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1. Introduction

We shall consider the Cauchy problem

\[\begin{align*}
u_{tt}(t, x) + \Phi((Au(t, \cdot), u(t, \cdot))) Au(t, x) &= f(t, x) \quad (t > 0, x \in \mathbb{R}^n), \\
u(0, x) &= u_0(x), \quad u_t(0, x) = u_1(x),
\end{align*}\]

where \(Au(t, x) = \sum_{hk} D_{x_h} (a_{hk}(x) D_{x_k} u(t, x)), \quad D_{x_h} = \partial \partial_{x_h} (h = 1, \ldots, n),\)

\((Au(t, \cdot), u(t, \cdot))\) is the inner product of \(Au(x)\) and \(u(x)\) in \(L^2(\mathbb{R}^n),\) and \(\Phi(\eta)\) is a nonnegative function in \(C^1([0, \infty)),\) When \(A = \sum_{h=1}^{n} D_{x_h}^2 = -\Delta\) (i.e. \(a_{hk}(x) = \delta_{hk},\) Kronecker’s \(\delta)\) equation (1) is called the Kirchhoff equation, which has been studied by many authors (cf. [2], [3], [5], [12], [13], [14], [16], etc.). The problem which we shall treat in this paper is a generalization of \(-\Delta\) to a degenerate elliptic operator \(A,\) where \([a_{hk}(x)]_{hk}\) is a real symmetric matrix which satisfies

\[\sum_{hk} a_{hk}(x) \xi_h \xi_k \geq 0 \quad (\forall x \in \mathbb{R}^n_x, \forall \xi = (\xi_1, \ldots, \xi_n) \in \mathbb{R}^n_\xi).
\]

We know some results for the problem (1), that are the following. When the coefficients \(a_{hk}\) and the Cauchy data \(u_0\) and \(u_1\) belong to real analytic class, Kajitani-Yamaguti [9] proved global existence and uniqueness for the solution of (1) in case that \(\Phi(\eta) \in C^1([0, \infty))\) and \(\Phi(\eta) \geq 0,\) and later Hirosawa [7] relaxed the assumption on \(\Phi\) to \(\Phi(\eta) \in C^0([0, \infty)).\) In case of quasi-analytic class data, Yamaguti [17] proved the global solvability for (1) in case that \(u_0\) and \(u_1\) belong to a certain subclass of quasi-analytic functions (for the definition of quasi-analytic class, see [10]) under the assumptions that \(\Phi(\eta) \in C^1([0, \infty)),\)

\(\Phi(\eta) \geq \exists \Phi_0 > 0\) and that \([a_{hk}(x)]_{hk}\) is real analytic. Actually in case \(A = -\Delta,\)

Nishihara [13] investigated the global solvability for data in the quasi-analytic class. The class of functions which was introduced in [13] is more general than

Yamaguti's one in [17]. This paper enhances Nishihara's work to a general
degenerate elliptic operator $A$ whose coefficients satisfy a particular condition,
specified below, with respect to the space variable $x$. We remark that the
problem which was treated in [13] is strictly hyperbolic, and the author showed
a sufficient condition for the existence of a local solution in a more general
class. But our problem is weakly hyperbolic, so we can't get the same result
for local existence in general.

At first we define some function spaces to state our main theorem.

**Definition.** Let $M_0, M_1, \ldots$ be a sequence of positive real numbers and
$\rho$ a positive constant.

(i) A $C^\infty(\mathbb{R}^n)$ function $f$ said to belong to the class $C((M_j)_\rho)$ if there exists
a $C > 0$ independent of $\alpha$, $\rho$ and $x$ such that

$$|D_{\alpha}^f(x)| \leq C \rho^{-|\alpha|} M_{|\alpha|},$$

where $\alpha = (\alpha_1, \ldots, \alpha_n)$ is a multi-index of nonnegative integers, $D_{\alpha}^f =
D_{\alpha_1}^f \cdots D_{\alpha_n}^f$ and $|\alpha| = \alpha_1 + \cdots + \alpha_n$. For $f(x) \in C((M_j)_\rho)$ we define the
norm $|f|_{C((M_j)_\rho)}$ by

$$|f|_{C((M_j)_\rho)} = \sup_{\alpha \in \mathbb{N}^n, x \in \mathbb{R}^n} \left\{ \rho^{\langle \alpha \rangle} M_{|\alpha|} |D_{\alpha}^f(x)| \right\}.$$ (ii) Let $\{M_j\}$ be a monotone increasing, logarithmically convex sequence, that
is, $\frac{M_j}{M_{j-1}} \leq \frac{M_k}{M_{k-1}}$ for any $j \leq k$. A $C^\infty(\mathbb{R}^n)$ function $f$
said to belong to the class $C^2((M_j)_\rho)$ if there is a $C > 0$ independent of $\alpha$ and $\rho$
such that

$$\|D_{\alpha}^f\| \leq C \rho^{-|\alpha|} M_{|\alpha|},$$

where $\|\cdot\|$ is the usual norm in $L^2(\mathbb{R}^n)$. We set $C^2((M_j)_\rho) \equiv \bigcup_{\rho > 0} C^2((M_j)_{\rho}).$
Moreover if the summation

$$\sum_{j=0}^{\infty} \frac{\rho^j}{M_j} \left\{ \sum_{|\alpha| = j} \|D_{\alpha}^f\| \right\}^{\frac{1}{2}}$$

is finite, then we say that $f$ belongs to the class $C^2((M_j)_\rho)$ and define the
norm $\|f\|_{C^2((M_j)_\rho)}$ by

$$\|f\|_{C^2((M_j)_\rho)} = \sum_{j=0}^{\infty} \frac{\rho^j}{M_j} \left\{ \sum_{|\alpha| = j} \|D_{\alpha}^f\| \right\}^{\frac{1}{2}}.$$
MAIN THEOREM. Let $\rho_0$ and $\rho_1$ be positive numbers satisfying $\rho_0 < \rho_1$, $\{M_j\}$ be a positive, logarithmically convex sequence and $\sup_j Q(j) \equiv \sup_j \{M_j/j^{\frac{3}{2}}M_{j-1}\} < \infty$. Assume that $[a_{hk}(x)] \in [C([M_j])_{\rho_1}]$ satisfies (2), $u_0(x), u_1(x) \in C^0([M_j])_{\rho_0}$, $f(t, x) \in C^0([0, T]; C^2([M_j])_{\rho_0})$ and that $\Phi(\eta) \in C^1([0, \infty))$ satisfies $\Phi(\eta) \geq \exists \Phi_0 > 0$ for all $\eta \in [0, \infty)$. Then there are a positive number $T_0([M_j]) = T_0 \leq T$, a monotone decreasing positive function $\rho(t) \in C^1([0, T_0])$ and the unique solution $u(t, x)$ of (1) such that

$$u(t, x) \in C^2([0, T_0]; C^2([M_j])_{\rho(t)})$$

Moreover, if $\{M_j\}$ satisfies a quasi-analytic condition, that is, $\sum_{j=1}^\infty M_j/M_{j+1} = \infty$ (see Theorem A) and $Q(j) \to 0$ as $\eta \to \infty$, then we can take $T_0 = T$.

REMARK 1. Quasi-analytic condition holds for $\{M_j\} = \{1 (j = 1), \prod_{k=2}^j k \log k (j \geq 2)\}, \{M_j\} = \{j^{\frac{1}{2}}(s \leq 1) and so on. The condition $\sup_j Q(j) < \infty$ holds for $\{M_j\} = \{j^{\frac{1}{2}}\} s \leq 3/2$, and so on. In the linear weakly hyperbolic case, that is for $\Phi(\eta) \equiv$ constant, the condition $\sup_j Q(j) < \infty$ is optimal (see [4]).

REMARK 2. Roughly speaking, the class of the solution $\mathcal{H}(M)$ introduced in [13] is defined as follows. For a strictly increasing nonnegative continuous function $M(r)$, $\mathcal{H}(M)$ is the set of functions $f(x)$ satisfy $\|M(\cdot)\hat{f}(\cdot)\|_{L^2} < \infty$, where $|\xi| = \sqrt{\xi_1^2 + \ldots + \xi_n^2}$ and $\hat{f}(\cdot)$ is the Fourier image of $f(x)$. Then the condition for $M(r)$ which ensures the existence of a global solution is the following:

$$\int_c^\infty \frac{ds}{s(d_0 + M^{-1}(s))} = \infty,$$

for some positive constants $c$ and $d_0$, where $M^{-1}$ is the inverse function of $M$. On the other hand, by the Denjoy-Carleman theorem (Theorem A), we can easily see that the quasi-analytic condition $\sum M_j/M_{j+1} = \infty$ is equivalent to (3) by considering $M(r) \equiv \sup\{r_j/M_j\}$, here $M(r)$ is the associated function of $\{M_j\}$.

REMARK 3. In strictly hyperbolic case (i.e., when the matrix $[a_{hk}]$ is strictly positive definite), we can omit the assumption $\sup_j Q(j) < \infty$ to prove the local existence of the solution.

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2. – Proof of Main Theorem

In case that the coefficients \( a_{hk}(x) \) \((h, k = 1, \cdots, n)\) and the Cauchy data are real analytic, we know the following result:

**Theorem 1** (K. Kajitani-K. Yamaguti [9]). Let \( \rho_0 \) and \( \rho_1 \) be positive constants satisfying \( \rho_0 < \rho_1 \). Assume that \( \Phi(\eta) \in C^1([0, \infty)) \), \( \Phi(\eta) \geq 0 \) and \([a_{hk}(x)] \in [C([j!]])_{\rho_1}\). Then if \( u_0(x), u_1(x) \in C^2([j!])_{\rho_0} \) and \( f(t, x) \in C_0([0, T]; C^2([j!]_{\rho_0}) \) for some \( T > 0 \), there is a nonnegative function \( \rho(t) \in C^1([0, T]) \) such that the Cauchy problem (1) has the unique solution \( u(t, x) \in C^2([0, T]; C^2([j!]_{\rho(t)}) \).

We shall relax the assumption of Theorem 1 from \([j!]\) to general \(\{M_j\}\).

For given \( u_0, u_1, a_{hk} \) and \( f \) satisfying the assumptions of Main Theorem, we define the real analytic functions \( u_0^{(v)}, u_1^{(v)}, a_{hk}^{(v)} \) and \( f^{(v)} \) as

\[
 u_0^{(v)}(x) \equiv \chi_{\frac{1}{v}} \ast u_0(x), \quad u_1^{(v)}(x) \equiv \chi_{\frac{1}{v}} \ast u_1(x), \\
 a_{hk}^{(v)}(x) \equiv \chi_{\frac{1}{v}} \ast a_{hk}(x) \quad (h, k = 1, \cdots, n), \quad f^{(v)}(x) \equiv \chi_{\frac{1}{v}} \ast f(x), \quad (v = 1, 2, \cdots),
\]

where \( \chi_{\frac{1}{v}} \ast \) is Friedrichs’ mollifier and \( \chi(y) \) satisfies \( \chi(y) \geq 0 \) and \( \int |D^\alpha_y \chi(y)|dy \leq \rho_0 |\alpha|! \).

Note that, \( \chi_{\frac{1}{v}} \ast \) is an analytic convolution, so \( u_0^{(v)}(x), u_1^{(v)}(x), a_{hk}^{(v)}(x) \) and \( f^{(v)}(x) \) are real analytic functions, namely, \( u_0^{(v)}(x), u_1^{(v)}(x) \in C^2([j!])_{\rho_0}, \ \ [a_{hk}^{(v)}(x)] \in [C([j!]_{\rho_1}) \) and \( f^{(v)}(t, x) \in C_0([0, T]; C^2([j!]_{\rho_0}) \) for any fixed \( v \).

These regularized functions satisfy the following properties:

**Lemma 1.**

(i) \( D_s^\alpha a_{hk}^{(v)}(x) \) converges to \( D_s^\alpha a_{hk}(x) \) in \( C([M_j])_{\rho_1} \) \((h, k = 1, \cdots, n)\).

(ii) \( u_0^{(v)}(x) \) and \( u_1^{(v)}(x) \) converge to \( u_0(x) \) and \( u_1(x) \) respectively in \( C^2([M_j])_{\rho_0} \) as \( v \rightarrow \infty \).

(iii) \( f^{(v)}(t, x) \) converges to \( f(t, x) \) uniformly with respect to \( t \in [0, T] \) in \( C^2([M_j])_{\rho_0} \) as \( v \rightarrow \infty \).

**Proof.** (i) can be easily seen by using the Lebesgue convergence theorem. We can prove (ii) by using a property of Fourier transformation. Indeed, the difference of the \( \alpha \)-derivative of \( u_0^{(v)} \) and that of \( u_0 \) can be estimated as

\[
 \left\| D^\alpha u_0^{(v)}(\cdot) - D^\alpha u_0(\cdot) \right\| = \left\| \chi_{\frac{1}{v}} \ast D^\alpha u_0(\cdot) - D^\alpha u_0(\cdot) \right\| \\
 = \left\{ \int \left| \xi^\alpha \hat{u}_0(\xi) \left\{ (2\pi)^{\frac{\alpha}{2}} \hat{\chi}_{\frac{1}{v}}(\xi) - 1 \right\} \right|^2 d\xi \right\}^{\frac{1}{2}} \\
 \leq \sup_{\xi \in \mathbb{R}^n} \left| (2\pi)^{\frac{\alpha}{2}} \hat{\chi}(\xi/v) - 1 \right| \left\| D^\alpha u_0(\cdot) \right\|. 
\]
Using the property
\[(2\pi)^\frac{n}{2} \hat{\chi}(\xi/\nu) = \int e^{-ix \cdot \xi/\nu} \chi(x) dx \rightarrow \int \chi(x) dx = 1 \quad (\nu \rightarrow \infty),\]
we have
\[
\sum_{j=0}^{\infty} \frac{\rho_j}{M_j} \left\{ \sum_{|\alpha|=j} \left\| D^\alpha u_0^{(v)}(\cdot) - D^\alpha u_0(\cdot) \right\| \right\}^{\frac{1}{2}} \leq C_{u_0} \sup_{\xi \in \mathbb{R}^n} \left| (2\pi)^\frac{n}{2} \hat{\chi}(\xi/\nu) - 1 \right| \rightarrow 0 \quad (\nu \rightarrow \infty).
\]

Applying the same argument to (ii), we have (iii). \qed

For the real analytic functions \(u_0^{(v)}, u_1^{(v)}, a_{hk}^{(v)} (h, k = 1, \ldots, n)\) and \(f^{(v)}\) constructed above, we shall consider the following Cauchy problem
\[(4) \quad \begin{cases}
u^{(v)}(t, x) + \Phi \left( (A_v u^{(v)}(t, \cdot), u^{(v)}(t, \cdot)) A_v u^{(v)}(t, x) = f^{(v)}(t, x) \quad (t > 0), \\ u^{(v)}(0, x) = u_0^{(v)}(x), \quad u^{(v)}_1(0, x) = u_1^{(v)}(x), \end{cases}\]
where \(A_v u(t, x) = \sum_{hk} D_{xh} a_{hk}^{(v)}(x) D_{xk} u(t, x)\). Here we remark that condition (2) is satisfied by the coefficients \([a_{hk}^{(v)}(x)]\).

Suppose now that \(u^{(v)}(t, x)\) is a solution of (4). We define the infinite order energy \(E_v(t)\) and its \(j\)-th order element \(e_j^{(v)}(t)\) as
\[(5) \quad E_v(t) = \sum_{j=1}^{\infty} \frac{\rho(t)^j}{M_j} e_j^{(v)}(t),\]
\[
e_j^{(v)}(t)^2 = \sum_{|\alpha|=j-1} \left\{ \Psi_v(t)(A_v D^\alpha u^{(v)}(t, \cdot), D^\alpha u^{(v)}(t, \cdot)) + j^2 \| D^\alpha u^{(v)}(t, \cdot) \|^2 + \| D^\alpha u^{(v)}(t, \cdot) \|^2 \right\} + j^{-1} \sum_{|\alpha|=j} \| D^\alpha u^{(v)}(t, \cdot) \|^2,
\]
where \(\Psi_v(t) \equiv \Phi \left( (A_v u^{(v)}(t, \cdot), u^{(v)}(t, \cdot)) \right)\). (From now on we write \(u^{(v)}(t, \cdot) = u\) in the norm and inner product.)

We prove here some estimates to the solution of (4), but, in order to obtain them, we shall introduce a lemma for the Kirchhoff type problem.

**Lemma 2** (Proposition 6.1 [9]). If \(u_0 \in H^1, u_1 \in L^2\) and \(f(t, x) \in C^0([0, T]; L^2)\) for \(T > 0\), then there is a constant \(C_T\) independent of \(t \in [0, T]\) such that the solution of the Cauchy problem (1) satisfies
\[
\| u(t, \cdot) \| \leq C_T, \quad \| u(t, \cdot) \| \leq C_T, \quad \| A u(t, \cdot), u(t, \cdot) \| \leq C_T,
\]
where \(H^1\) is the Sobolev space defined by \(H^1 = \{ f \in L^2(\mathbb{R}^n); \sum_{|\alpha| \leq 1} \| D^\alpha f \|^2 < \infty \}. \)
PROOF. Let \( F(\eta) = \int_0^\eta \Phi(s)ds \). Define \( e(t) \) for the solution of (1) as follows:

\[
e(t)^2 = \frac{1}{2} \left\{ \|u_t(t, \cdot)\|^2 + F ((Au(t, \cdot), u(t, \cdot)) \right\}.
\]

Then computing the first order derivative of \( e(t)^2 \) we have

\[
2e'(t)e(t) = \Re(u_t(t, \cdot), u_t(t, \cdot)) + \Phi ((Au(t, \cdot), u(t, \cdot)) (Au(t, \cdot), u_t(t, \cdot))
\]

\[
= \Re(f(t, \cdot), u_t(t, \cdot))
\]

\[
\leq \sqrt{2}e(t)\|f(t, \cdot)\|.
\]

Applying Gronwall’s lemma, we obtain \( e(t) \leq e(0) + \frac{1}{\sqrt{2}} \int_0^t \|f(\tau, \cdot)\|d\tau \leq CT \)
for \( \forall t \in [0, T] \), that is, \( \|u_t\| \leq CT \). Since \( \Phi(\eta) \) is positive, we have

\[
F((Au, u)) = \int_0^{(Au, u)} \Phi(s)ds \geq \Phi_0(Au, u),
\]

therefore we get \((Au, u) \leq F((Au, u))/\Phi_0 \leq CT\).  

Now we shall estimate the first order derivative of \( e_j^{(v)}(t)^2 \). Adopting equation (4), we have

\[
\frac{d}{dt}e_j^{(v)}(t)^2 \leq \frac{|\Psi_v'(t)|}{\Psi_v(t)} e_j^{(v)}(t)^2 + 2\Psi_v(t) \sum_{|\alpha|=j-1} \Re([A_v, D^\alpha]u, D^\alpha u_t)
\]

\[
+ 2 \sum_{|\alpha|=j-1} \Re(D^\alpha f^{(v)}, D^\alpha u_t) + 2j^2 \left( \sum_{|\alpha|=j-1} \|D^\alpha u\|^2 \right)^{\frac{1}{2}}
\]

\[
\times \left( \sum_{|\alpha|=j-1} \|D^\alpha u_t\|^2 \right)^{\frac{1}{2}} + 2(j-1) \sum_{|\alpha|=j} \Re(D^\alpha u, D^\alpha u_t)
\]

\[
\leq \frac{|\Psi_v'(t)|}{\Psi_v(t)} e_j^{(v)}(t)^2 + 2\Psi_v(t) \left( \sum_{|\alpha|=j-1} \|[A_v, D^\alpha]u\|^2 \right)^{\frac{1}{2}} e_j^{(v)}(t)
\]

\[
+ 2 \left( \sum_{|\alpha|=j-1} \|D^\alpha f^{(v)}\|^2 \right)^{\frac{1}{2}} e_j^{(v)}(t) + 2j e_j^{(v)}(t)^2 + 2j^{-\frac{1}{2}} e_j^{(v)}(t)e_j^{(v)}(t+1) .
\]

Dividing by \( 2e_j^{(v)}(t) \), we get

\[
e_j^{(v)}(t) \leq \frac{|\Psi_v'(t)|}{2\Psi_v(t)} e_j^{(v)}(t) + j e_j^{(v)}(t) + \Psi_v(t) \left( \sum_{|\alpha|=j-1} \|[A_v, D^\alpha]u\|^2 \right)^{\frac{1}{2}}
\]

\[
+ j^{-\frac{1}{2}} e_{j+1}^{(v)}(t) + \left( \sum_{|\alpha|=j-1} \|D^\alpha f^{(v)}\|^2 \right)^{\frac{1}{2}} .
\]

(7)
Now, differentiating $E_v(t)$, and applying the above estimate of $e_j^{(v)'}(t)$, we obtain

$$\frac{d}{dt} E_v(t) = \sum_{j=1}^{\infty} \left\{ \frac{j \rho'(t) \rho(t)^{j-1}}{M_j} e_j^{(v)}(t) + \frac{\rho(t)^j}{M_j} e_j^{(v)'}(t) \right\}$$

$$\leq \sum_{j=1}^{\infty} \frac{\rho(t)^j}{M_j} \left( j \frac{\rho'(t)}{\rho(t)} + j \right) e_j^{(v)}(t) + \frac{|\Psi_v'(t)|}{2\Psi_v(t)} \sum_{j=1}^{\infty} \frac{\rho(t)^j}{M_j} e_j^{(v)}(t)$$

$$+ \Psi_v(t) \sum_{j=1}^{\infty} \frac{\rho(t)^j}{M_j} \left( \sum_{|\alpha|=j-1} [[A_v, D^\alpha]u]^2 \right)^{1/2}$$

$$+ \sum_{j=1}^{\infty} \frac{\rho(t)^j}{M_j} e_{j+1}^{(v)}(t) + \sum_{j=1}^{\infty} \frac{\rho(t)^j}{M_j} \left( \sum_{|\alpha|=j-1} \| D^\alpha f^{(v)} \|^2 \right)^{1/2}.$$

(8)

Now we shall calculate the commutator $[A_v, D^\alpha]$ in order to estimate the third term in (8). Applying Leibniz' rule, $[A_v, D^\alpha]u$ can be rewritten as follows:

$$[A_v, D^\alpha]u = A_v D^\alpha u - D^\alpha A_v u$$

$$= \sum_{h,k} \left( D^{e_h} a^{(v)}_{h,k} D^{\alpha+e_k} u - D^{\alpha+e_h} a^{(v)}_{h,k} D^{e_k} u \right)$$

$$= -\sum_{h,k} \left\{ \sum_{\beta<\alpha} \left( \begin{array}{c} \alpha \\ \beta \end{array} \right) (D^{\alpha-\beta+e_h} a^{(v)}_{h,k}) D^{\beta+e_k} u \right\}$$

$$+ \sum_{\beta<\alpha} \left( \begin{array}{c} \alpha \\ \beta \end{array} \right) (D^{\alpha-\beta} a^{(v)}_{h,k}) D^{\beta+e_h+e_k} u \right\}$$

$$= I_\alpha + II_\alpha + III_\alpha,$$

(9)

where

$$I_\alpha = -\sum_{h,k} \sum_{\beta<\alpha} \left( \begin{array}{c} \alpha \\ \beta \end{array} \right) (D^{\alpha-\beta+e_h} a^{(v)}_{h,k}) D^{\beta+e_k} u,$$

$$II_\alpha = -\sum_{h,k} \sum_{\beta<\alpha \atop |\beta| \leq |\alpha|-2} \left( \begin{array}{c} \alpha \\ \beta \end{array} \right) (D^{\alpha-\beta} a^{(v)}_{h,k}) D^{\beta+e_h+e_k} u,$$

$$III_\alpha = -\sum_{h,k} \sum_{\beta<\alpha \atop |\beta|=|\alpha|-1} \left( \begin{array}{c} \alpha \\ \beta \end{array} \right) (D^{\alpha-\beta} a^{(v)}_{h,k}) D^{\beta+e_h+e_k} u,$$

and $e_h = (0, \cdots, 1, \cdots, 0)$. 

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Using the equality $|\alpha - \beta + \mathbf{e}_h| = j - |\beta|$ and the inequality $\binom{\alpha}{\beta} \leq \binom{|\alpha|}{|\beta|}$, we have

\[
(10) \left( \sum_{|\alpha| = j-1} \|I_\alpha\|^2 \right)^{\frac{1}{2}} \leq C_1 \left\{ \sum_{|\alpha| = j-1} \left( \sum_{h \geq 0} \frac{1}{\beta - \alpha} \left( j - 1 \right)^{\frac{1}{2}} M_{j-|\beta|} \rho_1^{-(j-|\beta|)} \|D^{\beta + \mathbf{e}_h} u\| \right)^2 \right\}^{\frac{1}{2}},
\]

where $C_1$ is a constant independent of $j$.

We now introduce the following lemma to estimate (10).

**Lemma 3 (Lemma 2.1 [4]).** Let $\{X_\alpha\} (\alpha \in \mathbb{N}^n)$ be a sequence of non-negative real numbers. Then, for every integer $l$ and $l' (\leq l)$, and every real number $0 < \kappa < 1$, there is a constant $C(n, \kappa)$ such that

\[
\left\{ \sum_{|\alpha| = l} \left( \sum_{\beta \leq \alpha \atop |\beta| \leq l'} X_\beta \right)^2 \right\}^{\frac{1}{2}} \leq C(n, \kappa) \sum_{r=0}^{l'} \kappa^{-(l-r)} \left( \sum_{|\beta| = r} X_\beta^2 \right)^{\frac{1}{2}}.
\]

Applying this lemma, (10) can be estimated by

\[
(10) \leq C_2(n, \kappa) \sum_{r=0}^{j-2} \kappa^{-(j-r-1)} \left\{ \sum_{|\beta| = r} \left( \sum_{h \geq 0} \frac{1}{\beta - \alpha} \left( j - 1 \right)^{\frac{1}{2}} M_{j-r} \rho_1^{-(j-r)} \|D^{\beta + \mathbf{e}_h} u\| \right)^2 \right\}^{\frac{1}{2}}
\]

\[
= C_2 \sum_{r=1}^{j-1} \kappa^{-(j-r)} \left\{ \sum_{|\beta| = r-1} \left( \sum_{h \geq 0} \frac{1}{\beta - \alpha} \left( j - 1 \right)^{\frac{1}{2}} M_{j-r+1} \rho_1^{-(j-r+1)} \|D^{\beta + \mathbf{e}_h} u\| \right)^2 \right\}^{\frac{1}{2}}
\]

\[
\leq C_3 \sum_{r=1}^{j-1} (\kappa \rho_1)^{-(j-r+1)} \left( \frac{j}{r-1} \right)^{\frac{1}{2}} M_{j-r+1} \left( \sum_{|\beta| = r} \|D^\beta u\|^2 \right)^{\frac{1}{2}},
\]

where $C_2$ and $C_3$ are constants independent of $j$.

For the term $II_\alpha$, using an analogous inequality with $\binom{j-1}{r-2}$ instead of $\binom{j-1}{r-1}$, we get

\[
(11) \left( \sum_{|\alpha| = j-1} \|II_\alpha\|^2 \right)^{\frac{1}{2}} \leq C_4 \sum_{r=2}^{j-1} (\kappa \rho_1)^{-(j-r+1)} \left( \frac{j}{r-2} \right)^{\frac{1}{2}} M_{j-r+1} \left( \sum_{|\beta| = r} \|D^\beta u\|^2 \right)^{\frac{1}{2}}.
\]

Hence we have

\[
\left( \sum_{|\alpha| = j-1} \|I_\alpha\|^2 \right)^{\frac{1}{2}} + \left( \sum_{|\alpha| = j-1} \|II_\alpha\|^2 \right)^{\frac{1}{2}} \leq C_5 \sum_{r=1}^{j-1} (\kappa \rho_1)^{-(j-r+1)} \left( \frac{j}{r-1} \right)^{\frac{1}{2}} M_{j-r+1} \left( \sum_{|\beta| = r} \|D^\beta u\|^2 \right)^{\frac{1}{2}}.
\]
for some constant $C_5$ independent of $j$, where we have used the identity \( \binom{j-1}{r-2} + \binom{j-1}{r-1} = \binom{j}{r-1} \).

Finally, to estimate the term $III_\alpha$, we apply the following lemma due to O. A. Oleinik.

**LEMMA 4 (Lemma 4 [15]).** Let $[a_{hk}^{(v)}(x)]$ be a Hermitian non-negative matrix of functions in $B^2(\mathbb{R}^n)$. Then, for every $n \times n$ symmetric matrix $[\xi_{hk}]$, for $\mu = 1, \ldots, n$, there is a constant $C$ such that

\[
\left\| \sum_{hk} D_{x_\mu} a_{hk}^{(v)}(x) \xi_{hk} \right\|^2 \leq C \sum_{hkq} a_{hk}^{(v)}(x) \xi_{hq} \xi_{kq}.
\]

Applying this lemma, we have

\[
\left( \sum_{|\alpha|=j-1} \|III_\alpha\|^2 \right)^{\frac{1}{2}} \leq \left( \sum_{|\alpha|=j-1} \sum_{|\beta|\leq|\alpha|-1} (j - 1)^2 \left\| \sum_{hk} (D^{\alpha-\beta} a_{hk}^{(v)}) D^{\beta+e_h+e_k} u \right\|^2 \right)^{\frac{1}{2}}
\]

\[
\leq C_6 (j - 1) \left( \sum_{|\alpha|=j-1} \sum_{|\beta|\leq|\alpha|-1} \sum_{q=1}^n \left( \sum_{hk} (D^{e_k} a_{nk}^{(v)} D^{e_h} D^{\beta+e_h} u, D^{\beta+e_k} u) \right) \right)^{\frac{1}{2}}
\]

\[
= C_6 (j - 1) \left( \sum_{|\alpha|=j-1} (A_v D^\alpha u, D^\alpha u) \right)^{\frac{1}{2}} \leq \frac{C_6 (j - 1)}{\sqrt{\Psi_v(t)}} e_j^{(v)}(t).
\]

Hence the third term of (8) can be estimated as follows

\[
\Psi_v(t) \sum_{j=1}^\infty \frac{\rho(t)^j}{M_j} \left( \sum_{|\alpha|=j-1} \|A_v, D^\alpha u\|^2 \right)^{\frac{1}{2}}
\]

\[
\leq C_7 \Psi_v(t) \sum_{j=1}^\infty \frac{\rho(t)^j}{M_j} \left\{ \sum_{r=1}^{j-1} (\kappa \rho_1)^{-j+r+1} \binom{j}{r-1} M_{j-r+1} \left( \sum_{|\beta|=r} \|D^\beta u\|^2 \right)^{\frac{1}{2}} \right\}
\]

(12)

\[
+ C_8 \sqrt{\Psi_v(t)} \sum_{j=1}^\infty \frac{\rho(t)^j}{M_j} (j - 1) e_j^{(v)}(t)
\]

\[
= C_7 \Psi_v(t) \rho(t)^{-2} \sum_{r=1}^\infty \sum_{j=r+1}^{\infty} \left( \frac{\rho(t)}{\kappa \rho_1} \right)^{j-r+1} M_{j-r+1} \left( \binom{j}{r-1} \rho(t)^{r+1} \left( \sum_{|\alpha|=r} \|D^\alpha u\|^2 \right)^{\frac{1}{2}} \right)
\]

\[
+ C_8 \sqrt{\Psi_v(t)} \sum_{j=1}^\infty \frac{\rho(t)^j}{M_j} (j - 1) e_j^{(v)}(t).
\]
Let us consider $\rho(t) \leq \rho_0$. By using Lemma B (ii), the first term of (12) can be estimated as

$$\text{first term of (12)} \leq \frac{C_7 \Psi_v(t)}{\kappa \rho_1 (\kappa \rho_1 - \rho_0)} \sum_{j=1}^{\infty} \frac{\rho(t)^j}{M_j} (j - 1) e^{(v)}_j(t)$$

$$\leq C_9 \Psi_v(t) \sum_{j=1}^{\infty} \frac{\rho(t)^j}{M_j} (j - 1) e^{(v)}_j(t),$$

where $C_9$ is a constant independent of $t$.

Applying the assumption $\sup_j Q(j) = \sup_j \{M_j/j^2M_{j-1}\} < \infty$, we obtain

$$\text{the fourth term of (8)} \leq C_{10} \rho(t)^{-1} \sum_{j=1}^{\infty} \frac{\rho(t)^j}{M_j} j e^{(v)}_j(t) Q(j),$$

where $C_{10}$ is some constant independent of $t$.

Moreover we have

$$\text{the fifth term of (8)} \leq C_{11}$$

for some constant $C_{11}$ independent of $v$ (by the definition of $f^{(v)}$).

Then, taking together the preceding estimates for the derivative of $E_v(t)$, we get the following bound

$$\frac{d}{dt} E_v(t) \leq \sum_{j=1}^{\infty} \frac{\rho(t)^j}{M_j} \left\{ j \rho'(t) \rho(t) + C_8 \sqrt{\Psi_v(t)} (j - 1) + C_9 \Psi_v(t)(j - 1) + C_{10} \frac{j Q(j)}{\rho(t)} \right\} e^{(v)}_j(t) + \frac{|\Psi_v'(t)|}{2 \Psi_v(t)} E_v(t) + C_{11}.$$ (13)

We now proceed to a choice of $\rho(t)$. Applying Lemma 2, in the parenthesis of the first term of (13) there is a constant $C_{12}$ independent of $j$ and $t$ such that

$$\text{the first term in (13)} \leq j \rho(t)^{-1} \left( \rho'(t) + C_{12} \rho(t) + C_{10} Q_0 \right),$$

where $Q_0 = \sup_j Q(j)$. Let us assume that $\rho(0) > 0$, then we can easily see that there is $T_1 > 0$ such that the ordinary differential inequality

$$\rho'(t) + C_{12} \rho(t) + C_{10} Q_0 \leq 0,$$ (14)

has a positive decreasing solution on $[0, T_1]$. Moreover if $\lim_{j \to \infty} Q(j) = 0$, let $j_0(T)$ large enough such that the ordinary differential inequality

$$\rho'(t) + C_{12} \rho(t) + C_{10} Q(j) \leq 0,$$
has a positive decreasing solution on $[0, T]$ for any $j \geq j_0$. On the other hand, if $j < j_0$, we obtain
\[ j \rho(t)^{-1}(\rho'(t) + C_{12} \rho(t) + C_{10} Q(j)) \leq \rho(T)^{-1}(C_{12} \rho(0) + C_{10} Q_0) \equiv C_{13} \]
for some constant $C_{13}$ independent of $j$ and $t$. Thus we get the following estimate
\[ \frac{d}{dt} E_v(t) \leq \left( \frac{|\Psi'_v(t)|}{2 \Psi_v(t)} + C_{13} \right) E_v(t) + C_{11} \quad \forall t \in [0, T]. \]
Note that $Q(j) \to 0$ as $j \to \infty$ and then we can take $T_1 = T$.

Next we estimate the nonlinear part of (15), $|\Psi'_v(t)| = |\frac{d}{dt} \Phi((A_v u(t), \cdot), u(t), \cdot))|.$

We now introduce a lemma, which is classical in convex analysis.

**Lemma 5 (Theorem 243 [6]).** Let $\mu$ and $\sigma$ be continuous and strictly increasing. We define $\mathcal{M}_\mu(f)$ by
\[ \mathcal{M}_\mu(f) = \mu^{-1} \left( \int \mu(f(x)) p(x) dx \right), \]
where $f$ and $p$ are nonnegative functions such that $\int p(x) dx = 1$ and $\int \mu(f(x)) p(x) dx$ exists. Then, in order that $\mathcal{M}_\mu(f) \leq \mathcal{M}_\sigma(f)$ for all $f$, it is necessary and sufficient that $\sigma \circ \mu^{-1}$ should be convex.

Using the inequality $|((A_v u, u_t))| \leq (A_v u, u_t)^{\frac{1}{2}} (A_v u_t, u_t)^{\frac{1}{2}}$, the Plancherel theorem and Lemma 2, we have
\[ |\Psi'_v(t)| = 2 |\Re(A_v u, u_t) \Phi'((A_v u, u))| \leq 2(A_v u, u_t)^{\frac{1}{2}} \left( A_v u_t, u_t^{\frac{1}{2}} \right) \Phi'((A_v u, u)) \leq C_{T, A, \Phi} \| \cdot \| \| \hat{u}_t(t, \cdot) \|, \]
where $C_{T, A, \Phi}$ is a constant independent of $u$.

When $\| \hat{u}_t(t, \cdot) \| \geq 1$, we see $1/\| \hat{u}_t(t, \cdot) \| \leq 1$. So
\[ \| 1/|\hat{u}_t(t, \cdot)\| = \| \hat{u}_t(t, \cdot) \| \left( \int_{\mathbb{R}^q} \frac{|\hat{u}_t(t, \xi)|^2}{\| \hat{u}_t(t, \cdot) \|^2} |\xi|^2 d\xi \right)^{\frac{1}{2}} \leq C_T N^{-1} \left( \int_{\mathbb{R}^q} \| \hat{u}_t(t, \xi) \|^2 N(|\xi|) d\xi \right), \]
where $N$ is a nonnegative function such that $\lambda \to N(\lambda^{\frac{1}{2}})$ is convex, having used Lemma 5 for $\mu = s^2$, $\sigma = N(s)$ and $p(t, \xi) = |\hat{u}_t(t, \xi)|^2/\| \hat{u}_t(t, \cdot) \|^2$. When
\\(|\hat{u}_t(t, \cdot)| < 1\), let us take \(p_\theta(t, \xi) \equiv (1 - \|\hat{u}_t(t, \cdot)\|^2)\chi_\theta(\xi)\), where \(\chi_\theta(\xi) \equiv \theta^{-n} \chi(\theta \xi)\), with \(\int \chi(\xi) d\xi = 1\) and \(0 < \theta < 1\), is Friedrichs’ mollifier. It is easy to see that \(p_\theta(t, \xi)\) satisfies \(0 < \int p_\theta(t, \xi) d\xi\) and \(\int \{\|\hat{u}_t(t, \xi)\|^2 + p_\theta(t, \xi)\}\) = 1. Now, by using Lemma 5 with \(p(t, \xi) = \|\hat{u}_t(t, \xi)\|^2 + p_\theta(t, \xi)\), we have

\[
\left\| P \cdot \|\hat{u}_t(t, \cdot)\| \right\| \leq \left\{ \int_{\mathbb{R}^n_+} \|\hat{u}_t(t, \xi)\|^2 \left(\|\hat{u}_t(t, \xi)\|^2 + p_\theta(t, \xi)\right) d\xi \right\}^{1/2} \\
\leq N^{-1} \left( \int_{\mathbb{R}^n_+} N(|\xi|) \left(\|\hat{u}_t(\xi)\|^2 + p_\theta(t, \xi)\right) d\xi \right).
\]

In addition, we choose \(\chi_\theta(\xi)\) satisfies that \(\text{supp} p_\theta(t, \xi) \subset \mathbb{R}_+ \times \{\xi \in \mathbb{R}^n; |\xi| \leq \theta\}\) and so we have

\[
\int_{\mathbb{R}^n_+} N(|\xi|) p_\theta(t, \xi) d\xi = \int_{|\xi| \leq \theta} N(|\xi|) p_\theta(t, \xi) d\xi \\
\leq \left( \int_{|\xi| \leq \theta} N(|\xi|)^2 d\xi \right)^{1/2} \left( \int_{|\xi| \leq \theta} p_\theta(t, \xi)^2 d\xi \right)^{1/2} \rightarrow 0, \; (\theta \rightarrow 0).
\]

Hence we obtain

\[
\left\| P \cdot \|\hat{u}_t(t, \cdot)\| \right\| \leq C_T N^{-1} \left( \left\| N(|\cdot|)^{1/2} \hat{u}_t(t, \cdot) \right\|^2 \right).
\]

Now we introduce the following lemma.

**Lemma 6.** Let \(v(x) \in C^\infty(\{M_j\})\) and let \(M(r)\) be the associated function of \(\{M_j\}\). If \(\sup_j Q(j) < \infty\), then for any \(\varepsilon > 0\), there is a constant \(C_{\varepsilon,n}\) such that

\[
\left\| M(\tilde{\rho}) \cdot \hat{v}(\cdot) \right\| \leq C_{\varepsilon,n} \sum_{j=1}^\infty \frac{\rho^j}{M_j} \left( \sum_{|\alpha| = j-1} \|D^\alpha v\|^2 \right)^{1/2},
\]

where \(\tilde{\rho} = \rho/(1+\varepsilon)\).

**Proof.** The right hand side of (18) can be estimated as the following:

\[
\sum_{j=1}^\infty \frac{\rho^j}{M_j} \left( \sum_{|\alpha| = j-1} \|D^\alpha v\|^2 \right)^{1/2} \geq \sum_{j=1}^\infty \frac{\rho^j}{M_j} \left( j - 2 + n \right)^{-1} \sum_{|\alpha| = j-1} \|D^\alpha v\|
\]

\[
\geq C'_{\varepsilon,n} \sum_{j=1}^\infty \frac{\rho^j(1+\varepsilon)^j}{M_j} \left( \int_{\mathbb{R}^n} \left( \sum_{|\alpha| = j-1} |\xi|^{\alpha} \right)^2 |\hat{v}(\xi)|^2 d\xi \right)^{1/2}
\]

\[
\geq C''_{\varepsilon,n} \sum_{j=1}^\infty \left( 1+\varepsilon \right)^{j/2} \frac{\rho^j}{M_{j+1}} \frac{j^{1/2} M_j}{M_{j+1}} \left\| \cdot \cdot |\hat{v}(\cdot)\| \right\|
\]

\[
\geq C_{\varepsilon,n} \sum_{j=1}^\infty \frac{\tilde{\rho}^j}{M_j} \left\| \cdot \cdot |\hat{v}(\cdot)\| \right\|,
\]
for any \( \varepsilon > 0 \), where we have applied the boundedness of \( Q(j) \) and the following inequalities

\[
\sum_{|\alpha|=N} 1 = \binom{N + n - 1}{n - 1}, \quad \left( \sum_{i=1}^{M} a_i^2 \right)^{\frac{1}{2}} \geq M^{-\frac{1}{2}} \sum_{i=1}^{M} a_i, \quad (a_i \geq 0),
\]

\[
\binom{N + n}{n} \leq \frac{(1 + \varepsilon)^{N+n}}{\varepsilon^n} \quad (\forall \varepsilon > 0), \quad \sum_{|\alpha|=N} |x^\alpha| \geq |x|^N, \quad \frac{(1 + \varepsilon)^j}{j} \geq \exists C_\varepsilon > 0.
\]

Now let \( R_k (k = 0, 1, \ldots) \) and \( \Omega_k (k = 1, 2, \ldots) \) be chosen as follows:

\[
R_k \equiv \max \left\{ r \geq 0; \sup_{j \geq 1} \left\{ \frac{(\tilde{\rho}r)^j}{M_j} \right\} = \frac{(\tilde{\rho}r)^k}{M_k} \right\}, \quad (k \geq 1), \quad R_0 \equiv 0,
\]

\[
\Omega_k \equiv \{ \xi \in \mathbb{R}^n; R_{k-1} \leq |\xi| < R_k \},
\]

where we note that \( \bigcup_{k=1}^{\infty} \Omega_k = \mathbb{R}^n \). Hence we have

\[
\sum_{j=1}^{\infty} \frac{\tilde{\rho}^j}{M_j} \left\| \cdot \right\|_j \hat{\cdot} \geq \left\{ \sum_{k=1}^{\infty} \int_{\Omega_k} \left| \frac{(\tilde{\rho} |\xi|)^k}{M_k} \right|^2 \hat{\cdot}^2 d\xi \right\}^{\frac{1}{2}}
\]

\[
= \left( \sum_{k=1}^{\infty} \int_{\Omega_k} \left| \sup_{j \geq 1} \left\{ \frac{(\tilde{\rho} |\xi|)^j}{M_j} \right\} \right|^2 \hat{\cdot}^2 d\xi \right)^{\frac{1}{2}}
\]

\[
= \left\| M(\tilde{\rho} |\cdot|) \hat{\cdot} \right\|.
\]

Here we remark that the estimate (17) remains valid for any continuous strictly increasing function \( N \) such that \( \lambda \rightarrow N(\lambda^{\frac{1}{2}}) \) is convex. Now, considering that \( N(r) = \left\{ M \left( \frac{\rho(t)r}{1+\varepsilon} \right) / C_{e,n} \right\}^2 \) for any fixed \( t \in [0, T] \) and \( \varepsilon > 0 \), then we see that \( N(r) \) is a continuous strictly increasing function such that \( \lambda \rightarrow N(\lambda^{\frac{1}{2}}) \) is convex by definition of \( M(r) \), where \( C_{e,n} \) is the constant in Lemma 6. Finally, applying Lemma 6, we obtain

\[
\left\| N(\cdot) \left\|^{\frac{1}{2}} \hat{u}_t(t, \cdot) \right\| \leq \sum_{j=1}^{\infty} \frac{\rho(t)^j}{M_j} \left( \sum_{|\alpha|=j-1} \left\| D^\alpha u_t(t, \cdot) \right\|^2 \right)^{\frac{1}{2}} \leq E_v(t).
\]

Therefore we have the estimate

\[
\frac{d}{dt} E_v(t)^2 \leq C N^{-1} \left( E_v(t)^2 \right) E_v(t)^2 + C' E_v(t).
\]
Now, without loss of generality let $E_v(t) \geq 1$, and consider the inequality
\begin{equation}
\frac{d}{dt} E_v(t)^2 \leq CN^{-1} \left( E_v(t)^2 \right) E_v(t)^2 + C'E_v(t)^2.
\end{equation}

Next we shall investigate the condition which must be satisfied by $N$ in order that some estimates on the solution of (4) hold. For this purpose we introduce the following lemmas.

**Lemma 7.** Let $f : [0, \infty) \to [0, \infty)$ be continuous, $g : (0, \infty) \to (0, \infty)$ continuous and nondecreasing, and $c$ a positive constant. Then the inequality
\[
f(t) \leq c + \int_0^t g(f(s))ds \quad (0 \leq t < \infty),
\]
implies that
\[
f(t) \leq G^{-1}(G_0) < +\infty \quad (0 \leq t \leq G_0),
\]
for any fixed number $G_0$ less than $G(\infty)$, where $G(t) = \int_t^\infty 1/g(s)ds$ for $t \geq c$. Moreover, if $G(\infty) = \infty$, then the inequality
\[
f(t) \leq G^{-1}(t),
\]
is valid for all $t \geq 0$.

The proof of Lemma 3 is given in [13].

**Lemma 8.** Let $M(r)$ be a continuous nondecreasing function such that
\[
\int_c^\infty \frac{ds}{s(d_0 + M^{-1}(s))} = \infty,
\]
for some positive constant $c$ and $d_0$, then for any positive constant $\rho$, the function $N(r)$ defined by $N(r) \equiv M(\rho r)^2$ satisfies
\[
\int_c^{\infty} \frac{ds}{s(\tilde{\rho}^{-1}d_0 + N^{-1}(s))} = \infty,
\]
for some positive constant $c'$ and for any $0 < \tilde{\rho} < \rho$.

Now we introduce a lemma to prove the Lemma 8.

Let $M(r)$ be the associated function of $\{M_j\}$. We define the regularized associated function $M_\varepsilon(r)$ of $M(r)$, written in the form $M_\varepsilon(r) \equiv \chi_\varepsilon * M(r)$ for $\chi(r) \in C_0^\infty(\mathbb{R})$, such that $\mathrm{supp} \chi \subset [-\delta, \delta]$, $\int \chi(r)dr = 1$ and $\chi_\varepsilon(r) \equiv \varepsilon^{-1} \chi(\varepsilon^{-1}r)$ for $\varepsilon > 0$ and $\delta > 0$. Then we have the following lemma.

**Lemma 9 (W. Matsumoto [11]).** For $0 < \forall \varepsilon < 1$ and $\forall \delta > 0$, a regularized associated function $M_\varepsilon(r) \in C^\infty(\mathbb{R})$ satisfies the inequality
\[
M((1 - \varepsilon)r) \leq M_\varepsilon(r) \leq M((1 + \varepsilon)r)
\]
for any $r \geq \delta$. 
PROOF. For \(0 < \forall \varepsilon < 1\),

\[
\frac{M_{\varepsilon}(r)}{M((1 + \varepsilon)r)} = \int_{-\delta}^{\delta} \frac{M(r - \varepsilon s)}{M((1 + \varepsilon)r)} ds \geq \int_{-\delta}^{\delta} \chi(s) ds = 1,
\]

where we used \(M(r - \varepsilon s) \geq M((1 + \varepsilon)r)\) for \(-\delta \leq \forall s \leq \delta\). \(\square\)

PROOF OF LEMMA 8. Let \(N(r) = M(\rho r)^2\) and \(N_{\varepsilon}(r) = M_{\varepsilon}(\rho(1 - \varepsilon)r)^2\), where \(M_{\varepsilon}\) is a regularized associated function of \(M\) defined above. Then we see that \(N^{-1}(s) \leq N_{\varepsilon}^{-1}(s)\), and hence we have

\[
\int_c^\infty \frac{ds}{s(\tilde{\rho}^{-1}d_0 + N^{-1}(s))} \geq \int_c^\infty \frac{ds}{s(\tilde{\rho}^{-1}d_0 + N_{\varepsilon}^{-1}(s))} = 2\rho(1-\varepsilon) \int_{N_{\varepsilon}^{-1}(c)}^\infty \frac{M_{\varepsilon}'(\rho(1-\varepsilon)r)}{M_{\varepsilon}(\rho(1-\varepsilon)r)(\tilde{\rho}^{-1}d_0 + r)} dr \\
\geq 2\rho(1-\varepsilon) \int_{M_{\varepsilon}(\rho(1-\varepsilon)N_{\varepsilon}^{-1}(c))}^\infty \frac{ds}{s(\rho(1-\varepsilon)\tilde{\rho}^{-1}d_0 + M_{\varepsilon}^{-1}(s))},
\]

where we used the equality \(\frac{d\varepsilon}{d\varepsilon} = 2\rho(1-\varepsilon)M_{\varepsilon}'(\rho(1-\varepsilon)r)M_{\varepsilon}(\rho(1-\varepsilon)r)\) and the inequality \(M_{\varepsilon}^{-1}(s) \leq M^{-1}(s/(1-\varepsilon))\). Putting \(\varepsilon = 1 - \tilde{\rho}/\rho\), we get Lemma 8. \(\square\)

Let \(C = C'\tilde{\rho}/d_0\) in (20) and then (20) can be rewritten as

\[
(21) \quad \frac{d}{dt} E_v(t)^2 \leq C'E_v(t)^2 \left(N^{-1} \left(E_v(t)^2 \right) + \tilde{\rho}^{-1}d_0\right).
\]

Then, regarding \(g(f)\) in Lemma 7 as \(g(f) = C_1 f(N^{-1}(f) + \tilde{\rho}^{-1}d_0)\) and applying Lemma 8, we see that \(\int_0^\infty \frac{ds}{g(s)} = \infty\), where \(N(r) \equiv M(\rho(t)r)^2\) and \(\tilde{\rho}\) is a constant such that \(0 < \tilde{\rho} < \rho(T)\). Hence, if \(\{M_j\}\) satisfies a quasi-analytic condition, by applying Lemma 7, we have the energy estimate

\[
(22) \quad E_v(t) \leq C_T (0 \leq \forall t \leq T),
\]

where \(C_T\) is a constant independent of \(v\).

By the foregoing arguments, finally we have the following proposition.

PROPOSITION 1. Let \(\rho_0\) and \(\rho_1\) be positive numbers such that \(\rho_0 < \rho_1\), \(\{M_j\}\) be positive, logarithmically convex and \(\sup_j Q(j) \equiv \sup_j \{M_j/j^{\frac{3}{2}}M_{j-1}\} < \infty\). Assume that \([a_{jk}(x)] \in [C((M_j))]_{\rho_1}\) satisfies (2), \(u_0\) and \(u_1 \in C^0([0, T]; C^2(M_j))_{\rho_0}\) and that \(\Phi(\eta) \in C^1([0, \infty])\) satisfies the inequality \(\Phi \geq \exists \Phi_0 > 0\). Then we get the following estimates of \(E_v(t)\) to the solution of (4).

i) There exist positive constants \(T_0(\leq T)\) and \(C\) independent of \(v\), and a positive function \(\rho(t)\) on \([0, T_0]\) such that

\[
E_v(t) \leq C_{T_0} (0 \leq \forall t \leq T_0).
\]
ii) If \( \{M_j\} \) satisfies a quasi-analytic condition and \( Q(j) \to 0 \) as \( j \to \infty \), there exists a constant \( C_T \) independent of \( v \) such that
\[
E_v(t) \leq C_T \quad (0 \leq \forall t \leq T).
\]

Now we shall show the existence of the solution for the nonlinear Cauchy problem (1) in case that the data and the coefficients satisfies the assumption of the Main Theorem.

Let \( u^{(v)}(t, x) \) and \( u^{(v')} (t, x) \) be solutions of (4). Define \( v^{(v,v')}(t, x) \equiv u^{(v)}(t, x) - u^{(v')}(t, x) \). We shall consider the following Cauchy problem for \( v^{(v,v')}(t, x) \)
\[
\begin{cases}
\partial_t v^{(v,v')}(t, x) + \psi_v(t) A_v v^{(v,v')}(t, x) = f^{(v)}(t, x) - f^{(v')}(t, x) \\
\quad + (\psi_v(t) A_v - \psi_{v'}(t) A_v) u^{(v')}(t, x) \quad (t > 0, x \in \mathbb{R}^n),
\end{cases}
\]
\[
v^{(v,v')}(0, x) = u_0^{(v)}(x) - u_0^{(v')}(x), \quad v^{(v,v')}(0, x) = u_1^{(v)}(x) - u_1^{(v')}(x).
\]

From now on we will show that the solution of the Cauchy problem (23) \( v^{(v,v')}(t, x) \) converges to 0 in \( C^2([0, T_0]; C^2([M_j]) \hat{\rho}(t) \) for some \( \hat{\rho}(t) > 0 \).

We define the infinite order energy \( E_{v,v'}(t) \) and its j-th order element \( e^{(v,v')}_j(t) \) as
\[
E_{v,v'}(t) \equiv \sum_{j=1}^{\infty} \frac{\hat{\rho}(t)^j}{M_j} e^{(v,v')}_j(t)
\]
with
\[
e^{(v,v')}_{j}(t)^2 \equiv \sum_{|\alpha|=j-1} \left\{ \psi_v(t) (A_v D^\alpha v^{(v,v')}(t, \cdot), D^\alpha v^{(v,v')}(t, \cdot)) + j^2 \|D^\alpha v^{(v,v')}(t, \cdot)\|^2 \right. \]
\[
\left. \quad + \|D^\alpha v^{(v,v')}_t(t, \cdot)\|^2 \right\} + j^{-1} \sum_{|\alpha|=j} \|D^\alpha v^{(v,v')}(t, \cdot)\|^2.
\]

Then, applying Proposition 1 and the estimate obtained for \( E_v(t) \), we conclude that there is a constant \( C_1 \) independent of \( v \) and \( v' \) such that
\[
\frac{d}{dt} E_{v,v'}(t) \leq \sum_{j=1}^{\infty} \frac{\hat{\rho}(t)^j}{M_j} \left\{ \sum_{|\alpha|=j-1} \|D^\alpha \left\{ (\psi_v(t) A_v - \psi_{v'}(t) A_v) u^{(v')} + f^{(v)} - f^{(v')} \right\} \|^2 \right\}^{\frac{1}{2}}
\]
\[
\quad + C_1 E_{v,v'}(t)
\]
on \([0, T_0]\).
As to the first term of (26), we see that

\[ \text{first term of (26)} \leq \sqrt{3} |\Psi'(t) - \Psi(t)| \sum_{j=1}^{\infty} \frac{\tilde{\rho}(t)^j}{M_j} \left\{ \sum_{|\alpha| = j-1} \|D^\alpha A_{\nu'} u^{(\nu')}\|_2 \right\}^{\frac{1}{2}} \]

\[ + \sqrt{3} \Psi(t) \sum_{j=1}^{\infty} \frac{\tilde{\rho}(t)^j}{M_j} \left\{ \sum_{|\alpha| = j-1} \|D^\alpha (A_{\nu'} - A_{\nu}) u^{(\nu')}\|_2 \right\}^{\frac{1}{2}} \]

\[ \text{Next we shall estimate (28) as follows:} \]

\[ \text{first term of (26)} \leq \sqrt{6} \Psi(t) \sum_{j=1}^{\infty} \frac{\tilde{\rho}(t)^j}{M_j} \left\{ \sum_{|\alpha| = j-1} \|D^\alpha (A_{\nu'} - A) u^{(\nu')}\|_2 \right\}^{\frac{1}{2}} \]

\[ + \sqrt{6} \Psi(t) \sum_{j=1}^{\infty} \frac{\tilde{\rho}(t)^j}{M_j} \left\{ \sum_{|\alpha| = j-1} \|D^\alpha (f^{(\nu)} - f^{(\nu')}\|_2 \right\}^{\frac{1}{2}} \]

Applying Lemma 1 and Lemma 2 to (27) and (29), we find a constant $C_2$, independent of $\nu$ and $\nu'$, such that

\[ (27) + (29) \leq C_2 E_{\nu, \nu'}(t) + \varepsilon_1(\nu, \nu'), \]

where $\varepsilon_1(\nu, \nu') \to 0$ as $j \to \infty$ for any $t \in [0, T_0]$.

Next we shall estimate (28) as follows:

\[ (28) \leq \sqrt{6} \Psi(t) \sum_{j=1}^{\infty} \frac{\tilde{\rho}(t)^j}{M_j} \left\{ \sum_{|\alpha| = j-1} \|D^\alpha (A_{\nu'} - A) u^{(\nu')}\|_2 \right\}^{\frac{1}{2}} \]

\[ + \sqrt{6} \Psi(t) \sum_{j=1}^{\infty} \frac{\tilde{\rho}(t)^j}{M_j} \left\{ \sum_{|\alpha| = j-1} \|D^\alpha (f^{(\nu)} - f^{(\nu')}\|_2 \right\}^{\frac{1}{2}} \]

Now, applying Lemma 1, Lemma C and the fact $C(\{M_j\}_\rho) \subset C(\{M_{j+1}\}_\rho)$, there are constants $C_3$ and $\varepsilon_2(\nu)$ such that $\|D^\alpha (A_{\nu'} - A) u^{(\nu')}\|$ can be estimated by

\[ \|D^\alpha (A_{\nu'} - A) u^{(\nu')}\| \leq C_3 \varepsilon_2(\nu)(\kappa \rho(t))^{-|\alpha|+2} M_{|\alpha|+2} \]

for $0 < \forall \kappa < 1$, where $\varepsilon_2(\nu) \to 0$ as $\nu \to \infty$ and $C_3$ is a constant independent of $\nu$. Similarly, we have

\[ \|D^\alpha (A_{\nu'} - A) u^{(\nu')}\| \leq C_3 \varepsilon_2(\nu')(\kappa \rho(t))^{-|\alpha|+2} M_{|\alpha|+2} . \]

Hence we get the following estimate

\[ (28) \leq C_4 (\varepsilon_2(\nu) + \varepsilon_2(\nu')) \sum_{j=1}^{\infty} \left( \frac{\tilde{\rho}(t)}{\kappa \rho(t)} \right)^j \]
for $0 < \forall \kappa' < 1$, where we have applied the same arguments as the proof of Lemma 6. Here let us consider that $\rho(t) < \kappa' \rho(t)$, so that we have
\[ \sum_{j=1}^{\infty} (\rho(t)/\kappa')^{j} < \infty. \]
Therefore by taking together the previous estimates, there are constants $C_5$ and $C_6$ such that we get the following estimate of $E_{u,v}(t)$:
\[ (31) \quad \frac{d}{dt} E_{u,v}(t) \leq C_5 \left( \varepsilon_1(v, v') + \varepsilon_2(v) + \varepsilon_2(v') \right) + C_6 E_{u,v}(t) \quad \forall t \in [0, T_0]. \]
Hence by applying Gronwall's lemma, Lemma 1 and the definition of $\varepsilon_1$ and $\varepsilon_2$, there are $C$ and $C'$ such that
\[ E_{u,v}(t) \leq CE_{u,v}(0) + C' \left( \varepsilon_1(v, v') + \varepsilon_2(v) + \varepsilon_2(v') \right) \rightarrow 0 \]
as $v, v' \rightarrow \infty$ on $[0, T_0]$.

3. Appendix

**THEOREM A (Denjoy-Carleman).** The following three statements are equivalent:

(i) $C^\mathbb{Z}([M_j])$ is a quasi-analytic class,

(ii) \[ \int_{0}^{\infty} \frac{1}{r^2} \log M(r) \, dr = \infty, \]

(iii) \[ \sum_{j=0}^{\infty} \frac{M_j}{M_{j+1}} = \infty, \]

where $M(r) \equiv \sup_j \{ r^j / M_j \}$.

The proof is given in [10].

**LEMMA B (Kinoshita [8]).** Let $\{M_j\}$ be a positive sequence of real numbers. Assume that $\{M_j\}$ is logarithmically convex. Then the following inequalities are established.

(i) $\left( \binom{j}{v} \right) M_{j-v} M_v \leq M_j$.

(ii) $\left( \binom{j}{v-1} \right) M_{j-v+1} M_{v+1} \leq C M_j v(v+1)$ for $\forall j \geq 2v$, where $C$ is a constant independent of $j$. 
PROOF. (i) can be proved by the same argument to (ii). For (ii), we shall prove that there is a constant $C$ independent of $j$ and $v$ such that

$$(\frac{j}{v-1}) \frac{M_{j-v+1}M_{v+1}}{M_j} \leq C(v+1).$$

In case that $j - v + 1 > v + 1$, we see that the following inequality holds:

$$M_{j-v+1} \cdot M_{j-v} \cdot \frac{M_{v+1}}{2M_1} \times \frac{M_{v+1}}{M_1} \leq \frac{1}{2M_1}.$$
PROOF. Let \( f(x) \) be a function belonging the class \( C^2(\{M_j\})_p' \). Then we can get the following estimates:

\[
\sum_{j=0}^{\infty} \frac{\rho_j}{M_j} \left( \sum_{|\alpha|=j} \|D^\alpha f\|^2 \right)^{\frac{1}{2}} \leq C \sum_{j=0}^{\infty} \frac{\rho_j}{M_j} \left\{ \sum_{|\alpha|=j} \left( \rho'^{-|\alpha|} M_{|\alpha|} \right)^2 \right\}^{\frac{1}{2}}
\]

\[
= C \sum_{j=0}^{\infty} \left( \frac{\rho}{\rho'} \right)^j \left( j + n - 1 \right)^{\frac{1}{2}} \frac{1}{n - 1}
\]

\[
\leq C_{\epsilon,n} \sum_{j=0}^{\infty} \left( \frac{\rho}{\rho'} \right)^j (1 + \epsilon)^\frac{j}{2},
\]

where we used the relations \( \sum_{|\alpha|=j} 1 = \binom{j+n-1}{n-1} \) and \( \binom{j+n-1}{n-1} \leq (1+\epsilon)^{j+n-1}/\epsilon^{n-1} \leq C_{\epsilon,n}(1+\epsilon)^j \). Hence, by choosing \( 0 < \epsilon < (\rho'/\rho)^2 - 1 \), we get the lemma. \( \square \)

REFERENCES


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