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Compactness of Conformal Metrics with Positive Gaussian Curvature in $\mathbb{R}^2$

KUO-SHUNG CHENG – CHANG-SHOU LIN

Abstract. In this paper we consider the compactness of a sequence of solutions $u_n$ of

\[(0.1) \quad \Delta u + K(x)e^{2u} = 0 \quad \text{in} \quad \mathbb{R}^2,\]

where $K(x)$ is positive in $\mathbb{R}^2$ and decays like $|x|^{-b}$ at $\infty$ for some $b > 0$. Assuming that the limit of the total curvature of $u_n$ satisfies

\[(0.2) \quad 2 - b \neq \lim_{n \to +\infty} \frac{1}{2\pi} \int_{\mathbb{R}^2} K(x)e^{2u_n(x)} \, dx < 2,\]

we prove that $u_n$ must be bounded in $W^{2,p}_{\text{loc}}(\mathbb{R}^2)$ for any $p > 1$. We also construct a specific $K(x) = K(|x|)$ to show that the total curvature of any solution $u$ of equation (0.1) with this $K(|x|)$ must satisfy

\[(0.3) \quad (2 - b) < \frac{1}{2\pi} \int_{\mathbb{R}^2} K(x)e^{2u} \, dx < 2.\]

This appears to be in contrast with the statement of Theorem A\textsuperscript{1} in [A]. In this respect, we show that for any $K$ which decays like $|x|^{-b}$ for $0 < b < 2$, there exists $\alpha_0(K) > \frac{2-b}{2}$ such that the total curvature of any solution $u$ of (0.1) must satisfy

\[\frac{1}{2\pi} \int_{\mathbb{R}^2} Ke^{2u} \, dx \geq \alpha_0(K) > \frac{2-b}{2}.\]

1. – Introduction

In this paper, we consider the entire solution of the equation

\[(1.1) \quad \Delta u + K(x)e^{2u} = 0 \quad \text{in} \quad \mathbb{R}^2,\]

where $\Delta$ is the Laplacian operator of $\mathbb{R}^2$ and $K(x)$ is a given function in $\mathbb{R}^2$. Equation (1.1) arises in the problem of finding a Riemannian metric which is
conformal to the flat metric of $\mathbb{R}^2$ and realizes the given function $K(x)$ as its Gaussian curvature. We refer the reader to [CN1] for a brief description of the background and the history of this problem.

In case $K$ is nonpositive on $\mathbb{R}^2$, a fairly complete understanding of the the solution set of (1.1) was achieved in [CN1], [CN2]. To state the results in [CN2], we introduce $\alpha_1$ as

$$\alpha_1 = \sup \left\{ \alpha \in \mathbb{R} \mid \int_{\mathbb{R}^2} |K(x)|(1 + |x|^2)^\alpha \, dx < +\infty \right\}.$$  

Then the main result in [CN2] is

**Theorem A.** Suppose that $K \leq 0$ in $\mathbb{R}^2$ and that

$$|x|^{-m} \leq |K(x)| \leq |x|^m$$

for $|x|$ large and some positive constant $m$. Then we have:

(I) If $\alpha_1 \leq 0$, then (1.1) possesses no entire solution in $\mathbb{R}^2$.

(II) If $\alpha_1 > 0$, then the following conclusions hold:

(i) For each $\alpha \in (0, \alpha_1)$, (1.1) possesses a unique solution $u_\alpha$ such that

$$u_\alpha(x) = \alpha \log |x| + O(1) \text{ at } \infty.$$  

(ii) The function $U(x)$ given by

$$U(x) = \sup \{ u(x) \mid u \text{ is an entire solution of (1.1) in } \mathbb{R}^2 \}$$

is well-defined everywhere in $\mathbb{R}^2$ and is a solution of (1.1) in $\mathbb{R}^2$. Moreover, $K(x)e^{2u(x)} \in L^1(\mathbb{R}^2)$.

(iii) Let $u$ be an arbitrary solution of (1.1) in $\mathbb{R}^2$. Then either $u \equiv U$ or $u \equiv u_\alpha$ for some $\alpha \in (0, \alpha_1)$.

(iv) If $0 < \alpha < \beta < \alpha_1$, then $u_\alpha(x) < u_\beta(x) < U(x)$ for all $x \in \mathbb{R}^2$. Furthermore, for any given $\varepsilon > 0$, there exists a constant $R = R(\varepsilon)$ such that for $|x| > R$,

$$(\alpha_1 - \varepsilon) \log |x| - C \leq U(x) \leq \alpha_1 \log |x| + C.$$  

In this paper, $K$ is always assumed locally bounded and positive in $\mathbb{R}^2$. A solution $u$ means $u \in W^{2,p}_{\text{loc}}(\mathbb{R}^2)$ for any $p > 1$ and satisfies (1.1) in the distributional sense. For the case $K(x)$ is positive in $\mathbb{R}^2$, it is not expected that results similar to Theorem A should hold. However, for some special $K(x)$ as stated in Theorem 1.1 below, we have the following result in the spirit of Theorem A.
THEOREM 1.1. Let $K(x) \equiv 1$ for $|x| \leq 1$ and $K(x) \equiv |x|^{-b}$ for $|x| \geq 1$ for some constant $b > 0$. Then the following statements hold:

(i) For every $\alpha$ satisfying $-2 < \alpha < \min\{0, b - 2\}$, (1.1) possesses a unique $C^2$ radial solution $u_\alpha(r)$ satisfying (1.4).

(ii) Let $u$ be an arbitrary solution of (1.1) satisfying (1.4) for some $\alpha$, then $\alpha$ satisfies $-2 < \alpha < \min\{0, b - 2\}$ and $u(x) \equiv u_\alpha(x)$ where $u_\alpha(x)$ is the solution in (i) above.

REMARK 1.2. On the contrast to the case $K \leq 0$, the family of solution $u_\alpha(x)$ in Theorem 1.1 does not have the monotone property in $\alpha$ as the case in Theorem A. In fact, by the concrete construction of solutions in the proof of Theorem 1.1, it can be seen that $u_\alpha(r)$ and $u_\beta(r)$ exactly intersects once for $\alpha \neq \beta$. We hope that it will be useful in a future study.

Although Theorem 1.1 are only concerned with some specific $K(x)$, it still provides an interesting example to the situation when $K(x)$ is positive in $\mathbb{R}^2$. In [A], Aviles proved the following theorem, (See Theorem A1 in [A]).

THEOREM B. Assume $K(x) > 0$ in $\mathbb{R}^2$ and $\lim_{|x| \to +\infty} K(x)|x|^b = 1$ for some positive constant $b > 0$. Then, for any $\alpha$ satisfying

\begin{equation}
-2 < \alpha < \min\left(0, \frac{b - 2}{2}\right),
\end{equation}

there exists a solution $u$ of (1.1) satisfying

\begin{equation}
\alpha \log |x| + O(1) \quad \text{at } \infty.
\end{equation}

Let $K(x)$ be the specific function given in Theorem 1.1 with $0 < b < 2$. Then Theorem 1.1 contradicts to the result of Theorem B. In fact, Theorem 1.1 is not an isolated case to show that Theorem B does not hold. For a general $K(x)$, set

\begin{equation}
\alpha_0 = \sup\{\alpha \mid \text{there is an entire solution } u \text{ of (1.1) such that } u(x) = \alpha \log |x| + O(1) \text{ at } \infty\}.
\end{equation}

Our main result is

THEOREM 1.2. Suppose that $K(x)$ is positive and locally bounded in $\mathbb{R}^2$ and satisfies

\begin{equation}
B|x|^{-b} \leq K(x) \leq A|x|^{-b}
\end{equation}

for $|x| \geq 1$ and for positive constants $A$, $B$ and $0 < b < 2$. Then $\alpha_0 < -\frac{2 - b}{2}$, where $\alpha_0$ is given in (1.6).
Obviously, Theorem 1.2 implies that Theorem B does not hold in general. We note that the real number $\alpha_1$ in (1.2) is $-\frac{2-b}{2}$ if $K(x)$ satisfies (1.7). Theorem 1.2 provides a major contrast to Theorem A for the case $K(x) \leq 0$. We would like to remark that solutions possessing the asymptotic behavior (1.4) have a geometric meaning. Following conventional notations, a solution $u(x)$ of (1.1) is said to have a finite total curvature if $K(x)e^{2u(x)} \in L^1(\mathbb{R}^2)$, and the quantity $\frac{1}{2\pi} \int_{\mathbb{R}^2} K(x)e^{2u(x)} \, dx$ is called the total curvature of $u$. Assume $K(x)$ satisfies (1.7). A consequence of our previous results in [CLn] is that a solution $u$ has a finite total curvature if and only if $u$ possesses the asymptotic behavior (1.4), or more precisely, $\lim_{|x| \to +\infty} \frac{u(x)}{\log |x|}$ exists, and the identity

$$-\frac{1}{2\pi} \int_{\mathbb{R}^2} K(x)e^{2u(x)} \, dx = \lim_{|x| \to +\infty} \frac{u(x)}{\log |x|}$$

are always true. Please see Lemma 2.1 in Section 2. Thus, it is interesting to know what is the possible range of $\alpha$ or equivalently, the possible range of the total curvature of solutions. In [M], McOwen proved that if $0 < K(x) \leq C|x|^{-b}$ at $\infty$, then for every $\alpha \in (-2, (b-2)^-)$ where $(b-2)^- = \min(0, b-2)$, there exists a solution of (1.1) satisfying (1.4). Together with Theorem 1.1, we see that the result of McOwen is the best possible for a general $K$ which decays like $|x|^{-b}$ at $\infty$.

**Theorem 1.3.** Suppose $K(x)$ is a positive continuous function in $\mathbb{R}^2$ and satisfies $\lim_{|x| \to +\infty} K(x)|x|^b = 1$ for some $0 \leq b < 2$. Assume $u_n$ is a sequence of solutions of (1.1) such that

$$2 - b \neq \lim_{n \to +\infty} \frac{1}{2\pi} \int_{\mathbb{R}^2} K(x)e^{2u_n(x)} \, dx < 2$$

Then $u_n$ is bounded in $W^{2,p}_{\text{loc}}(\mathbb{R}^2)$ for any $p > 1$. Furthermore if $u_n$ converges to $u$ in $W^{2,p}_{\text{loc}}(\mathbb{R}^2)$, then

$$\lim_{n \to +\infty} \int_{\mathbb{R}^2} K(x)e^{2u_n(x)} \, dx = \int_{\mathbb{R}^2} K(x)e^{2u(x)} \, dx .$$

**Corollary 1.4.** Suppose $K$ satisfies the assumption of Theorem 1.3 and $u_n$ is a sequence of solutions of (1.1). If $|u_n(0)| \to +\infty$ as $n \to +\infty$ and $\frac{1}{2\pi} \int_{\mathbb{R}^2} K(x)e^{2u_n(x)} \, dx \leq 2 - \epsilon_0$ for some $\epsilon_0 > 0$, then we always have

$$\lim_{n \to +\infty} \int_{\mathbb{R}^2} K(x)e^{2u_n(x)} \, dx = 2\pi(2-b).$$

**Corollary 1.5.** Suppose $K$ satisfies the assumption of Theorem 1.3 and $\alpha_0(K)$ is defined in (1.6). If $\alpha_0(K) > -(2-b)$, then $\alpha_0(K)$ is achieved, i.e. there exists a solution $u$ of (1.1) with

$$-\alpha_0(K) = \frac{1}{2\pi} \int_{\mathbb{R}^2} K(x)e^{2u(x)} \, dx .$$
Remark 1.6. When \( K(x) \) decays like \(|x|^{-b}\) for \( b \geq 2 \), and \( u_n \) is a sequence of solutions of (1.4) satisfying
\[
0 < \varepsilon_0 \leq \frac{1}{2\pi} \int_{\mathbb{R}^2} K(x)e^{2u_n(x)} \, dx \leq 2 - \varepsilon_0
\]
for some \( \varepsilon_0 > 0 \), then \( u_n \) is bounded in \( L^\infty_{\text{loc}}(\mathbb{R}^2) \). The proof is easy, and will be omitted.

The paper is organized as follows. In Section 2, we will give a proof of Theorem 1.1. Both Theorem 1.2 and Theorem 1.3 will be proved in Section 3.

2. – Proof of Theorem 1.1

Let \( K \) be positive in \( \mathbb{R}^2 \) and satisfy
\[
|x|^{-m} \leq K(x) \leq |x|^m
\]
for \( |x| \) large, where \( m \) is a positive constant. A solution \( u \) of (1.1) is said to have a finite total curvature if \( Ke^{2u} \in L^1(\mathbb{R}^2) \), and the quantity \( \frac{1}{2\pi} \int Ke^{2u} \, dx \) is called the total curvature of \( u \). Theorem 1.1 in [CLn] says that if \( u \) is a solution of (1.1) with a finite total curvature, then \( \lim_{|x| \to +\infty} \frac{u(x)}{\log |x|} \) exists and
\[
\lim_{|x| \to +\infty} \frac{u(x)}{\log |x|} = -\frac{1}{2\pi} \int \mathbb{R}^2 Ke^{2u} \, dx.
\]
Conversely, it is easy to see that if \( \lim_{|x| \to +\infty} \frac{u(x)}{\log |x|} \) exists, then \( Ke^{2u} \in L^1(\mathbb{R}^2) \) and (2.2) holds. Hence, we have

Lemma 2.1. Suppose \( K \) satisfies (2.1). Then \( K(x)e^{2u(x)} \in L^1(\mathbb{R}^2) \) if and only if \( \lim_{|x| \to +\infty} \frac{u(x)}{\log |x|} \) exists. Moreover, (2.2) always holds.

Remark 2.2. In fact, Theorem 1.1 in [CLn] also shows that for a solution \( u \) of (1.1) having a finite total curvature \( \alpha \), there exists a constant \( C \) such that
\[
\alpha \log |x| - C \leq u(x)
\]
holds. Hence, if \( C_2|x|^{-b} \leq K(x) \leq C_1|x|^{-b} \) for large \( |x| \), then \( \alpha < -\frac{(2-b)^+}{2} \) where \( (2-b)^+ = \max\{2-b, 0\} \).

Proof of Theorem 1.1. Let
\[
u_{\alpha}(r) = \frac{1}{2} \log(4B_1) - \log[1 + B_1r^2], \quad r \in [0, 1]
\]
and

\[ u_\alpha(r) = \frac{1}{2} \log(4A_2^2B_2) + \left( A_2 - 1 + \frac{b}{2} \right) \log r - \log[1 + B_2r^{2A_2}], \]

where \( B_1 > 0 \) is a constant and \( \alpha = -A_2 - 1 + \frac{b}{2} \). Then it is not very difficult to verify that \( u_\alpha \) is a \( C^2 \)-solution of (1.1) provided that

\[
A_2 = \left( \frac{4B_1 + B_1 \left( 1 + \frac{b}{2} \right) - \left( 1 - \frac{b}{2} \right)}{(1 + B_1)^2} \right)^{\frac{1}{2}},
\]

\[
B_2 = \frac{A_2(1 + B_1) + \left[ B_1 \left( 1 + \frac{b}{2} \right) - \left( 1 - \frac{b}{2} \right) \right]}{A_2(1 + B_1) - \left[ B_1 \left( 1 + \frac{b}{2} \right) - \left( 1 - \frac{b}{2} \right) \right]}.
\]

Since \( u_\alpha(0) = \frac{1}{2} \log(4B_1) \), we see that \( B_1 > 0 \) exhausts all radial solutions. It is easy to see that \( u_\alpha \) satisfies (1.4) with \( \alpha = -A_2 - 1 + \frac{b}{2} \). Now \( A_2 \) is a monotonic function of \( B_1 \) satisfying

\[
\lim_{B_1 \to 0^+} A_2(B_1) = \left| \frac{b}{2} - 1 \right| \quad \text{and} \quad \lim_{B_1 \to \infty} A_2(B_1) = \frac{b}{2} + 1.
\]

Hence \( \alpha \) satisfies \( -2 < \alpha < \min\{0, b - 2\} \). This proves (i).

Now suppose that \( u \) be an arbitrary solution of (1.1) with finite total curvature. Since \( K(x) = K(|x|) \) is nonincreasing in \( r \) and \( K(r) \geq e^{-r^\beta} \) for any \( 0 < \beta < 1 \), then from Theorem 1.7 in [CLn], we conclude that \( u \) must be a radial function. Hence \( u \equiv u_\alpha \) for some \( \alpha \) in the range \( -2 < \alpha < \min\{0, b - 2\} \), where \( u_\alpha \) is defined in (2.4) and (2.5). This proves (ii).

\[ \square \]

3. – Proofs of compactness theorems

In this section, we begin with a proof of Theorem 1.2. First, we need the following result which was proved in [BM].

**Theorem 3.1 (Theorem 3 in [BM]).** Assume \( u_n \) is a sequence of solutions of

\[
\Delta u_n + K_n e^{2u_n} = 0 \quad \text{in} \ \Omega
\]
satisfying

\[ 0 \leq K_n \leq C_1 \quad \text{in } \Omega, \]

and

\[ \|e^{2u_n}\|_{L^1(\Omega)} \leq C_2 \]

for two constants \( C_1 \) and \( C_2 \). Then either \( u_n \) is bounded in \( L_\text{loc}^\infty(\Omega) \) or there exists a subsequence of \( u_n \) (still denoted by \( u_n \)) such that either \( u_n \to -\infty \) uniformly on any compact sets of \( \Omega \) or the blow-up set \( S \) is a set of finite number of points, \( u_n \to -\infty \) uniformly on any compact set of \( \Omega \setminus S \), and \( K_n e^{2u_n} \) converges to \( \sum_i \alpha_i \delta_{p_i} \) with \( \alpha_i \geq 2\pi \) and \( S = \bigcup_i \{ p_i \} \).

**Remark 3.2.** When either \( K_n \) is uniformly convergent or converges to a positive constant then Theorem 3.1 can be improved to have \( \alpha_i \geq 4\pi \).

**Proof of Theorem 1.2.** Suppose \( \alpha_0 = -\left(\frac{2-b}{2}\right) \). Since \( \alpha_0 = -\frac{2-b}{2} \) cannot be achieved by some solution of (1.1) by Remark 2.2, there exists a sequence of solutions of \( u_n \) such that the total curvature

\[ \lim_{n \to +\infty} \frac{1}{2\pi} \int_{\mathbb{R}^2} K(x)e^{2u_n(x)}dx = \frac{2-b}{2} < 1. \]

Since \( K \) has a lower positive bound in any compact set of \( \mathbb{R}^2 \), by Theorem 3.1, we have either \( u_n \) is uniformly bounded in any compact set or \( u_n \) is uniformly convergent to \( -\infty \) in any compact set of \( \mathbb{R}^2 \).

**Step 1.** We claim that \( u_n \to -\infty \) uniformly in any compact set of \( \mathbb{R}^2 \). Suppose \( u_n \) is uniformly bounded in any compact set of \( \mathbb{R}^2 \). By the elliptic estimates, we may assume \( u_n \to u \) in \( W^{2,p}_\text{loc}(\mathbb{R}^2) \) for any \( p > 1 \). In particular, \( u \) satisfies (1.1) and the total curvature

\[ \frac{1}{2\pi} \int K(x)e^{2u(x)}dx \leq \lim_{n \to +\infty} \frac{1}{2\pi} \int K(x)e^{2u_n(x)}dx = \frac{2-b}{2}, \]

which yields a contradiction by Remark 2.2. Hence, by Theorem 3.1, we have \( u_n \to -\infty \) uniformly in any compact set of \( \mathbb{R}^2 \).

**Step 2.** We claim there exists a constant \( C > 0 \) such that

\[ K(x)e^{2u_n(x)} \leq C|x|^{-2} \quad \text{for } x \in \mathbb{R}^2. \]

To prove the claim, we assume there exists \( x_n \in \mathbb{R}^2 \) such that \( u_n(x_n) + \frac{(2-b)}{2} \log |x_n| \to +\infty \). By Step 1, we have \( |x_n| \to +\infty \) as \( n \to +\infty \). Set

\[ v_n(y) = u_n(x_n + |x_n|y) + \frac{2-b}{2} \log |x_n|. \]
Then \( v_n \) satisfies

\[
\Delta v_n + K_n(y)e^{2v_n(y)} = 0 \quad \text{in} \quad |y| < \frac{1}{2},
\]

where \( K_n(y) = |x_n|^{\beta}K_n(x_n + |x_n|y) \). By the assumption on \( K, 0 < \tilde{C}_1 \leq K_n(y) \leq \tilde{C}_2 \) for \( |y| < \frac{1}{2} \), and

\[
\int_{|y| < \frac{1}{2}} K_n(y)e^{2v_n(y)} \, dy \leq \int_{\mathbb{R}^2} K(x)e^{2u_n(x)} \, dx < 2\pi.
\]

By Theorem 3.1, we conclude that \( v_n(0) \leq C \) for some constant \( C \), which yields a contradiction to the assumption.

**STEP 3.** There exists a positive constant \( C \) such that \( |\nabla u_n(x)| \leq C|x|^{-1} \) and \( |u_n(x) - u_n(y)| \leq C \) for \( |x| = |y| \).

In [CLn], we have proved that \( u_n \) has the following representation

\[
(3.7) \quad u_n(x) = u_n(0) + \frac{1}{2\pi} \int_{\mathbb{R}^2} \log \frac{|y|}{|x-y|} K(y)e^{2u_n(y)} \, dy.
\]

Thus, we have

\[
|\nabla u_n(x)| \leq \frac{1}{2\pi} \int_{\mathbb{R}^2} |x-y|^{-1} K(y)e^{2u_n(y)} \, dy
\]
\[
= \frac{1}{2\pi} \int_{|y-x| \leq \frac{|x|}{2}} |x-y|^{-1} K(y)e^{2u_n(y)} \, dy
\]
\[
+ \frac{1}{2\pi} \int_{|y-x| \geq \frac{|x|}{2}} |x-y|^{-1} K(y)e^{2u_n(y)} \, dy.
\]

By Step 2, the first integral can be estimated by

\[
\frac{1}{2\pi} \int_{|y-x| \leq \frac{|x|}{2}} |x-y|^{-1} K(y)e^{2u_n(y)} \, dy \leq C_1|x|^{-2} \int_{|y-x| \leq \frac{|x|}{2}} |y-x|^{-1} \, dy = C_2|x|^{-1}.
\]

For the second integral, we have

\[
\frac{1}{2\pi} \int_{|y-x| \geq \frac{|x|}{2}} |x-y|^{-1} K(y)e^{2u_n(y)} \, dy \leq \frac{1}{\pi|x|} \int_{\mathbb{R}^2} K(y)e^{2u_n(y)} \, dy.
\]

Combined these two estimates together, we have

\[
|\nabla u_n(x)| \leq C_3|x|^{-1}.
\]

Set \( w_n(x) = e^{2u_n(x)} \). Then \( w_n(x) \) satisfies

\[
(3.8) \quad \Delta w_n(x) + 4(Ke^{2u_n} + |\nabla u_n|^2)w_n = 0.
\]
Since $K(x)e^{2u_n(x)} + |\nabla u_n|^2 \leq C_4|x|^{-2}$ for some constant $C_4$, by Harnack inequality, for any $a \geq 1$, there exists a positive constant $C_5 = C_5(a)$ such that

$$\sup_{a^{-1}r \leq |x| \leq ar} w_n(x) \leq C_5 \inf_{a^{-1}r \leq |x| \leq ar} w_n(x).$$

Hence, Step 3 is proved.

**STEP 4.** For any $\varepsilon > 0$ there exists $R = R(\varepsilon) > 0$ such that $|u_n(x) - u_n(y)| \leq \varepsilon$ for $|x| = |y| \geq R_\varepsilon$ and large $n$.

Step 4 will be proved by contradiction. Suppose there exist a positive number $\varepsilon_0 > 0$ and $x_n, y_n$ with $r_n = |x_n| = |\tilde{x}_n| \to +\infty$ such that $u_n(\tilde{x}_n) - u_n(x_n) \geq \varepsilon_0$. Let

$$v_n(y) = u_n(r_ny) - u_n(x_n).$$

Then $v_n$ satisfies

$$\Delta v_n + K_n(y)e^{2v_n} = 0,$$

where $K_n(y) = e^{2u_n(x_n)}K(r_ny)r_n^2$. By Step 2,

$$K_n(y) \leq C_1 e^{2u_n(x_n)}r_n^{2-b}|y|^{-b} \leq C_2|y|^{-b}.$$

For $|y| \geq 1$, we have

$$K_n(y) \geq C_3 e^{2u_n(x_n)}r_n^{2-b}|y|^{-b}.$$

By Step 3 and the Harnack inequality (3.9), $v_n(y)$ is bounded in $L^\infty_{\text{loc}}(\mathbb{R}^2)$. By the elliptic estimates, we may assume $v_n(y) \to v_0(y)$ in $W^{2,p}_{\text{loc}}(\mathbb{R}^2)$ for any $p > 1$. Suppose there exists a subsequence of $x_n$ (still denoted by $x_n$) such that $\lim_{n \to +\infty} e^{2u_n(x_n)}r_n^{2-b} = S > 0$, then by (3.10) and (3.11), we may assume $K_n(y) \to K_0(y)$ weakly in $L^\infty_{\text{loc}}(\mathbb{R}^2 \setminus \{0\})$, where $K_0(y)$ satisfies

$$C_1|y|^{-b} \leq K_0(y) \leq C_2|y|^{-b}$$

for some positive constants $C_1$ and $C_2$, and $v_0(y)$ satisfies

$$\Delta v_0(y) + K_0(y)e^{2v_0(y)} = 0 \text{ in } \mathbb{R}^2 \setminus \{0\}.$$

For any $0 < r_0 < r_1$, we have

$$\int_{r_0 \leq |y| \leq r_1} K_n(y)e^{2v_0(y)}dy = \lim_{n \to +\infty} \int_{r_0 \leq |y| \leq r_1} K_n(y)e^{2v_n(y)}dy$$

$$= \int_{r_0 \leq |y| \leq r_1} K(x)e^{2u_n(x)}dx$$

$$\leq \int_{\mathbb{R}^2} K(x)e^{2u_n(x)}dx$$

$$\to \left(\frac{2-b}{2}\right)2\pi$$

$$\to \left(\frac{2-b}{2}\right)2\pi$$
Thus, the total curvature

\[ \frac{1}{2\pi} \int_{\mathbb{R}^2} K_0(y)e^{2\nu_0(y)}\,dy \leq \frac{2 - b}{2}. \]

Applying Corollary 1.4 in [CLn], \( \nu_0(y) \) in fact satisfies

(3.13) \[ \Delta \nu_0(y) + K_0(y)e^{2\nu_0(y)} = 2\pi \beta \delta(0) \quad \text{in} \quad \mathbb{R}^2 \]

for some \( \beta \in \mathbb{R} \), where \( \delta(0) \) is the Dirac measure at the origin, and the function \( \nu_1(y) = \nu_0(y) - \beta \log |y| \) satisfies

(3.14) \[ \Delta \nu_1(y) + K_0(y)|y|^{2\beta}e^{2\nu_1(y)}dy = 0 \quad \text{in} \quad \mathbb{R}^2. \]

It is easy to see that

\[
o(1) + \beta = \frac{1}{2\pi} \int_{|y|=r} \frac{\partial \nu_0}{\partial v}(y)d\sigma = \lim_{n \to +\infty} \frac{1}{2\pi} \int_{|y|=r} \frac{\partial \nu_0}{\partial v}(y)d\sigma = \lim_{n \to +\infty} \frac{-1}{2\pi} \int_{|x| \leq r} K_n(y)e^{2\nu_0(y)}dy = \lim_{n \to +\infty} \frac{-1}{2\pi} \int_{|x| \leq r} K_n(y)e^{2\nu_0(y)}dx, \]

where \( o(1) \) denotes \( o(1) \to 0 \) as \( r \to 0 \). Thus, putting (3.12) and the above together, we have

\[
\frac{1}{2\pi} \int_{\mathbb{R}^2} K_0(y)|y|^{2\beta}e^{2\nu_1(y)}dy = \frac{1}{2\pi} \int_{\mathbb{R}^2} K_0(y)e^{2\nu_0(y)}dy \leq \frac{2 - b}{2} + \beta = \frac{2 - b + 2\beta}{2}. \]

Obviously, \( 2 - b + 2\beta > 0 \). Since \( K_0(y)|y|^{2\beta} \sim |y|^{-b+2\beta} \) at \( \infty \), by Remark 2.2, there exists no entire solution (3.14) with the total curvature equal to \( \frac{2 - b + 2\beta}{2} \). Thus, it yields a contradiction. Hence we have proved

\[ \lim_{n \to +\infty} e^{2u_n(x)} = 0. \]

Since \( e^{2u_n(x)}r_n^{-2\beta} \to 0 \) as \( n \to +\infty \), then \( \nu_0(y) \) is harmonic in \( \mathbb{R}^2 \setminus \{0\} \).

By Step 3,

\[ |\nabla \nu_n(y)| = r_n|\nabla u_n(r_ny)| \leq C|y|^{-1}. \]

By Liouville’s theorem, we have

\[ \nu_0(y) = \alpha_0 \log |y| + C, \]
where both $\alpha_0$ and $C$ are constant. Since $v_0(y)$ is radially symmetric, it obviously yields a contradiction to the assumption. Hence, Step 4 is proved.

**STEP 5.** Set

$$F_n(r) = \int_{B_r} K(x) e^{2u_n(x)} dx,$$

and

$$\bar{u}_n(r) = \frac{1}{2\pi r} \int_{|x|=r} u(x) ds.$$  

Define $\tilde{K}_n(r)$ by

$$\tilde{K}_n(r) = (2\pi r)^{-1} e^{-2\bar{u}_n(r)} \int_{|x|=r} K(x) e^{2u_n(x)} ds.$$

Differentiating (3.15) and (3.16) with respect to $r$, we have

$$F'_n(r) = (2\pi r) \tilde{K}_n(r) e^{2\bar{u}_n(r)}$$

$$\bar{u}'_n(r) = \frac{1}{2\pi r} \int_{B_r} K(x) e^{2u_n(x)} dx = \frac{-F_n(r)}{2\pi r}.$$

Thus, we have

$$\left( \frac{r^{1-b} F'_n(r)}{\tilde{K}_n(r)} \right)' = (2\pi r^{2-b} e^{2\bar{u}_n(r)})'$$

$$= 2\pi [-(2-b)r^{1-b} e^{2\bar{u}_n(r)} + 2r^{2-b} e^{2\bar{u}_n(r)} \bar{u}'_n(r)]$$

$$= \frac{(2-b)F'_n}{r^b \tilde{K}_n} - \frac{F_n F'_n(r)}{\pi r^b \tilde{K}_n}$$

$$= \frac{-F'_n(r)}{r^b \tilde{K}_n} \left[ \frac{F_n(r)}{\pi} - (2-b) \right].$$  

Since $F_n(\infty) > \pi(2-b)$, set $r_n$ to satisfy $F_n(r_n) = \pi(2-b)$. Obviously, $\lim_{n \to +\infty} r_n = +\infty$. For any $\varepsilon > 0$, by Step 4, there exists $R = R(\varepsilon) > 0$ such that

$$A e^{2\varepsilon} r^{-b} \leq \tilde{K}_n(r) \leq e^{-2\varepsilon} B r^{-b} \quad \text{for} \quad r \geq R(\varepsilon).$$

Hence,

$$\left( \frac{r^{1-b} F'_n(r)}{\tilde{K}_n(r)} \right)' \geq \left\{ \begin{array}{ll} \frac{e^{2\varepsilon}}{B} \left( 2-b - \frac{F_n(r)}{\pi} \right) F'_n(r) & \text{for} \quad R \leq r \leq r_n, \\ \frac{e^{-2\varepsilon}}{A} \left( 2-b - \frac{F_n(r)}{\pi} \right) F'_n(r) & \text{for} \quad r \geq r_n. \end{array} \right.$$
Since \( \lim_{r \to +\infty} \frac{r F_n'(r)}{K_n(r)} = 0 \) for any \( n \), we have

\[
- \frac{r^{1-b} F_n'(r)}{K_n(r)} \bigg|_{r=R} \geq \frac{e^{-2\varepsilon}}{B} \int_R^{\infty} \left( 2 - b \right) - \frac{F_n(r)}{\pi} F_n'(r) dr \\
+ \frac{e^{2\varepsilon}}{A} \int_{r_n}^{\infty} \left( 2 - b \right) - \frac{F_n(r)}{\pi} F_n'(r) dr \\
= -e^{-2\varepsilon} B^{-1} \left[ \left( 2 - b \right) F_n(r) - \frac{F_n^2(r)}{2\pi} \right] \bigg|_{r=R} \\
+ \left( \frac{e^{2\varepsilon}}{B} - \frac{e^{-2\varepsilon}}{A} \right) \frac{\pi (2-b)^2}{2} + e^{-2\varepsilon} A^{-1} \left( 2 - b \right) F_n(\infty) - \frac{F_n^2(\infty)}{2\pi}.
\]

By Step 1, we note that the boundary term at \( R \) tends to 0 as \( n \to +\infty \). By letting \( n \to +\infty \) first and then \( \varepsilon \to 0 \) the above yields

\[
0 \geq \left( \frac{1}{B} - \frac{1}{A} \right) \frac{\pi (2-b)^2}{2} + \frac{1}{A} \left[ (2-b) \lim_{n \to +\infty} F_n(\infty) - \frac{\lim_{n \to +\infty} F_n^2(\infty)}{2\pi} \right] = \frac{\pi (2-b)^2}{2B},
\]

a contradiction, where \( \lim_{n \to +\infty} F_n(\infty) = (2-b)\pi \) is used. Therefore, the proof of Theorem 1.2 is completely finished.

**Proof of Theorem 1.3.** Suppose \( u_n \) is a sequence of solution of (1.1) and satisfies the assumption of Theorem 1.3. By Remark 3.2, we may assume that either \( u_n \) is uniformly bounded in any compact set or \( u_n \) uniformly converges to \( -\infty \) in any compact set of \( \mathbb{R}^2 \). By the the same reasoning of Step 1 and Step 2 of Theorem 1.2, there exists a constant \( C > 0 \) such that inequalities

\begin{align}
(3.20) \quad K(x) e^{2u_n(x)} & \leq C |x|^{-2}, \\
(3.21) \quad |\nabla u_n(x)| & \leq C |x|^{-1}, \\
(3.22) \quad |u_n(x) - u_n(y)| & \leq C \quad \text{whenever} \quad |x| = |y|
\end{align}

hold.

First, we want to prove \( u_n \) is bounded in \( L^\infty_{\text{loc}}(\mathbb{R}^2) \). Suppose the claim is not true. As before, we want to prove the asymptotic symmetry of \( u_n \), i.e. for any \( \varepsilon > 0 \), there exists \( R = R(\varepsilon) > 0 \) such that for \( |y| = |x| \geq R \), \( |u_n(x) - u_n(y)| \leq \varepsilon \). Assume the conclusion is not true. Then there exists \( r_n \to +\infty \) such that \( u_n(\tilde{x}_n) \geq u_n(x_n) + \varepsilon_0 \) with \( |\tilde{x}_n| = |x_n| = r_n \) for some positive constant \( \varepsilon_0 > 0 \). Let

\[
u_n(y) = u_n(r_n y) - u_n(x_n).
\]
Then \( v_n \) is bounded in \( L^\infty_{\text{loc}}(\mathbb{R}^2 \setminus \{0\}) \) by Harnack inequality and satisfies

\[
\Delta v_n + K_n(y)e^{2vn} = 0 \quad \text{in} \quad \mathbb{R}^2,
\]

where \( K_n(y) = e^{2u_n(x_n)}K(r_n y) r_n^2 \). By the assumption on \( K \) and (3.20) for any \( r_0 > 0 \), we have for \( |y| \geq r_0 \),

\[
K_n(y) \leq 2e^{2u_n(x_n)}r_n^{2-b}|y|^{-b}
\]

for large \( n \). If \( \lim_{n \to +\infty} e^{u_n(x_n)}r_n^{2-b} = 0 \), then using (3.21) and the same argument of Step 4 of Theorem 1.2, \( v_n(y) \) converges to \( v_0(y) = \alpha_0 \log |y| + C_0 \) in \( L^\infty_{\text{loc}}(\mathbb{R}^2 \setminus \{0\}) \) where \( \alpha_0 \) and \( C_0 \) are constant. Since \( v_0(y) \) is radially symmetric, it yields a contradiction.

If \( \lim_{n \to +\infty} e^{2u_n(x_n)}r_n^{2-b} = s > 0 \), then \( K_n(y) \to s|y|^{-b} \) uniformly in any compact set of \( \mathbb{R}^2 \). Then \( v_0(y) \) satisfies

\[
\begin{cases}
\Delta v_0(y) + s|y|^{-b}e^{2v_0(y)} = \beta\delta(0) & \text{in} \quad \mathbb{R}^2, \\
v_0(y) = \frac{\beta}{2\pi} \log |y| + O(1) & \text{as} \quad y \to 0,
\end{cases}
\]

where \( \beta \in \mathbb{R} \) and \( \delta(0) \) is the Dirac measure. For any \( r_0 > 0 \),

\[
\int_{|y|=r_0} \frac{\partial v_0(y)}{\partial n} d\sigma = \lim_{n \to +\infty} \int_{|y|=r_0} \frac{\partial v_n}{\partial n} d\sigma = - \lim_{n \to +\infty} \int_{|y|\leq r_0} K_n(y)e^{2vn} dy \leq 0.
\]

Thus either \( v_0(y) \) is regular at 0 or \( v_0(y) \to +\infty \) as \( |y| \to +\infty \). Since \( |y|^{-b}e^{2v_0(y)} \in L^1(\mathbb{R}^2) \) and \( v_0(y) = \alpha \log |y| + O(1) \) as \( |y| \to +\infty \), for some \( \alpha \in \mathbb{R} \), we have \( 2\alpha - b < -2 \), i.e. \( |y|^{-b}e^{2v_0(y)} = o(1)|y|^{-2} \) as \( |y| \to +\infty \). Hence, we can apply the method of moving planes as in [CL] and [CLn] to prove \( v_0(y) \) is radially symmetric with respect to the origin, which obviously yields a contradiction. Hence the uniformly asymptotic symmetry of \( u_n \) is proved.

To finish the proof of Theorem 1.3, we set

\[
F_n(r) = \int_{B_r} K(x)e^{2u_n(x)} dx,
\]

and,

\[
\tilde{K}_n(r) = (2\pi r)^{-1}e^{-2u_n(r)} \int_{|x|=r} K(x)e^{2u_n(x)} dx.
\]

As in (3.19), we have

\[
(3.23) \quad \left( \frac{r^{1-b}F'_n(r)}{K_n(r)} \right)' = \frac{-F'_n(r)}{r^b K_n(r)} \left[ \frac{F_n(r)}{\pi} - (2 - b) \right].
\]
For any $\varepsilon > 0$, let $R = R(\varepsilon)$ be large such that

$$e^{-3\varepsilon} r^{-b} \leq \tilde{K}_n(r) \leq e^{3\varepsilon} r^{-b}$$

holds for $r \geq R$. This immediately follows from the uniformly asymptotic symmetry of $u_n$ and the assumption on $K$. Let $r_n$ satisfy $F_n(r_n) = \pi(2 - b)$. Suppose $\lim_{n \to +\infty} \frac{1}{2\pi} \int_{\mathbb{R}^2} K(x)e^{2u_n(x)}dx < (2 - b)$ first. Then we can follow the same proof as Step 5 in Theorem 1.2 to obtain

$$(2 - b)F(\infty) - (2\pi)^{-1} F^2(\infty) \leq 0,$$

where $F(\infty) = \lim_{n \to +\infty} \int_{\mathbb{R}^2} K(x)e^{2u_n(x)}dx$. Obviously, the above yields $F(\infty) \geq 2\pi(2 - b)$, a contradiction.

Suppose $\lim_{n \to +\infty} \int_{\mathbb{R}^2} K(x)e^{2u_n(x)}dx > 2\pi(2 - b)$. Then by (3.23), we have the reverse inequality

$$\left( \frac{r^{1-b} F_n'(r)}{\tilde{K}_n(r)} \right)' \leq \begin{cases} e^{3\varepsilon} [(2 - b) - F_n(r)/\pi] F_n'(r) & \text{for } R \leq r \leq r_n, \\ e^{-3\varepsilon} [(2 - b) - F_n(r)/\pi] F_n'(r) & \text{for } r \geq r_n. \end{cases}$$

Integrating the above and letting $n \to +\infty$ first and then $\varepsilon \to 0$, we have

$$(2 - b)F(\infty) - \frac{F^2(\infty)}{2\pi} \geq 0,$$

which implies

$$F(\infty) \leq 2\pi(2 - b).$$

Obviously, it yields a contradiction. Hence the boundedness of $u_n$ in $L^\infty_{\text{loc}}(\mathbb{R}^2)$ is proved.

To prove (1.9), we may assume $u_n \to u_0$ in $W^{2,p}_{\text{loc}}(\mathbb{R}^2)$ for any $p > 1$. Obviously, $u_0$ satisfies (1.1) and has a finite total curvature. In particular,

$$\frac{1}{2\pi} \int_{\mathbb{R}^2} K(x)e^{2u_0(x)}dx > \frac{2 - b}{2}.$$

Hence, there exists $R_0 > 0$, $\varepsilon_0 > 0$ and $n_0$ such that

$$\frac{1}{2\pi} \int_{B_r} K(x)e^{2u_n(x)}dx > \left( \frac{2 - b}{2} + \varepsilon_0 \right)$$

for all $r \geq R_0$ and $n \geq n_0$. Integrating (1.1), we have

$$\frac{d}{dr} \tilde{u}_n(r) < - \left( \frac{2 - b}{2} + \varepsilon_0 \right) r^{-1}.$$
for all $r \geq R_0$ and $n \geq n_0$, where $ar{u}_n(r) = \frac{1}{2\pi r} \int_{|x|=r} u_n(x) d\sigma$. Thus,

$$
\bar{u}_n(r) \leq \bar{u}_n(R_0) - \left( \frac{2 - b}{2} + \varepsilon_0 \right) \log r / R_0.
$$

Applying the Harnack inequality, we have

$$
u_n(x) \leq \bar{u}_n(|x|) + C_1 \leq C_2 - \left( \frac{2 - b}{2} + \varepsilon_0 \right) \log r
$$

for $r \geq R_0$ and $n \geq n_0$ where $C_1$ and $C_2$ are constants independent of $n$ and $r$. In particular,

$$
\int_{|y| \geq r} K(x) e^{2\bar{u}_n(x)} dx \leq C_3 \int_{|y| \geq r} |x|^{-(2+2\varepsilon_0)} dx
$$

could be arbitrarily small provided that $r$ is large. Thus, (1.9) follows immediately. And the proof of Theorem 1.3 is finished. \qed

\begin{thebibliography}{99}


[H] Z.-C. Han, Prescribing Gaussian curvature on $S^2$, Duke Math. J. 61 (1990), 679-703.


\end{thebibliography}