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Compactness of Conformal Metrics with Positive Gaussian Curvature in $\mathbb{R}^2$

KUO-SHUNG CHENG – CHANG-SHOU LIN

Abstract. In this paper we consider the compactness of a sequence of solutions $u_n$ of

$$\Delta u + K(x)e^{2u} = 0 \quad \text{in} \ \mathbb{R}^2,$$

where $K(x)$ is positive in $\mathbb{R}^2$ and decays like $|x|^{-b}$ at $\infty$ for some $b > 0$. Assuming that the limit of the total curvature of $u_n$ satisfies

$$2 - b \neq \lim_{n \to +\infty} \frac{1}{2\pi} \int_{\mathbb{R}^2} K(x)e^{2u_n(x)}dx < 2,$$

we prove that $u_n$ must be bounded in $W^{2,p}_{\text{loc}}(\mathbb{R}^2)$ for any $p > 1$. We also construct a specific $K(x) = K(|x|)$ to show that the total curvature of any solution $u$ of equation (0.1) with this $K(|x|)$ must satisfy

$$2 - b < \frac{1}{2\pi} \int_{\mathbb{R}^2} K(x)e^{2u}dx < 2.$$

This appears to be in contrast with the statement of Theorem A in [A]. In this respect, we show that for any $K$ which decays like $|x|^{-b}$ for $0 < b < 2$, there exists $\alpha_0(K) > \frac{2-b}{2}$ such that the total curvature of any solution $u$ of (0.1) must satisfy

$$\frac{1}{2\pi} \int_{\mathbb{R}^2} Ke^{2u}dx \geq \alpha_0(K) > \frac{2-b}{2}.$$

1. – Introduction

In this paper, we consider the entire solution of the equation

$$\Delta u + K(x)e^{2u} = 0 \quad \text{in} \ \mathbb{R}^2,$$

where $\Delta$ is the Laplacian operator of $\mathbb{R}^2$ and $K(x)$ is a given function in $\mathbb{R}^2$. Equation (1.1) arises in the problem of finding a Riemannian metric which is
conformal to the flat metric of $\mathbb{R}^2$ and realizes the given function $K(x)$ as its Gaussian curvature. We refer the reader to [CN1] for a brief description of the background and the history of this problem.

In case $K$ is nonpositive on $\mathbb{R}^2$, a fairly complete understanding of the the solution set of (1.1) was achieved in [CN1], [CN2]. To state the results in [CN2], we introduce $\alpha_1$ as

\begin{equation}
\alpha_1 = \sup \left\{ \alpha \in \mathbb{R} \mid \int_{\mathbb{R}^2} |K(x)| (1 + |x|^2)^{q} dx < +\infty \right\}.
\end{equation}

Then the main result in [CN2] is

**Theorem A.** Suppose that $K \leq 0$ in $\mathbb{R}^2$ and that

\begin{equation}
|x|^{-m} \leq |K(x)| \leq |x|^m
\end{equation}

for $|x|$ large and some positive constant $m$. Then we have:

(I) If $\alpha_1 \leq 0$, then (1.1) possesses no entire solution in $\mathbb{R}^2$.

(II) If $\alpha_1 > 0$, then the following conclusions hold:

(i) For each $\alpha \in (0, \alpha_1)$, (1.1) possesses a unique solution $u_\alpha$ such that

\begin{equation}
\text{for } |x| \to \infty, \quad u_\alpha(x) = \alpha \log |x| + O(1).
\end{equation}

(ii) The function $U(x)$ given by

\[ U(x) = \sup \{u(x) \mid u \text{ is an entire solution of } (1.1) \text{ in } \mathbb{R}^2 \} \]

is well-defined everywhere in $\mathbb{R}^2$ and is a solution of (1.1) in $\mathbb{R}^2$. Moreover, $K(x) e^{2u(x)} \in L^1(\mathbb{R}^2)$.

(iii) Let $u$ be an arbitrary solution of (1.1) in $\mathbb{R}^2$. Then either $u \equiv U$ or $u \equiv u_\alpha$ for some $\alpha \in (0, \alpha_1)$.

(iv) If $0 < \alpha < \beta < \alpha_1$, then $u_\alpha(x) < u_\beta(x) < U(x)$ for all $x \in \mathbb{R}^2$. Furthermore, for any given $\varepsilon > 0$, there exists a constant $R = R(\varepsilon)$ such that for $|x| > R$,

\[ (\alpha_1 - \varepsilon) \log |x| - C \leq U(x) \leq \alpha_1 \log |x| + C. \]

In this paper, $K$ is always assumed locally bounded and positive in $\mathbb{R}^2$. A solution $u$ means $u \in W^{2,p}_{\text{loc}}(\mathbb{R}^2)$ for any $p > 1$ and satisfies (1.1) in the distributional sense. For the case $K(x)$ is positive in $\mathbb{R}^2$, it is not expected that results similar to Theorem A should hold. However, for some special $K(x)$ as stated in Theorem 1.1 below, we have the following result in the spirit of Theorem A.
THEOREM 1.1. Let $K(x) \equiv 1$ for $|x| \leq 1$ and $K(x) \equiv |x|^{-b}$ for $|x| \geq 1$ for some constant $b > 0$. Then the following statements hold:

(i) For every $\alpha$ satisfying $-2 < \alpha < \min\{0, b - 2\}$, (1.1) possesses a unique $C^2$ radial solution $u_\alpha(r)$ satisfying (1.4).

(ii) Let $u$ be an arbitrary solution of (1.1) satisfying (1.4) for some $\alpha$, then $\alpha$ satisfies $-2 < \alpha < \min\{0, b - 2\}$ and $u(x) \equiv u_\alpha(x)$ where $u_\alpha(x)$ is the solution in (i) above.

REMARK 1.2. On the constrast to the case $K \equiv 0$, the family of solution $u_\alpha(x)$ in Theorem 1.1 does not have the monotone property in $\alpha$ as the case in Theorem A. In fact, by the concrete construction of solutions in the proof of Theorem 1.1, it can be seen that $u_\alpha(r)$ and $u_\beta(r)$ exactly intersects once for $\alpha \neq \beta$. We hope that it will be useful in a future study.

Although Theorem 1.1 are only concerned with some specific $K(x)$, it still provides an interesting example to the situation when $K(x)$ is positive in $\mathbb{R}^2$. In [A], Aviles proved the following theorem, (See Theorem A1 in [A]).

THEOREM B. Assume $K(x) > 0$ in $\mathbb{R}^2$ and $\lim_{|x| \to +\infty} K(x)|x|^b = 1$ for some positive constant $b > 0$. Then, for any $\alpha$ satisfying

$$-2 < \alpha < \min\left(0, \frac{b - 2}{2}\right),$$

there exists a solution $u$ of (1.1) satisfying

$$u(x) = \alpha \log |x| + O(1) \quad \text{at } \infty.$$

Let $K(x)$ be the specific function given in Theorem 1.1 with $0 < b < 2$. Then Theorem 1.1 contradicts to the result of Theorem B. In fact, Theorem 1.1 is not an isolated case to show that Theorem B does not hold. For a general $K(x)$, set

$$\alpha_0 = \sup\{\alpha | \text{there is an entire solution } u \text{ of (1.1) such that } u(x) = \alpha \log |x| + O(1) \quad \text{at } \infty\}.$$

Our main result is

THEOREM 1.2. Suppose that $K(x)$ is positive and locally bounded in $\mathbb{R}^2$ and satisfies

$$B|x|^{-b} \leq K(x) \leq A|x|^{-b}$$

for $|x| \geq 1$ and for positive constants $A, B$ and $0 < b < 2$. Then $\alpha_0 < -\frac{2 - b}{2}$, where $\alpha_0$ is given in (1.6).
Obviously, Theorem 1.2 implies that Theorem B does not hold in general. We note that the real number $a_I$ in (1.2) is $-\frac{2-b}{2}$ if $K(x)$ satisfies (1.7). Theorem 1.2 provides a major contrast to Theorem A for the case $K(x) \leq 0$. We would like to remark that solutions possessing the asymptotic behavior (1.4) have a geometric meaning. Following conventional notations, a solution $u(x)$ of (1.1) is said to have a finite total curvature if $K(x)e^{2u(x)} \in L^1(\mathbb{R}^2)$, and the quantity \[ \frac{1}{2\pi} \int_{\mathbb{R}^2} K(x)e^{2u(x)} \, dx \] is called the total curvature of $u$. Assume $K(x)$ satisfies (1.7). A consequence of our previous results in [CLn] is that a solution $u$ has a finite total curvature if and only if $u$ possesses the asymptotic behavior (1.4), or more precisely, $\lim_{|x| \to +\infty} u(x)/\log |x|$ exists, and the identity \[ -\frac{1}{2\pi} \int_{\mathbb{R}^2} K(x)e^{2u(x)} \, dx = \lim_{|x| \to +\infty} \frac{u(x)}{\log |x|} \] are always true. Please see Lemma 2.1 in Section 2. Thus, it is interesting to know what is the possible range of $a$ or equivalently, the possible range of the total curvature of solutions. In [M], McOwen proved that if $0 < K(x) \leq C|x|^{-b}$ at $\infty$, then for every $a \in (-2, (b-2)^{-})$ where $(b-2)^{-} = \min(0, b-2)$, there exists a solution of (1.1) satisfying (1.4). Together with Theorem 1.1, we see that the result of McOwen is the best possible for a general $K$ which decays like $|x|^{-b}$ at $\infty$.

**Theorem 1.3.** Suppose $K(x)$ is a positive continuous function in $\mathbb{R}^2$ and satisfies $\lim_{|x| \to +\infty} K(x)|x|^b = 1$ for some $0 \leq b < 2$. Assume $u_n$ is a sequence of solutions of (1.1) such that

(1.8) \[ 2 - b \neq \lim_{n \to +\infty} \frac{1}{2\pi} \int_{\mathbb{R}^2} K(x)e^{2u_n(x)} \, dx < 2 \]

Then $u_n$ is bounded in $W^{2,p}_{\text{loc}}(\mathbb{R}^2)$ for any $p > 1$. Furthermore if $u_n$ converges to $u$ in $W^{2,p}_{\text{loc}}(\mathbb{R}^2)$, then

(1.9) \[ \lim_{n \to +\infty} \int_{\mathbb{R}^2} K(x)e^{2u_n(x)} \, dx = \int_{\mathbb{R}^2} K(x)e^{2u(x)} \, dx. \]

**Corollary 1.4.** Suppose $K$ satisfies the assumption of Theorem 1.3 and $u_n$ is a sequence of solutions of (1.1). If $|u_n(0)| \to +\infty$ as $n \to +\infty$ and \[ \frac{1}{2\pi} \int_{\mathbb{R}^2} K(x)e^{2u_n(x)} \, dx \leq 2 - \varepsilon_0 \text{ for some } \varepsilon_0 > 0, \] then we always have

(1.10) \[ \lim_{n \to +\infty} \int_{\mathbb{R}^2} K(x)e^{2u_n(x)} \, dx = 2\pi(2-b). \]

**Corollary 1.5.** Suppose $K$ satisfies the assumption of Theorem 1.3 and $a_0(K)$ is defined in (1.6). If $a_0(K) > -(2-b)$, then $a_0(K)$ is achieved, i.e. there exists a solution $u$ of (1.1) with

(1.11) \[ -a_0(K) = \frac{1}{2\pi} \int_{\mathbb{R}^2} K(x)e^{2u(x)} \, dx. \]
REMARK 1.6. When $K(x)$ decays like $|x|^{-b}$ for $b \geq 2$, and $u_n$ is a sequence of solutions of (1.4) satisfying
\[ 0 < \epsilon_0 \leq \frac{1}{2\pi} \int_{\mathbb{R}^2} K(x)e^{2u_n(x)} \, dx \leq 2 - \epsilon_0 \]
for some $\epsilon_0 > 0$, then $u_n$ is bounded in $L^\infty_{\text{loc}}(\mathbb{R}^2)$. The proof is easy, and will be omitted.

The paper is organized as follows. In Section 2, we will give a proof of Theorem 1.1. Both Theorem 1.2 and Theorem 1.3 will be proved in Section 3.

2. – Proof of Theorem 1.1

Let $K$ be positive in $\mathbb{R}^2$ and satisfy
\begin{equation}
|x|^{-m} \leq K(x) \leq |x|^m
\end{equation}
for $|x|$ large, where $m$ is a positive constant. A solution $u$ of (1.1) is said to have a finite total curvature if $Ke^{2u} \in L^1(\mathbb{R}^2)$, and the quantity $\frac{1}{2\pi} \int_{\mathbb{R}^2} Ke^{2u} \, dx$ is called the total curvature of $u$. Theorem 1.1 in [CLn] says that if $u$ is a solution of (1.1) with a finite total curvature, then $\lim_{|x| \to +\infty} \frac{u(x)}{\log |x|}$ exists and
\begin{equation}
\lim_{|x| \to +\infty} \frac{u(x)}{\log |x|} = -\frac{1}{2\pi} \int_{\mathbb{R}^2} Ke^{2u} \, dx.
\end{equation}
Conversely, it is easy to see that if $\lim_{|x| \to +\infty} \frac{u(x)}{\log |x|}$ exists, then $Ke^{2u} \in L^1(\mathbb{R}^2)$ and (2.2) holds. Hence, we have

\textbf{LEMMA 2.1.} Suppose $K$ satisfies (2.1). Then $K(x)e^{2u(x)} \in L^1(\mathbb{R}^2)$ if and only if $\lim_{|x| \to +\infty} \frac{u(x)}{\log |x|}$ exists. Moreover, (2.2) always holds.

\textbf{REMARK 2.2.} In fact, Theorem 1.1 in [CLn] also shows that for a solution $u$ of (1.1) having a finite total curvature $\alpha$, there exists a constant $C$ such that
\begin{equation}
\alpha \log |x| - C \leq u(x)
\end{equation}
holds. Hence, if $C_2|x|^{-b} \leq K(x) \leq C_1|x|^{-b}$ for large $|x|$, then $\alpha < -\frac{(2-b)^+}{2}$ where $(2-b)^+ = \max[2-b, 0]$.

\textbf{PROOF OF THEOREM 1.1.} Let
\begin{equation}
u_\alpha(r) = \frac{1}{2} \log(4B_1) - \log[1 + B_1r^2], \quad r \in [0, 1]
\end{equation}
and

\[ u_a(r) = \frac{1}{2} \log(4A_2^2B_2) + \left( A_2 - 1 + \frac{b}{2} \right) \log r - \log[1 + B_2r^{2A_2}], \]

where \( B_1 > 0 \) is a constant and \( \alpha = -A_2 - 1 + \frac{b}{2} \). Then it is not very difficult to verify that \( u_a \) is a \( C^2 \)-solution of (1.1) provided that

\[ A_2 = \left\{ \frac{4B_1 + \left[ B_1 \left( 1 + \frac{b}{2} \right) - \left( 1 - \frac{b}{2} \right) \right]^2}{(1 + B_1)^2} \right\}^{\frac{1}{2}}, \]

\[ B_2 = \frac{A_2(1 + B_1) + \left[ B_1 \left( 1 + \frac{b}{2} \right) - \left( 1 - \frac{b}{2} \right) \right]}{A_2(1 + B_1) - \left[ B_1 \left( 1 + \frac{b}{2} \right) - \left( 1 - \frac{b}{2} \right) \right]}. \]

Since \( u_a(0) = \frac{1}{2} \log(4B_1) \), we see that \( B_1 > 0 \) exhausts all radial solutions. It is easy to see that \( u_a \) satisfies (1.4) with \( \alpha = -A_2 - 1 + \frac{b}{2} \). Now \( A_2 \) is a monotonic function of \( B_1 \) satisfying

\[ \lim_{B_1 \to 0^+} A_2(B_1) = \left| \frac{b}{2} - 1 \right| \quad \text{and} \quad \lim_{B_1 \to \infty} A_2(B_1) = \frac{b}{2} + 1. \]

Hence \( \alpha \) satisfies \(-2 < \alpha < \min\{0, b - 2\} \). This proves (i).

Now suppose that \( u \) be an arbitrary solution of (1.1) with finite total curvature. Since \( K(x) = K(|x|) \) is nonincreasing in \( r \) and \( K(r) \geq e^{-r^\beta} \) for any \( 0 < \beta < 1 \), then from Theorem 1.7 in [CLn], we conclude that \( u \) must be a radial function. Hence \( u \equiv u_a \) for some \( \alpha \) in the range \(-2 < \alpha < \min\{0, b - 2\} \), where \( u_a \) is defined in (2.4) and (2.5). This proves (ii).

\[ \square \]

### 3. Proofs of compactness theorems

In this section, we begin with a proof of Theorem 1.2. First, we need the following result which was proved in [BM].

**Theorem 3.1 (Theorem 3 in [BM]).** Assume \( u_n \) is a sequence of solutions of

\[ \Delta u_n + K_ne^{2u_n} = 0 \quad \text{in} \ \Omega \]
satisfying

\begin{equation}
0 \leq K_n \leq C_1 \quad \text{in } \Omega,
\end{equation}

and

\begin{equation}
\|e^{2u_n}\|_{L^1(\Omega)} \leq C_2
\end{equation}

for two constants \(C_1\) and \(C_2\). Then either \(u_n\) is bounded in \(L^\infty_{\text{loc}}(\Omega)\) or there exists a subsequence of \(u_n\) (still denoted by \(u_n\)) such that either \(u_n \to -\infty\) uniformly on any compact sets of \(\Omega\) or the blow-up set \(S\) is a set of finite number of points, \(u_n \to -\infty\) uniformly on any compact set of \(\Omega \setminus S\), and \(K_n e^{2u_n}\) converges to \(\sum_i \alpha_i \delta_{p_i}\) with \(\alpha_i \geq 2\pi\) and \(S = \bigcup_i \{p_i\}\).

**Remark 3.2.** When either \(K_n\) is uniformly convergent or converges to a positive constant then Theorem 3.1 can be improved to have \(\alpha_i \geq 4\pi\).

**Proof of Theorem 1.2.** Suppose \(\alpha_0 = -(\frac{2-b}{2})\). Since \(\alpha_0 = -\frac{2-b}{2}\) cannot be achieved by some solution of (1.1) by Remark 2.2, there exists a sequence of solutions of \(u_n\) such that the total curvature

\begin{equation}
\lim_{n \to +\infty} \frac{1}{2\pi} \int_{\mathbb{R}^2} K(x)e^{2u_n(x)} \, dx = \frac{2-b}{2} < 1.
\end{equation}

Since \(K\) has a lower positive bound in any compact set of \(\mathbb{R}^2\), by Theorem 3.1, we have either \(u_n\) is uniformly bounded in any compact set or \(u_n\) is uniformly convergent to \(-\infty\) in any compact set of \(\mathbb{R}^2\).

**Step 1.** We claim that \(u_n \to -\infty\) uniformly in any compact set of \(\mathbb{R}^2\). Suppose \(u_n\) is uniformly bounded in any compact set of \(\mathbb{R}^2\). By the elliptic estimates, we may assume \(u_n \to u\) in \(W^{2,p}_{\text{loc}}(\mathbb{R}^2)\) for any \(p > 1\). In particular, \(u\) satisfies (1.1) and the total curvature

\[
\frac{1}{2\pi} \int K(x)e^{2u(x)} \, dx \leq \lim_{n \to +\infty} \frac{1}{2\pi} \int K(x)e^{2u_n(x)} \, dx = \frac{2-b}{2},
\]

which yields a contradiction by Remark 2.2. Hence, by Theorem 3.1, we have \(u_n \to -\infty\) uniformly in any compact set of \(\mathbb{R}^2\).

**Step 2.** We claim there exists a constant \(C > 0\) such that

\begin{equation}
K(x)e^{2u_n(x)} \leq C|x|^{-2} \quad \text{for } x \in \mathbb{R}^2.
\end{equation}

To prove the claim, we assume there exists \(x_n \in \mathbb{R}^2\) such that \(u_n(x_n) + \frac{(2-b)}{2} \log |x_n| \to +\infty\). By Step 1, we have \(|x_n| \to +\infty\) as \(n \to +\infty\). Set

\[
v_n(y) = u_n(x_n + |x_n|y) + \frac{2-b}{2} \log |x_n|.
\]
Then \( v_n \) satisfies

\[
\Delta v_n + K_n(y)e^{2u_n(y)} = 0 \quad \text{in} \quad |y| < \frac{1}{2},
\]

where \( K_n(y) = |x_n|^b K_n(x_n + |x_n|y) \). By the assumption on \( K \), \( 0 < \tilde{C}_1 \leq K_n(y) \leq \tilde{C}_2 \) for \( |y| < \frac{1}{2} \), and

\[
\int_{|y| < \frac{1}{2}} K_n(y)e^{2u_n(y)} dy \leq \int_{\mathbb{R}^2} K(x)e^{2u_n(x)} dx < 2\pi.
\]

By Theorem 3.1, we conclude that \( u_n(0) \leq C \) for some constant \( C \), which yields a contradiction to the assumption.

**Step 3.** There exists a positive constant \( C \) such that \( |\nabla u_n(x)| \leq C|x|^{-1} \) and \( |u_n(x) - u_n(y)| \leq C \) for \( |x| = |y| \).

In [CLn], we have proved that \( u_n \) has the following representation

\[
u_n(x) = u_n(0) + \frac{1}{2\pi} \int_{\mathbb{R}^2} \log \frac{|y|}{|x-y|} K(y)e^{2u_n(y)} dy.
\]

Thus, we have

\[
|\nabla u_n(x)| \leq \frac{1}{2\pi} \int_{\mathbb{R}^2} |x-y|^{-1} K(y)e^{2u_n(y)} dy
\]

\[
= \frac{1}{2\pi} \int_{|y-x| \leq \frac{|x|}{2}} |x-y|^{-1} K(y)e^{2u_n(y)} dy
\]

\[
+ \frac{1}{2\pi} \int_{|y-x| \geq \frac{|x|}{2}} |x-y|^{-1} K(y)e^{2u_n(y)} dy
\]

By Step 2, the first integral can be estimated by

\[
\frac{1}{2\pi} \int_{|y-x| \leq \frac{|x|}{2}} |x-y|^{-1} K(y)e^{2u_n(y)} dy \leq C_1|x|^{-2} \int_{|y-x| \leq \frac{|x|}{2}} |x-y|^{-1} dy = C_2|x|^{-1}.
\]

For the second integral, we have

\[
\frac{1}{2\pi} \int_{|y-x| \geq \frac{|x|}{2}} |x-y|^{-1} K(y)e^{2u_n(y)} dy \leq \frac{1}{\pi|x|} \int_{\mathbb{R}^2} K(y)e^{2u_n(y)} dy.
\]

Combined these two estimates together, we have

\[
|\nabla u_n(x)| \leq C_3|x|^{-1}.
\]

Set \( w_n(x) = e^{2u_n(x)} \). Then \( w_n(x) \) satisfies

\[
\Delta w_n(x) + 4(Ke^{2u_n} + |\nabla u_n|^2)w_n = 0.
\]
Since $K(x)e^{2u_n(x)} + |\nabla u_n|^2 \leq C_4|x|^{-2}$ for some constant $C_4$, by Harnack inequality, for any $a \geq 1$, there exists a positive constant $C_5 = C_5(a)$ such that
\[
\sup_{a^{-1}r \leq |x| \leq ar} w_n(x) \leq C_5 \inf_{a^{-1}r \leq |x| \leq ar} w_n(x).
\]

Hence, Step 3 is proved.

**Step 4.** For any $\varepsilon > 0$ there exists $R = R(\varepsilon) > 0$ such that $|u_n(x) - u_n(y)| \leq \varepsilon$ for $|x| = |y| \geq R\varepsilon$ and large $n$.

Step 4 will be proved by contradiction. Suppose there exist a positive number $\varepsilon_0 > 0$ and $x_n, y_n$ with $r_n = |x_n| = |y_n| \to +\infty$ such that $u_n(\bar{x}_n) - u_n(x_n) \geq \varepsilon_0$. Let
\[
v_n(y) = u_n(r_ny) - u_n(x_n).
\]

Then $v_n$ satisfies
\[
\Delta v_n + K_n(y)e^{2v_n} = 0,
\]
where $K_n(y) = e^{2u_n(x_n)}K(r_ny)r_n^2$. By Step 2,
\[
K_n(y) \leq C_1e^{2u_n(x_n)}r_n^{2-b}|y|^{-b} \leq C_2|y|^{-b}.
\]

For $|y| \geq 1$, we have
\[
K_n(y) \geq C_3e^{2u_n(x_n)}r_n^{2-b}|y|^{-b}.
\]

By Step 3 and the Harnack inequality (3.9), $v_n(y)$ is bounded in $L_\text{loc}^\infty(\mathbb{R}^2)$. By the elliptic estimates, we may assume $v_n(y) \to v_0(y)$ in $W^{2,p}_\text{loc}(\mathbb{R}^2)$ for any $p > 1$. Suppose there exists a subsequence of $x_n$ (still denoted by $x_n$) such that $\lim_{n \to +\infty} e^{2u_n(x_n)}r_n^{2-b} = S > 0$, then by (3.10) and (3.11), we may assume $K_n(y) \to K_0(y)$ weakly in $L_\text{loc}^\infty(\mathbb{R}^2 \setminus \{0\})$, where $K_0(y)$ satisfies
\[
C_1|y|^{-b} \leq K_0(y) \leq C_2|y|^{-b}
\]
for some positive constants $C_1$ and $C_2$, and $v_0(y)$ satisfies
\[
\Delta v_0(y) + K_0(y)e^{2v_0(y)} = 0 \quad \text{in} \quad \mathbb{R}^2 \setminus \{0\}.
\]

For any $0 < r_0 < r_1$, we have
\[
\int_{r_0 \leq |y| \leq r_1} K_n(y)e^{2v_0(y)} dy = \lim_{n \to +\infty} \int_{r_0 \leq |y| \leq r_1} K_n(y)e^{2v_n(y)} dy
= \int_{r_0r_n \leq |y| \leq r_1r_n} K(x)e^{2u_n(x)} dx
\leq \int_{\mathbb{R}^2} K(x)e^{2u_n(x)} dx
\to \left(\frac{2 - b}{2}\right)2\pi.
\]
Thus, the total curvature
\[ \frac{1}{2\pi} \int_{\mathbb{R}^2} K_0(y)e^{2\nu_0(y)}dy \leq \frac{2 - b}{2}. \]

Applying Corollary 1.4 in [CLn], \( \nu_0(y) \) in fact satisfies
\[ \Delta \nu_0(y) + K_0(y)e^{2\nu_0(y)} = 2\pi \beta \delta(0) \quad \text{in} \quad \mathbb{R}^2 \]
for some \( \beta \in \mathbb{R} \), where \( \delta(0) \) is the Dirac measure at the origin, and the function \( \nu_1(y) = \nu_0(y) - \beta \log |y| \) satisfies
\[ \Delta \nu_1(y) + K_0(y)|y|^{2\beta}e^{2\nu_1(y)}dy = 0 \quad \text{in} \quad \mathbb{R}^2. \]

It is easy to see that
\[ o(1) + \beta = \frac{1}{2\pi} \int_{|y|=r} \frac{\partial \nu_0}{\partial \nu}(y)d\sigma \]
\[ = \lim_{n \to +\infty} \frac{1}{2\pi} \int_{|y|=r} \frac{\partial \nu_n}{\partial \nu}(y)d\sigma \]
\[ = \lim_{n \to +\infty} -\frac{1}{2\pi} \int_{|y|\leq r} K_n(y)e^{2\nu_n(y)}dy \]
\[ = \lim_{n \to +\infty} -\frac{1}{2\pi} \int_{|x|\leq nr} K(x)e^{2\nu_n(x)}dx, \]
where \( o(1) \) denotes \( o(1) \to 0 \) as \( r \to 0 \). Thus, putting (3.12) and the above together, we have
\[ \frac{1}{2\pi} \int_{\mathbb{R}^2} K_0(y)|y|^{2\beta}e^{2\nu_1(y)}dy = \frac{1}{2\pi} \int_{\mathbb{R}^2} K_0(y)e^{2\nu_0(y)}dy \]
\[ \leq \frac{2 - b}{2} + \beta = \frac{2 - b + 2\beta}{2}. \]

Obviously, \( 2 - b + 2\beta > 0 \). Since \( K_0(y)|y|^{2\beta} \sim |y|^{-b+2\beta} \) at \( \infty \), by Remark 2.2, there exists no entire solution (3.14) with the total curvature equal to \( \frac{2 - b + 2\beta}{2} \). Thus, it yields a contradiction. Hence we have proved
as \( n \to +\infty \), then \( \nu_0(y) \) is harmonic in \( \mathbb{R}^2 \setminus \{0\} \).

By Step 3,
\[ |
\nabla \nu_n(y)| = |n| \nabla u_n(r_n y)| \leq C |y|^{-1}. \]

By Liouville’s theorem, we have
\[ \nu_0(y) = \alpha_0 \log |y| + C, \]
where both $\alpha_0$ and $C$ are constant. Since $v_0(y)$ is radially symmetric, it obviously yields a contradiction to the assumption. Hence, Step 4 is proved.

**STEP 5.** Set

$$F_n(r) = \int_{B_r} K(x) e^{2u_n(x)} \, dx,$$

and

$$\bar{u}_n(r) = \frac{1}{2\pi r} \int_{|x|=r} u_n(x) \, ds.$$

Define $\tilde{K}_n(r)$ by

$$\tilde{K}_n(r) = (2\pi r)^{-1} e^{-2\bar{u}_n(r)} \int_{|x|=r} K(x) e^{2u_n(x)} \, ds.$$

Differentiating (3.15) and (3.16) with respect to $r$, we have

$$F'_n(r) = (2\pi r) \tilde{K}_n(r) e^{2\bar{u}_n(r)},$$

$$\bar{u}'_n(r) = -\frac{1}{2\pi r} \int_{B_r} K(x) e^{2u_n(x)} \, dx = \frac{-F_n(r)}{2\pi r}.$$

Thus, we have

\[
\left(\frac{r^{1-b} F'_n(r)}{\tilde{K}_n(r)}\right)' = (2\pi r^{2-b} e^{2\bar{u}_n(r)})'.
\]

\[
= 2\pi [(2-b)r^{1-b} e^{2\bar{u}_n(r)} + 2r^{2-b} e^{2\bar{u}_n(r)} \bar{u}'_n(r)]
\]

\[
= \frac{(2-b)F'_n(r)}{r^b \tilde{K}_n} - \frac{F'_n(r)}{r^b \tilde{K}_n} \frac{F_n(r)}{\pi} + \frac{2-b F_n(r)}{\pi} + \frac{2-b F_n(r)}{\pi}
\]

\[
= -\frac{F'_n(r)}{r^b \tilde{K}_n} \left[ \frac{F_n(r)}{\pi} - (2-b) \right].
\]

Since $F_n(\infty) > \pi(2-b)$, set $r_n$ to satisfy $F_n(r_n) = \pi(2-b)$. Obviously, $\lim_{r \to +\infty} r_n = +\infty$. For any $\varepsilon > 0$, by Step 4, there exists $R = R(\varepsilon) > 0$ such that

$$A e^{2\varepsilon r^{-b}} \leq \tilde{K}_n(r) \leq e^{-2\varepsilon Br^{-b}} \quad \text{for} \quad r \geq R_\varepsilon.$$

Hence,

$$\left(\frac{r^{1-b} F'_n(r)}{\tilde{K}_n(r)}\right)' \geq \begin{cases} 
\frac{e^{2\varepsilon}}{B} \left( 2-b - \frac{F_n(r)}{\pi} \right) F'_n(r) & \text{for} \quad R \leq r \leq r_n, \\
\frac{e^{-2\varepsilon}}{A} \left( 2-b - \frac{F_n(r)}{\pi} \right) F'_n(r) & \text{for} \quad r \geq r_n.
\end{cases}$$
Since \( \lim_{x \to +\infty} r F_n(r) = 0 \) for any \( n \), we have

\[
- \frac{r^{1-b} F_n'(r)}{K_n(r)} |_{r=R} = \frac{e^{-2b}}{B} \int_R^\infty \left[ (2 - b) - \frac{F_n(r)}{\pi} \right] F_n'(r) dr + \frac{e^{2b}}{A} \int_R^\infty \left( 2 - b \right) F_n'(r) dr
\]

\[
= -e^{-2b} B^{-1} \left[ (2 - b) F_n(r) - \frac{F_n^2(r)}{2\pi} \right] |_{r=R} + \frac{e^{2b}}{B} - \frac{e^{-2b}}{A} \frac{\pi (2-b)^2}{2} + e^{-2b} A^{-1} \left( 2 - b \right) F_n(\infty) - \frac{F_n^2(\infty)}{2\pi}.
\]

By Step 1, we note that the boundary term at \( R \) tends to 0 as \( n \to +\infty \). By letting \( n \to +\infty \) first and then \( \epsilon \to 0 \), the above yields

\[
0 \geq \left( \frac{1}{B} - \frac{1}{A} \right) \frac{\pi (2-b)^2}{2} + \frac{1}{A} \left[ (2 - b) \lim_{n \to +\infty} F_n(\infty) - \lim_{n \to +\infty} \frac{F_n^2(\infty)}{2\pi} \right]
\]

\[
= \frac{\pi (2-b)^2}{2B},
\]

a contradiction, where \( \lim_{n \to +\infty} F_n(\infty) = (2-b)\pi \) is used. Therefore, the proof of Theorem 1.2 is completely finished.

**Proof of Theorem 1.3.** Suppose \( u_n \) is a sequence of solution of (1.1) and satisfies the assumption of Theorem 1.3. By Remark 3.2, we may assume that either \( u_n \) is uniformly bounded in any compact set or \( u_n \) uniformly converges to \(-\infty\) in any compact set of \( \mathbb{R}^2 \). By the same reasoning of Step 1 and Step 2 of Theorem 1.2, there exists a constant \( C > 0 \) such that inequalities

\[
(3.20) \quad K(x)e^{2\alpha_n(x)} \leq C|x|^{-2},
\]

\[
(3.21) \quad |\nabla u_n(x)| \leq C|x|^{-1},
\]

\[
(3.22) \quad |u_n(x) - u_n(y)| \leq C \text{ whenever } |x| = |y|
\]

hold.

First, we want to prove \( u_n \) is bounded in \( L^\infty_{loc}(\mathbb{R}^2) \). Suppose the claim is not true. As before, we want to prove the asymptotic symmetry of \( u_n \), i.e. for any \( \epsilon > 0 \), there exists \( R = R(\epsilon) > 0 \) such that for \( |y| = |x| \geq R \),

\[
|u_n(x) - u_n(y)| \leq \epsilon.
\]

Assume the conclusion is not true. Then there exists \( r_n \to +\infty \) such that

\[
u_n(x) \geq u_n(x_n) + \epsilon_0\]

with \( |x_n| = |x_n| = r_n \) for some positive constant \( \epsilon_0 > 0 \). Let

\[
u_n(y) = u_n(r_n y) - u_n(x_n).
\]
Then \( v_n \) is bounded in \( L^\infty(\mathbb{R}^2 \setminus \{0\}) \) by Harnack inequality and satisfies

\[
\Delta v_n + K_n(y)e^{2v_n} = 0 \quad \text{in} \quad \mathbb{R}^2,
\]

where \( K_n(y) = e^{2u_n(x_n)} K(r_n y)^2 \). By the assumption on \( K \) and (3.20) for any \( r_0 > 0 \), we have for \( |y| \geq r_0 \),

\[
K_n(y) \leq 2e^{2u_n(x_n)} r_n^{2-b} |y|^{-b}
\]

for large \( n \). If \( \lim_{n \to +\infty} e^{u_n(x_n)} r_n^{2-b} = 0 \), then using (3.21) and the same argument of Step 4 of Theorem 1.2, \( v_n(y) \) converges to \( v_0(y) = a_0 \log |y| + C_0 \) in \( L^\infty(\mathbb{R}^2 \setminus \{0\}) \) where \( a_0 \) and \( C_0 \) are constant. Since \( v_0(y) \) is radially symmetric, it yields a contradiction.

If \( \lim_{n \to +\infty} e^{u_n(x_n)} r_n^{2-b} = s > 0 \), then \( K_n(y) \to s |y|^{-b} \) uniformly in any compact set of \( \mathbb{R}^2 \setminus \{0\} \). Then \( v_0(y) \) satisfies

\[
\begin{cases} \\
\Delta v_0(y) + s |y|^{-b}e^{2v_0(y)} = \beta \delta(0) \quad \text{in} \quad \mathbb{R}^2, \\
v_0(y) = \frac{\beta}{2\pi} \log |y| + O(1) \quad \text{as} \quad y \to 0,
\end{cases}
\]

where \( \beta \in \mathbb{R} \) and \( \delta(0) \) is the Dirac measure. For any \( r_0 > 0 \),

\[
\int_{|y|=r_0} \frac{\partial v_0(y)}{\partial \nu} d\sigma = \lim_{n \to +\infty} \int_{|y|=r_0} \frac{\partial v_n}{\partial \nu} d\sigma = -\lim_{n \to +\infty} \int_{|y| \leq r_0} K_n(y)e^{2v_n} dy \leq 0.
\]

Thus either \( v_0(y) \) is regular at 0 or \( v_0(y) \to +\infty \) as \( |y| \to +\infty \). Since \( |y|^{-b}e^{2v_0(y)} \in L^1(\mathbb{R}^2) \) and \( v_0(y) = \alpha \log |y| + O(1) \) as \( |y| \to +\infty \), for some \( \alpha \in \mathbb{R} \), we have \( 2\alpha - b < -2 \), i.e. \( |y|^{-b}e^{2v_0(y)} = o(1)|y|^{-2} \) as \( |y| \to +\infty \). Hence, we can apply the method of moving planes as in [CL] and [CLn] to prove \( v_0(y) \) is radially symmetric with respect to the origin, which obviously yields a contradiction. Hence the uniformly asymptotic symmetry of \( u_n \) is proved.

To finish the proof of Theorem 1.3, we set

\[
F_n(r) = \int_{B_r} K(x)e^{2u_n(x)} dx,
\]

and,

\[
\tilde{K}_n(r) = (2\pi r)^{-1} e^{-2u_n(r)} \int_{|x|=r} K(x)e^{2u_n(x)} dx.
\]

As in (3.19), we have

\[
(3.23) \quad \left( \frac{r^{1-b} F'_n(r)}{\tilde{K}_n(r)} \right)' = -\frac{F'_n(r)}{r^b \tilde{K}_n(r)} \left[ \frac{F_n(r)}{\pi} - (2-b) \right].
\]
For any $\varepsilon > 0$, let $R = R(\varepsilon)$ be large such that

$$e^{-3\varepsilon} r^{-b} \leq K_n(r) \leq e^{3\varepsilon} r^{-b}$$

holds for $r \geq R$. This immediately follows from the uniformly asymptotic symmetry of $u_n$ and the assumption on $K$. Let $r_n$ satisfy $F_n(r_n) = \pi(2 - b)$. Suppose $\lim_{n \to +\infty} \int_{\mathbb{R}^2} K(x) e^{2u_n(x)} dx < (2 - b)$ first. Then we can follow the same proof as Step 5 in Theorem 1.2 to obtain

$$(2 - b) F(\infty) - (2\pi)^{-1} F^2(\infty) \leq 0,$$

where $F(\infty) = \lim_{n \to +\infty} \int_{\mathbb{R}^2} K(x) e^{2u_n(x)} dx$. Obviously, the above yields $F(\infty) \geq 2\pi(2 - b)$, a contradiction.

Suppose $\int_{\mathbb{R}^2} K(x) e^{2u_n(x)} dx > 2\pi(2 - b)$. Then by (3.23), we have the reverse inequality

$$\left( \frac{r^{1-b} F_n'(r)}{K_n(r)} \right) \leq \begin{cases} e^{3\varepsilon} [(2 - b) - F_n(r)/\pi] F_n'(r) & \text{for } R \leq r \leq r_n, \\ e^{-3\varepsilon} [(2 - b) - F_n(r)/\pi] F_n'(r) & \text{for } r \geq r_n. \end{cases}$$

Integrating the above and letting $n \to +\infty$ first and then $\varepsilon \to 0$, we have

$$(2 - b) F(\infty) - \frac{F^2(\infty)}{2\pi} \geq 0,$$

which implies

$$F(\infty) \leq 2\pi(2 - b).$$

Obviously, it yields a contradiction. Hence the boundedness of $u_n$ in $L^\infty_{\text{loc}}(\mathbb{R}^2)$ is proved.

To prove (1.9), we may assume $u_n \to u_0$ in $W^{2,p}_{\text{loc}}(\mathbb{R}^2)$ for any $p > 1$. Obviously, $u_0$ satisfies (1.1) and has a finite total curvature. In particular,

$$\frac{1}{2\pi} \int_{\mathbb{R}^2} K(x) e^{2u_0(x)} dx > \frac{2 - b}{2}.$$ 

Hence, there exists $R_0 > 0$, $\varepsilon_0 > 0$ and $n_0$ such that

$$\frac{1}{2\pi} \int_{B_r} K(x) e^{2u_n(x)} dx > \left( \frac{2 - b}{2} + \varepsilon_0 \right)$$

for all $r \geq R_0$ and $n \geq n_0$. Integrating (1.1), we have

$$\frac{d}{dr} u_n(r) < - \left( \frac{2 - b}{2} + \varepsilon_0 \right) r^{-1}$$
for all $r \geq R_0$ and $n \geq n_0$, where $\tilde{u}_n(r) = \frac{1}{2\pi r} \int_{|x|=r} u_n(x) d\sigma$. Thus,

$$\tilde{u}_n(r) \leq \tilde{u}_n(R_0) - \left( \frac{2 - b}{2} + \varepsilon_0 \right) \log r / R_0.$$  

Applying the Harnack inequality, we have

$$u_n(x) \leq \tilde{u}_n(|x|) + C_1 \leq C_2 - \left( \frac{2 - b}{2} + \varepsilon_0 \right) \log r$$

for $r \geq R_0$ and $n \geq n_0$ where $C_1$ and $C_2$ are constants independent of $n$ and $r$. In particular,

$$\int_{|y| \geq r} K(x) e^{2\tilde{u}_n(x)} dx \leq C_3 \int_{|y| \geq r} |x|^{-(2 + 2\varepsilon_0)} dx$$

could be arbitrarily small provided that $r$ is large. Thus, (1.9) follows immediately. And the proof of Theorem 1.3 is finished.

**REFERENCES**


