that $NS(A) \simeq \mathbb{Z}$ and hence also $NS(D) \simeq \mathbb{Z}$. Now consider the intersection of Δ with a hyperplane through P. This is an irreducible curve B of degree 32, non degenerate in $\mathbb{P}^6$, with at least 16 singular points on P. Let q be a point on B. Any hyperplane in $\mathbb{P}^6$ containing P and q has to contain B by Bezout's theorem, a contradiction. □

6. - The case $r = 9$

In this section we prove the following

THEOREM 6.1. *Let A be an abelian surface such that $NS(A) \simeq \mathbb{Z}$, linearly normally embedded in $\mathbb{P}^8$ via a line bundle $\mathcal{L}$ giving a polarization Θ in $NS(A)$ and let ϵ be any non trivial point of order 2 on A. Then the map $\phi : X \to Y$ is an embedding.*

PROOF. The polarization Θ is of type (d_1, d_2) with $9 = d_1 \cdot d_2$, hence we have only the two cases $d_1 = 1, d_2 = 9$ and $d_1 = d_2 = 3$. In both cases $\epsilon \notin K(\Theta)$. In the former case the assertion follows by Theorem 3.3. Hence we consider only the latter case, in which $\Theta = 3\theta$ in $NS(A)$, where θ is a principal polarization.

By Proposition 3.4, the morphism $\phi : X \to Y$ is birational. Suppose two distinct points z, z' of X are mapped to the same point w by ϕ. By Theorem 3.3, (ii), z and z' do not lie on A. Let F and F' be the two rulings of X through z and z' respectively, and let $x, x + \epsilon$ and $y, y + \epsilon$ the pair of points where F and F', respectively, meet A. Then the points $w, x, x + \epsilon, y, y + \epsilon$ are coplanar in $\mathbb{P}^8$, hence we have a 4-secant plane to A. By Proposition 3.2, (iii), there is an irreducible curve C, representing the polarization θ, passing through $x, x + \epsilon, y, y + \epsilon$. Consider the curve $C_\epsilon = t_\epsilon(C)$. Since $\epsilon \notin K(\theta) = 1$, it follows that $C_\epsilon \neq C$. On the other hand $x, x + \epsilon, y, y + \epsilon$ belong to both C and C_ϵ. Since $C \cdot C_\epsilon = C^2 = 2$, we get a contradiction.

Suppose $d\phi$ is not injective at $z \in X$. Again $z \notin A$. Let F be the ruling of X through z and let $x, x' = x + \epsilon$ be the pair of points where F meets A. The same argument we made in the proof of Theorem 3.3, (iii), shows that there are two tangent lines r and r' to A at x and x' which lie in a plane π which is therefore 4-secant. Once more by Proposition 3.2, (iii) there is a curve C representing θ, passing through x and x' and whose tangent cone at these points contains the lines r and r'.

Since $H^1(A, \mathcal{L}(-C)) = H^1(A, \mathcal{O}_A(2C)) = 0$, the map $H^0(A, \mathcal{L}) \to H^0(C, \mathcal{L}_{|C})$ is surjective, i.e. C is a curve of degree 6 which spans a $\mathbb{P}^4$. It follows that the linear system on C cut out by the hyperplanes through the plane π contains $2x + 2x'$ and a residual g^1_2 which must be the canonical g^1_2 of C, i.e. $\mathcal{L}_{|C} \simeq \mathcal{O}_C(2x+2x'+K_C)$. As before $C \cap C_\epsilon = \{x, x'\}$. Hence $x+x'$, as a divisor on C, is linearly equivalent to $K_C + \eta$, where η is a suitable point of order two in $\operatorname{Pic}^0(C)$. Therefore $2x + 2x' \equiv 2K_C$ and this yields that $\mathcal{L}_{|C} \simeq \mathcal{O}_C(3K_C)$. This implies that actually $\mathcal{L} \simeq \mathcal{O}_A(3C)$ (see [CFM, Proposition 1.6]).

This picks out exactly 81 curves C representing θ and therefore 81 rulings of X on which such a point z can lie. Note that these rulings form an orbit under the free action of the group $K(\Theta) \simeq \mathbb{Z}_3^4$ on the set of rulings of X. Hence the set of points Z of X where $d\phi$ is not injective is finite. The image of Z via ϕ is the set of singular points of Y, which is therefore also finite. Moreover Z is stable by the action of $K(\Theta) \simeq \mathbb{Z}_3^4$ on X. This implies that Z is of order $81n$, so that Z consists of 81 n-tuples of points, each n-tuple lying on one of the aforementioned 81 rulings of X.

Let us assume that $n \geq 1$. We consider the projection of Y in $\mathbb{P}^6$ from a general line R in $\mathbb{P}^8$. Since Y has finitely many singularities the same holds for Y'. The degree of $\mathbb{D}(X, 6)$ is 162 and since each point of Z clearly appears in $\mathbb{D}(X, 6)$ with multiplicity at least 2, we see that $\mathbb{D}(X, 6) = 2Z$. In particular $n = 1$. As a consequence, we also have that the secant variety of Y cannot meet the general line R of $\mathbb{P}^8$, hence it has dimension $\nu \leq 6$, i.e. it has dimension smaller than expected. This is excluded by a theorem of Scorza [S]. Hence we come to a contradiction, which proves that $n = 0$, thus proving that ϕ is an embedding.

Scorza's argument in [S] is long and rather complicated. We give here, for the reader's convenience, a shorter version of it, adapted to our case. Assume the secant variety of Y has dimension $\nu \leq 6$. By Terracini's lemma (see [LV, p. 18] or [Z, Prop. 1]), two general tangent spaces to Y meet in a subspace of dimension $6 - \nu$. Actually we see that $\nu = 6$. Otherwise the general surface section S is such that two general tangent spaces to it meet. Then it is well known that S lies in a $\mathbb{P}^r$, with $r \leq 5$ (see [LV] or [CC]), a contradiction.

Let us now make the projection ψ of Y to $\mathbb{P}^4$ from the tangent space π at a general point $y \in Y$. Since any other general tangent space to Y meets π at one point, we see that the differential of ψ has generic rank 2. Hence $W = \psi(Y)$ is a surface.

Notice that the general ruling of Y does not meet π. Otherwise the general ruling would be contained in the span of two general tangent spaces to Y, which is a $\mathbb{P}^6$. Then the whole Y would be contained in this $\mathbb{P}^6$, a contradiction. Therefore, since Y is a scroll, W is also a scroll. Notice that W has only a 1-dimensional system of lines, otherwise it would be a plane, contrary to the fact that it has to span a $\mathbb{P}^4$. Let R be a general line of W and let V be the closure of $\psi^{-1}(R)$, which is a surface in a $\mathbb{P}^5$. Then there is an irreducible component V' of V which is a scroll. The intersection of V' with A contains a curve C which is fixed by t_ϵ. Hence C represents $a\theta$ with a positive and even. Furthermore C, as well as V', spans at most a $\mathbb{P}^5$. Consider the exact sequence:

$$0 \to \mathcal{O}_A(-C) \otimes \mathcal{L} \to \mathcal{L} \to \mathcal{L}_{|C} \to 0.$$

Since $\mathcal{O}_A(-C)\otimes\mathcal{L}$ represents $(3-a)\theta$, we see that $H^1(A, \mathcal{O}_A(-C)\otimes\mathcal{L}) = 0$ (see [LB, p. 66]). Hence the restriction map $H^0(A, \mathcal{L}) \to H^0(C, \mathcal{L}_{|C})$ is surjective. Then we must have:

$$\begin{aligned} 6 \geq h^0(C, \mathcal{L}_{|C}) &= h^0(A, \mathcal{L}) - h^0(A, \mathcal{O}_A(-C) \otimes \mathcal{L}) \\ &= 9 - h^0(A, \mathcal{O}_A(-C) \otimes \mathcal{L}) = 9 - h^0(A, \mathcal{O}_A((3 - a)\theta)) \end{aligned}$$

i.e. $h^0(A, \mathcal{O}_A((3 - a)\theta)) \geq 3$. Since a is positive and even, $h^0(A, \mathcal{O}_A((3 - a)\theta)) \leq h^0(A, \mathcal{O}_A(\theta)) = 1$, a contradiction. $\square$

REFERENCES

[BS] M. Beltrametti – A. Sommese, *Zero-cycles and k-th order embeddings of smooth projective surfaces*, In: "Problems in the theory of surfaces and their classification" (F. Catanese, C. Ciliberto, M. Cornalba eds.), Symposia Math. 32 (1991), 33-48.

[CC] L. Chiantini – C. Ciliberto, *Weak defective surfaces*, in preparation.

[CFM] C. Ciliberto – P. Francia – M. Mendes Lopes, *Remarks on the bicanonical map for surfaces of general type*, Math. Z. **224** (1997), 137-166.

[CH] C. Ciliberto – K. Hulek, *A remark on the geometry of elliptic scrolls and bielliptic surfaces*, Manuscripta Math. **95** (1998), 213-224.

[EH] D. Eisenbud – J. Harris, "Curves in projective space", Montreal University Press, 1982.

[F] W. Fulton, "Intersection Theory", Ergebni. der Math., Springer-Verlag, 1984.

[LB] H. Lange – Ch. Birkenhake,"Complex Abelian Varieties", Grund. der Math., Springer-Verlag, 1992.

[LV] R. Lazarsfeld – A. van de Ven, "Topics in the Geometry of Projective Space", DMV Seminar, Band 4, Birkäuser Verlag, 1984.

[R] E. Rogora, *Varieties with many lines*, Manuscripta Math. **82** (1994), 207-226.

[S] G. Scorza, *Determinazione delle varietà a tre dimensioni di* S_r $(r \geq 7)$ *i cui* S_3 *tangenti si tagliano a due a due*, Rend. Circ. Mat. Palermo (2) **25** (1908), 193-204.

[Z] F. Zak, *Projections of algebraic varieties*, Math. USSR Sb. **44** (1983), 535-544.

Università di Roma "Tor Vergata"
Dipartimento di Matematica
Via della Ricerca Scientifica
I 00173 Roma, (Italy)
Cilibert@axp.mat.uniroma2.it

Institut für Mathematik
Universität Hannover
Post fach 30060
D 3167 Hannnover, (Germany)
Hulek@math.uni-hannover.de